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EVALUATION OF STRATEGIES OF PESTICIDE USE AND BIOLOGICAL CONTROL THROUGH LINEAR FEEDBACK CONTROL FOR CONTROLLING RAPIDLY GROWING PEST POPULATION

PANKAJ GULATI[1], SUDIPA CHAUHAN[1,*], ANUJ MUBAYI[2,3]

[1]Department of Mathematics, Amity Institute of Applied Science, Amity University, Noida, U.P, India
[2]The Public Health Company, Los Angeles, CA, USA
[3]Department of Mathematics, Illinois State University, Normal, Illinois, USA

Abstract. In agricultural management and ecological research, consideration of the impacts and risks of pests on the dynamics of crop growth has been introduced in the literature using process-based models at different ecological levels with varied usefulness. In this study, we attempt to overcome the selected limitations of some existing process-based models while (i) systematically developing coupled pest-crop systems, (ii) evaluating the results under the application of various types of interventions, and (iii) comparing the analysis with similar studies in the literature. The novelty of the paper lies in the consideration of a continuous system with discrete-time treatments. In particular, we have established the long-term behavior of two modeling frameworks capturing the growth of the crop infested with a different type of pests and different pesticide application strategies to control the exponentially growing pest population. In the first pesticide application strategy, pesticides are sprayed at fixed time intervals whereas, under the second strategy, pesticides are implemented when the pest population reaches Economic Threshold (ET) in pest abundance. Conditions on critical pest population size when single treatment and multiple treatments of pesticides in both the modeling frameworks have been discussed. The optimal timing of pesticide implementation, the optimal dosage of pesticide, the economic threshold of pests, and the threshold of pest survival rate have been obtained (both mathematically and numerically) to maximize the profit from crops.

*Corresponding author
E-mail address: sudipachauhan@gmail.com
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Further, we have also extended the model and the exponentially growing pest population is optimized by biological control linear feedback control to reduce the pest population to a desired economic threshold value. The numerical analysis validating analytical results is discussed for both the cases of chemical and biological control. Finally, using the sensitivity analysis technique, sensitive model parameters affecting the optimal dosage, optimal time, and optimal survival rate are identified in the case of chemical control. The results show that schematic implementation of complex pesticide control and biological control measures to reduce the harm brought by pests to crops will have significant implications in agriculture research.

Keywords: exponential growth of pests; optimal insecticide dosage; profit from crop yield; economic threshold of pest size; pest resistance; natural enemy; linear feedback control; dynamic model of pest; mathematical analysis; global sensitivity.

2010 AMS Subject Classification: 34D20, 92D40.

1. INTRODUCTION

Agricultural production has been on an exponential path reaching great heights every year. Benefits from such products have been enhanced by the use of pesticides. Higher agricultural production has gradually increased the economic growth of countries but studies have shown that excessive use of insecticides and pesticides by the farmers may have deteriorating health and financial impact. Thus, identifying the correct amount and frequency of pesticide use has been one of the big challenges for farmers, potentially due to unawareness of proper structure in use of insecticide for different types of crops.

1.1. Theoretical and empirical literature on pest control.

In the literature, various mathematical models have been studied to get insight into the dynamics of a pest-crop-insecticide system with limited success in some cases [1, 2, 3, 4]. For example, in a study by Talpaz et al. in [5], a mathematical model is analyzed on pest control in which the authors assumed exponential growth in pest population with multiple pesticide treatment regimes and Weibull distributed pest killing rate. The same authors proposed a different modeling-based study in which they estimated parents and analyzed the impact of boll weevil insect on cotton plant system [6]. They further evaluated different pesticide control schemes and numerically identified an optimal control strategy via a non-linear dynamic optimization model. Later, Liang et al(2010) proposed another dynamical model and identified the optimal timing and dosage of pesticide application under different scenarios while maximizing the profit (from crops) function in [3]. They considered logistic growth of pest population by spraying pesticides at regular intervals but ideally logistic growth rate is considered under data-driven laboratory scenario where food and environmental condition are kept constant [7]. In a real scenario, it is quite evident that the pest population grows exponentially in various places due to favorable (to the pest) environmental conditions and food availability for pests which eventually destroys the whole crop. This situation is significantly visible in crops like brinjal and potato which suffers from Eggplant fruit and shoot borer (EFSB) pest. The situation gets worsens if these pests additionally develop resistance to the pesticides. *There are several theoretical and empirical evidence that support our assumption that the pest population generally follows exponential growth rate Table [2].*

Further, generalizing a statement that exponential growth rate is a special case of logistic growth is an incorrect statement as both the cases may lead to different dynamics and further one of the cases can be more close to realism in comparison to the other and may have less mathematical complexity which can give a better insight of the system. Therefore, in this paper, the model is formulated with an assumption that abundant resources are present in favor of pest growth and hence there is no consideration of carrying capacity. For instance, Gordillo (2015) studied persistence and eradication conditions in a deterministic model for sterile insect release in an exponentially growing pest population. In this paper, authors derived and explored numerically deterministic population models that include sterile release together with scarce mating encounters in the particular case of species with long lifespan and multiple mating. In the presence of density dependence regulation, it is observed that sterile release might contribute to inducing sudden suppression of the pest population.

In another research by Bhattacharyya et al. (2013) in [8], oscillation in Pest Population and Its Management was studied by mathematical modeling where the logistic growth rate of pest population was considered. To quote a few more researchers, Grasman et al. (2001) developed a mathematical model in [9] where they explicitly mentioned that at the beginning of any season, the crop is always without pest population and the pest has to detect the crop and as soon as it detects, it enters the phase of exponential growth and that is the best time when the farmers should be able to detect it so that they can control their growth by either biological or chemical control. Again, in 2006, DeStefano et al.studied an exponentially growing sub-adult beaver's (Castor canadensis) transition to independence in [10]. Beaver feeds farm crops especially soybeans, and corn during the summer and spring seasons as mentioned in [11]. Later Costamagn et al.in [12] studied an exponential growth of Aphids glycines Matsumura (Hemiptera: Aphididae) where they mentioned its capacity towards rapid exponential growth which is mainly responsible to reach outbreak populations on crops. Again Andrea Maiorano in [1] studied a physiologically based approach for degree-day calculation in pest phonology models where he explicitly considered the case of European Corn Borer (Ostrinia nubilalis Hbn.) which is a moth pest of cereal crops, particularly and has invaded most of the United States and Canada. The major reason for its exponential growth is the favorable temperature. Further, the scenario of exponential growth rate can be visible significantly where the drastic climate factors are majorly contributing to the sudden increase in crop losses as the elevation in temperature not only boosts the reproductive rates of pests but also increases metabolic rates of pests exponentially leading to damage of crops.

According to the UN Food and Agriculture Organization (2016), the majority of crops affected this way are Corn, rice, and wheat which are stapled crops for about 4 billion people worldwide and United Nations has estimated that at least 815 million people worldwide may be deprived of food due to this problem. Another example is potato crops which are liable to be attacked by potato beetles, aphids spread viruses, and blight. Due to the favorable weather conditions of high humidity and abundant spring rains for the survival of pests, the risk of damage to crops is very high [10]. These uncertain variations in climatic factors prevailing worldwide, which are mostly uncontrolled factors via human interventions, make pesticide implementation strategies evaluation a serious part of modeling exercise. Hence, it becomes the key reason for studying the impact of various demographic and intervention-related mechanisms via systematic and proper mathematical modeling process and ensuring comprehensive evaluation of pest resistance management programs [13]. Therefore, there exist substantial evidence supporting our assumption of exponential growth rate. Later, several researchers [14, 1, 4, 15] also worked on the dynamics of various mathematical models using impulsive differential equations to study the control strategy via implementation of pesticides.

1.2. Goals of the present study. Motivated by the aforementioned challenges in the literature, in this study we aim to systematically evaluate the dynamics of pest population under different control strategies (chemical and biological) in reference to [3] using the exponential growth rate of the pest population. We will obtain the optimal timing and optimal dosage for each of the cases when the pesticides are sprayed once and multiple times. In addition, we will also obtain the optimal biological control using LQR. We will find the explicit solution for all the cases analytically. The highlighted differences in the analytical and numerical solutions for exponential and logistic growth in the case of chemical control will also be mentioned in Table 8 in the supplementary material for reference. Further, to an extension, we will not only calculate the results numerically but also show comparative results for each of the cases with the benefit of choosing one strategy in comparison to the other in the case of chemical control. Finally, the global sensitivity analysis for each of the cases by the method of Latin hypercube sampling (LHS) scheme will be done for the sensitive parameters of optimal dosage of pesticide and optimal time of spray.

1.3. Structure of the paper. The organization of the paper is as follows: In Section 2, the profit from crop yield and pest kill function is defined followed by a description of mathematical models. The mathematical analysis of the model for all scenarios is shown in Section 3. Further, numerical simulations for different pesticides and environmental conditions are discussed via hypothetical scenarios and the results of the models are compared in Section 4 followed by a global sensitivity analysis using the method of Latin hyper-cube sampling (LHS) scheme and the interpretation of the results and implications from optimal strategies are extracted in this section. Finally, the results are concluded in the discussion in Section 5. In the appendix, we have tried to give a detailed explanation of the uncertainty analysis mentioned in section 4.

TABLE 1. Literature supporting exponential growth rate of pest population

Model	Objective/Results	Excludes(in comparison to our paper)
1. Two-component exponential Model for a host-parasitoid interaction [9]	Transient dynamics and population crashes of this system are analysed using differential inequalities and has been found that population suppression of the host population can, in principle, be attained, but only if the initial ratio of hosts to parasitoids does not exceed the ratio of maximum host handling rate by one parasitoid and minimum host growth rate.	Optimal control
2. Exponential growth model of Aphis glycines [12]	Experimental study of A.glycines during population growth and decline under predator free condition in three soybean field from 2003-06 for five data sets have demonstrated that an exponential growth model, with r decreasing linearly with time, gives a much better description of A. glycines dynamics for all data sets than the exponential or logistic growth	Mathematical dynamical system Analysis and Optimal control

TABLE 2. Literature supporting exponential growth rate of pest population

Model and References	Objective/Results	Excludes(in comparison to our paper)
3. Importance of population dynamics in heterogeneous landscapes: Management of vertebrate pests and some other animals [16]	Demonstrated the importance of considering dynamic aspects of populations when attempting management (e.g. pest control) of vertebrate populations and have concluded a list of features which ought to be examined before species' pest status can be evaluated and management programs are planned	Optimal control
4. Managing Wildlife Damage from Beavers (Castor canadensis) [11]	Experimental study on beavers their biology and behavior, importance and various biological measures to control beavers population.	Optimal control
5. Experimental study of testing mechanistic growth model [17]	Increasing specific assimilation during the growth phase can explain the near-exponential growth trajectory of insects.	Mathematical dynamical system Analysis and Optimal control.
6. Pest persistence and eradication conditions in a deterministic model for sterile insect release [18]	Derived and explored numerically deterministic population models that include sterile release and it has been observed that sterile release might contribute to induce sudden suppression of the pest population.	Optimal control

2. MATHEMATICAL MODEL

The model is formulated with an assumption that all pests are able to reproduce, resources are unlimited and there is continuous reproduction. This assumption is due to favorable environmental conditions for the pest population which is enhancing their growth rate.

$$N'(t) = \frac{dN(t)}{dt} = rN(t), \tag{1}$$

where r denotes the growth rate of pest population , $N(t)$ is the pest population size at any time t.

It is generally acknowledged that the measure of pest killed by the use of the pesticide at time t depends on the pest population $N(t)$ and the number of pesticides, X, used. So, the total number of pest killed is represented by the kill function $K(X,N(t))$.

$$K(X,N(t)) = N(t)^{\gamma}F(X) \tag{2}$$

where γ is fixed and is greater than zero, the kill efficiency function is denoted by $F(X)$, which fulfills the conditions $F(0) = 0$, $\lim_{X\to\infty} F(X) \leq 1$. The parameters which would be used in this paper are explained in Table 3 .

TABLE 3. Meaning of variables/parameters

Variables / Parameters	**Meaning**
ET	Economic Threshold for pest population size
$K(X,N(t))$	Total number of pest killed
$F(X)$	Pest kill efficiency
P	Profit from crop yield ($)
H	Total harvest when pests have caused no damage (pounds)
D	Damage of crop due to pesticide (pounds)
t	Time variable in (days)
r	natural growth rate of pest population.
τ_k	Pesticide spraying time
X	Dosage of pesticides (pounds)
C	Pesticides set up cost for a single treatment in ($)
γ	a positive constant
β	Price per unit product ($)
α	Cost per unit pesticide ($)
N_0	Initial population of pest
t_h	Harvest time (days)
m	Per capita damage rate
p	Pest survival rate
P	Biological control
n	interaction rate in pest and biological control.
m_1	Death rate of Biological Control

The main aim of our model is to maximize the profit from the crop production while finding optimal timing and level of dosage of pesticide at which they should be sprayed. In our model, we have worked on exponential distribution function $F(X) = 1 - e^{-\lambda X}$ which is used as kill efficiency function. It is a continuous probability distribution which is used to model the time we need to wait before a given event occurs. This assumption is taken to model the time that when pesticide spray should be done. The profit function is given by

$$P = \beta H - \beta \sum_{0}^{n} D(\tau_i, \tau_{i+1}) - nC - \alpha \sum_{i=1}^{n} X_i \tag{3}$$

Here, $nC + \alpha \sum_{i=1}^{n} X_i$ is total cost of controlling the pest population. The optimal problem is $\max P\{\tau_i, X_i, ET\}$, here ET denotes the Economic Threshold for pest population size. i.e., we need to optimize the timing for spraying of pesticides, economic threshold value and dosage of pesticides in order to maximize our profit. In the next section, we will be studying the basic dynamical behavior of the model when the pesticides are sprayed at fixed time.

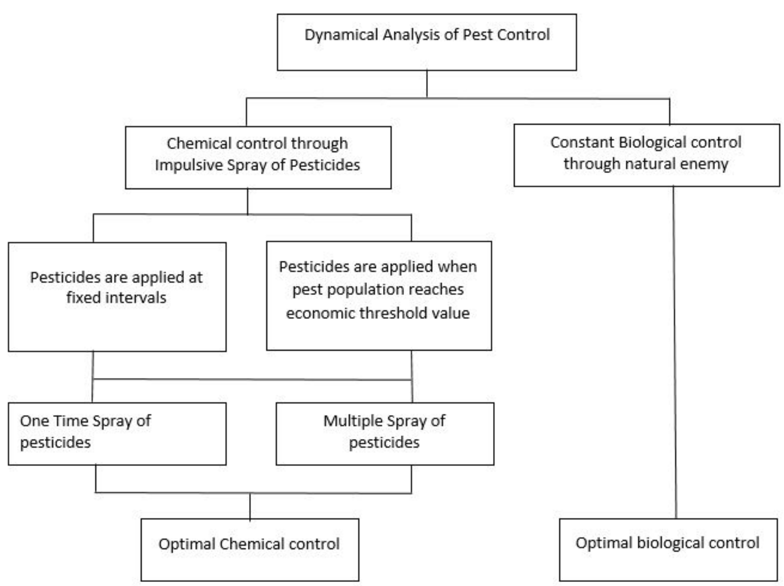


FIGURE 1. Flowchart

3. Mathematical Analysis

The derived model in Section 2 is analyzed here under different pesticide implementation strategies as described in the flow chart in the figure [1].

3.1. Time dependent implementation of pesticides. Basic dynamical behavior when the pesticides are sprayed at fixed intervals: We assume that the pest population is growing exponentially and at any time t and it is indicated by $N(t)$. We also assume that at time τ_k pests are killed by pesticides at relative rate q where ($0 \leq q \leq 1$). After the perturbation τ_k,the size of the pest population at time $N(\tau_k)$ becomes $N(\tau_k^+) = (1-q)N(\tau_k)$,where $k = 1,2,...(1-q)$. The pest survival rate is indicated by $(1-q)$ when pesticides are sprayed at time τ_k. Hence we have developed our model as :

$$\begin{cases} \frac{dN(t)}{dt} = rN(t), \; t \neq \tau_k \\ N(\tau_k^+) = (1-q)N(\tau_k), \qquad t = \tau_k \end{cases} \tag{4}$$

where r denotes the growth rate of pest population , q denotes the pest death rate and $N(0^+) = N_0$.Hence the solution of $N(t)$ starting with initial population N_0 as follows ;

$$N(t) = \begin{cases} N_0 e^{rt}, & 0 \leq t \leq \tau_1 \\ (1-q)N(\tau_k)e^{r(t-\tau_k)}, & \tau_k \leq t \leq \tau_{k+1}, \; k \in 1,2,... \end{cases} \tag{5}$$

If $\tau_{k+1} - \tau_k = T$ when $(\tau_0 = 0)$ for all $k \in W$, that is when pesticide are used with period T , then the sizes of pest population at control times τ_k fulfills the equation below:

$$N(\tau_{k+1}^+) = (1-q)N(\tau_k^+)e^{rT} \tag{6}$$

Now we will be discussing the case when pesticides are applied only once at fixed time.

3.1.1. ***Case1:*** *Pesticides are applied once at a fixed time.* In this case, we assume that the pesticides are applied only once with dosage X and at certain time τ_1 that is $k = 1$ in the system. From above section we can see that $qN(\tau_1) = K(X, N(\tau_1))$ and $N(\tau_1)$ is dependent on the parameters N_0, r, τ_1. Since, parameters N_0, r are generally considered to be constant for any given crop season and pests, at this point, kill function is indicated as follows:

$$K(X, N(\tau_1)) = K(X, \tau_1) \tag{7}$$

Hence, solution of (4) is ,

$$N(t) = \begin{cases} N_0 e^{rt}, & 0 \le t \le \tau_1 \\ (N_1 - K)e^{r(t-\tau_k)}, & \tau_1 \le t \le \tau_h \end{cases} \tag{8}$$

where $N_1 = N(\tau_1)$. Therefore, we get the damage as ,

$$D = \int_0^{\tau_1} mN(t)dt + \int_{\tau_1}^{t_h} mN(t)dt$$

$$= \frac{mN_0}{r}[e^{r\tau_1} - 1] + \frac{m(N_1 - K)}{r}[e^{r(t_h-\tau_1)} - 1] \tag{9}$$

Thus, profit using above equations becomes,

$$P = \beta H - \beta\left[\frac{mN_0}{r}[e^{r\tau_1} - 1] + \frac{m(N_1 - K)}{r}[e^{r(t_h-\tau_1)} - 1]\right] - \alpha X - C \tag{10}$$

Equation (10) shows that the maximal profit is dependent on two factors,that is,the optimal spraying time τ_1 and optimal dosage of pesticide X. To find both variables, we differentiate above equation with respect to X and τ_1 and hence the(10) becomes

$$P_X = \frac{m\beta}{r}\left[e^{(t_h-\tau_1)} - 1\right]K_X - \alpha \tag{11}$$

and

$$P_{\tau_1} = -\beta m\left\{N_0 e^{r\tau_1} + \frac{1}{r}\left(\frac{dN_1}{d\tau_1} - K_{\tau_1}\right)\left(e^{r(t_h-\tau_1)} - 1\right) - (N_1 - K)e^{r(t_h-\tau_1)}\right\} \tag{12}$$

where,

$$N_1 = N_0 e^{r\tau_1} \qquad therefore \qquad \frac{dN_1}{d\tau_1} = rN_0 e^{r\tau_1} = rN_1 \tag{13}$$

Solving the two equations taking $P_X = 0$ and $P_{\tau_1} = 0$ with respect to $K_X$and K_{τ_1} yields ,

$$K_X = \frac{r\alpha}{m\beta[e^{(t_h-\tau_1)} - 1]}$$

and ,

$$K_{\tau_1} = \frac{dN_1}{d\tau_1} - r\left(\frac{(N_1 - K)e^{r(t_h-\tau_1)} - N_0 e^{r\tau_1}}{e^{r(t_h-\tau_1)} - 1}\right)$$

Note that $K = K(X, \tau_1) = N_1^{\gamma} F(X)$ differentiating w.r.t. τ_1 gives

$$K_{\tau_1} = \frac{\gamma K}{N_1}\frac{dN_1}{d\tau_1} \tag{14}$$

Substituting the above values and letting $P_{\tau_1} = 0$, we get,

$$e^{r(t_h - \tau_1)} - 1 = \frac{1}{\gamma - 1} \tag{15}$$

Solving (15) for τ_1 shows $\tau_1^* = t_h - \frac{1}{r} \ln \frac{\gamma}{(\gamma - 1)}$

$$P_{\tau_1 \tau_1} = -\beta mrK[2\gamma - 1 - A(\gamma - 1)^2]$$

$$P_{\tau_1^* \tau_1^*} = -\beta mrK\gamma < 0$$

Further, $F(X) = 1 - e^{-\lambda X}$ and letting $P_X = 0$ and $t = t^*$ shows

$$X^* = \frac{-1}{\lambda} \ln \left[\frac{r\alpha(\gamma - 1)}{\lambda (N_1^*)^\gamma m\beta} \right]$$

where $N_1^* = N_0 e^{r\tau_1^*}$

$$P_{X^* X^*} = -r\lambda < 0$$

Therefore, $P(\tau_1^*, X^*)$ has the maximum value (2).

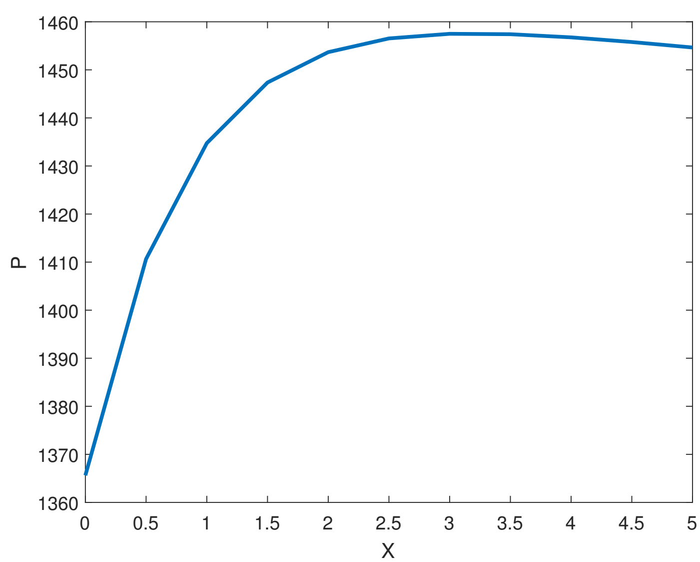


FIGURE 2. The existence of optimal dosage X^* and maximum profit P when pesticides are applied once at a fixed time.

3.1.2. ***Case 2:****When pesticides are applied n fixed times in harvest.* In this section we will be discussing the case when pesticides are applied at n fixed times. We assume that pesticides are sprayed at n times $\frac{t_h}{n+1}$,$\frac{2t_h}{n+1}$ .... $\frac{nt_h}{n+1}$. Then (4) model becomes,

$$\begin{cases} \frac{dN(t)}{dt} = rN(t), \quad t \neq \frac{it_h}{n+1} \quad where \quad i = 1,2...n \\ N\left(\frac{it_h^+}{n+1}\right) = p_i N\left(\frac{it_h}{n+1}\right), \quad t = \frac{it_h}{n+1} \quad where \quad i = 1,2...n \\ N(0^+) = N_0 \end{cases} \tag{16}$$

where $p_i = 1 - q_i$ is the pest survival rate after i^{th} use of the pesticides. At that point, the solution of above becomes

$$N(t) = \begin{cases} N_0 e^{rt}, \quad 0 \leq t \leq \frac{t_h}{n+1} \\ p_i N\left(\frac{it_h}{n+1}\right) e^{r\left(t - \frac{it_h}{n+1}\right)}, \quad \frac{it_h}{n+1} \leq t \leq \frac{(i+1)t_h}{n+1} \end{cases} \tag{17}$$

For simplification, we assume that the pest population level reaches the same level after each pest control. Since the control of pests depends upon the initial level because as per the initial pest population dosage must be applied and after that, we usually use an equal dosage. Thus we assume $(p_2 = p_3 = ... = p_n = p)$

$$p_1 N_1 = N\left(\frac{t_h^+}{n+1}\right) = N\left(\frac{2t_h^+}{n+1}\right) = ... = N\left(\frac{nt_h^+}{n+1}\right) = pp_1 N_1 e^{\frac{rt_h}{n+1}}$$

where $N_1 = N\left(\frac{t_h}{n+1}\right) = N_0 e^{\left(\frac{rt_h}{n+1}\right)}$. let $A_1 = e^{\left(\frac{rt_h}{n+1}\right)} > 1$. Then

$$p = \frac{1}{A_1} \tag{18}$$

Control factors p_1 and n determine maximization of profit.

$$p_1 = 1 - N_1^{\gamma-1} F(X(p_1)), \quad p = 1 - N_2^{\gamma-1} F(X(p)) \tag{19}$$

where $N_2 = N\left(\frac{2t_h}{n+1}\right)$ It follows from above that

$$X(p_1) = \frac{-1}{\lambda}[\ln(p_1 - 1 + N_1^{(\gamma-1)}) + (1-\gamma)\ln N_1]$$

$$X(p) = \frac{-1}{\lambda} \ln\left(\frac{p - 1 + N_2^{(\gamma-1)}}{N_2^{\gamma-1}}\right) \tag{20}$$

and ,

$D_i = D\left(\frac{it_h}{n+1}, \frac{(i+1)t_h}{n+1}\right), \qquad i = 0, 1, \ldots n$ .thus

$$D_0 = \frac{mN_0}{r}[A_1 - 1] \tag{21}$$

and

$$D_1 = \frac{mp_1N_1}{r}[A_1 - 1] \tag{22}$$

which implies that the total harvest damage is

$D = D_0 + nD_1$

therefore $S = nC + \alpha(X(p_1) + (n-1)X(p))$

which implies

$$P(p_1, n) = \beta H - nC - \alpha(X(p_1) + (n-1)X(p)) - \beta D \tag{23}$$

Using the above values profit becomes,

(24)

$$\begin{aligned} P(p_1, n) =& \beta H - nC + \frac{\alpha}{\lambda}\left[\ln(p_1 - 1 + N_1^{(\gamma-1)}) + (1-\gamma)\ln N_1\right] + \frac{(n-1)\alpha}{\lambda}\ln\left[\frac{p - 1 + N_2^{(\gamma-1)}}{N_2^{\gamma-1}}\right] \\ & - \beta\left[\frac{mN_0}{r}[A_1 - 1] + n\frac{mp_1N_1}{r}[A_1 - 1]\right] \end{aligned}$$

Therefore ,

$$P_{p_1} = \frac{\alpha}{\lambda}\left[\frac{1}{p_1 - 1 + N_1^{(\gamma-1)}} - \frac{(n-1)\alpha}{\lambda} \times \frac{1}{\left(1 + \frac{(p-1)}{(p_1N_1A_1)^{(\gamma-1)}}\right)} \times p_1N_1(p-1)(\gamma-1)(p_1N_1A_1)^{(-\gamma)}\right]$$

$$-\frac{\beta m}{r}[(A_1 - 1)nN_1] = 0 \tag{25}$$

Due to analytically complexity we will discuss the cases for $\gamma = 1$ and $n = 1$ Figure (3),

Case 1 : $\gamma = 1$

$$P_{p_1} = \frac{\alpha}{\lambda(p_1 - 1)} - \frac{\beta m}{r}(A_1 - 1)nN_1 = 0 \tag{26}$$

Solving w.r.t p_1 yields

$$p_1^* = \frac{\alpha r}{\lambda\beta m(A_1 - 1)N_1} \tag{27}$$

Since

$$P_{p_1p_1} = -\frac{\alpha}{\lambda p_1^2} < 0$$
$$P_{p_1^*p_1*} < 0$$

Substituting the value of p_1^* in (27)

$$X_1^* = \frac{-1}{\lambda}\ln p_1^*$$
$$X_2^* = \frac{-1}{\lambda}\ln\frac{1}{A_1}$$

Case 2 : When n = 1

Then

$$P_{p_1} = \frac{\alpha}{\lambda(N_1^{(\gamma-1)} + p_1 - 1)} - \frac{\beta m}{r}[A_1 - 1]N_1 = 0 \tag{28}$$

Solving w.r.t p_1

$$p_1^* = \frac{\alpha r}{\lambda\beta m(A_1-1)N_1} + 1 - N_1^{(\gamma-1)} \tag{29}$$

Since

$$P_{p_1p_1} = -\frac{\alpha}{\lambda(N_1^{(\gamma-1)} + p_1 - 1)^2} < 0 \tag{30}$$

Hence p_1^* is the optimal survival rate .

$$X_1^* = \frac{-1}{\lambda}\left[\ln\left(\frac{\alpha r}{\lambda\beta m(A_1-1)N_1}\right) + (1-\gamma)\ln N_1\right]$$
$$X_2^* = \frac{-1}{\lambda}\ln\left(\frac{\frac{1}{A_1} - 1 + N_2^{(\gamma-1)}}{N_2^{\gamma-1}}\right)$$

X_1^* denotes the first spray and X_2^* denotes the $n-1$ sprays.

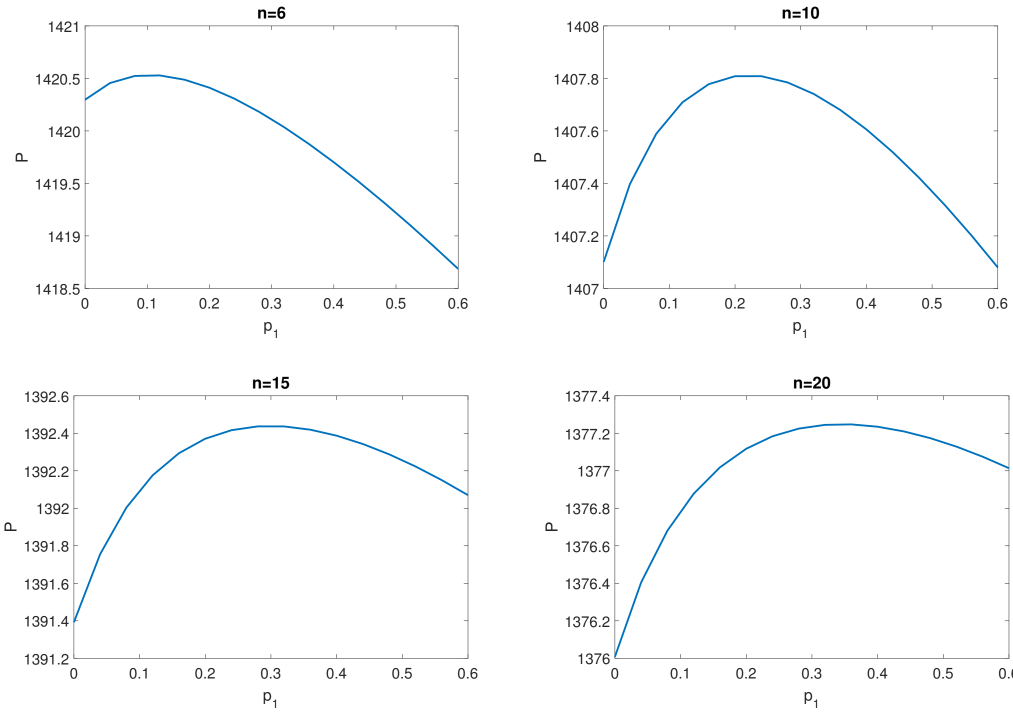


FIGURE 3. Profit and survival rate when pesticides are applied n times before harvest

3.2. Pest population dependent implementation of pesticides. Basic dynamical behavior of model when pesticides are applied when pest population reaches Economic Threshold: The major limitation of the previous model (section 3.2 and 3.3) is that the chemical pesticides are applied irrespective of their need. This practice is not only harmful for the crops but also results in reduction of profits and increase in pest resistance and therefore in the following model ET has also been incorporated. As noted in the introduction, we now form our model such that pesticides would be used only when the pest population reaches economic threshold.

$$\begin{cases} \frac{dN(t)}{dt} = rN(t), & N(t) \neq ET \\ N(t^+) = (1-q)N(t), & N(t) = ET \\ N(0^+) = N_0 \quad and & N_0 \leq ET \end{cases} \tag{31}$$

We assume that initial pest population N_0 attain ET value in some finite time. We assume that τ_1 is the first time at which the arrangement of (31) comes to ET. Hence the solution of (31) for the first time interval $t \in [0, \tau_1]$ becomes

$$N(t) = N_0 e^{rt} \qquad for \qquad t \in 0 \leq t \leq \tau_1$$

which satisfies $N(\tau_1) = ET = N_0 e^{r\tau_1}$ and $N(\tau_1^+) = (1-q)ET$. Thus τ_1 can be calculated as ,

$$\tau_1 = \frac{1}{r}\ln(\frac{ET}{N_0}) \tag{32}$$

The arrangement of the model (31) will come to the ET again at time τ_2. Correspondingly, τ_2 can be calculated as

$$\tau_2 = \tau_1 + \frac{1}{r}\ln\frac{1}{1-q} \tag{33}$$

Using induction process , the model starting with N_0 encounters impulsive impacts at fixed times $\tau_1, \tau_2, ..., \tau_{k-1}, \tau_k$. Hence the solution becomes ,

$$N(t) = \begin{cases} N_0 e^{rt}, 0 \le t \le \tau_1 \\ (1-q)ETe^{r(t-\tau_k)}, \tau_k \le t \le \tau_{k+1}, k = 1,2,.. \end{cases} \tag{34}$$

Here, any arrangement of (31) starting from N_0 and when population level reaches ET after pesticides spray follows a periodic solution.

$$N^T(t) = (1-q)ETe^{r(t-\tau_k)}$$

when $T = \tau_{k+1} - \tau_k$ which denotes the period

$$T = \frac{1}{r}\ln\frac{1}{1-q}$$

3.2.1. *When pesticides are applied once when pest population reaches economic threshold value.* Now we assume that pesticides are applied only once before harvest.

$$N(t) = \begin{cases} N_0 e^{rt},\ 0 \le t \le \tau_1 \\ \frac{N_0}{ET}(ET-K)e^{rt},\ \tau_1 \le t \le t_h \end{cases} \tag{35}$$

therefore

$$N_h = N(t_h) = \frac{N_0}{ET}(ET-K)e^{rt_h}$$

hence damage

$$D = \int_0^{\tau_1} mN(t)dt + \int_{\tau_1}^{t_h} mN(t)dt$$

$$= \frac{m}{r}[N_h - N_0 + K] \tag{36}$$

$$P = \beta H - \beta \frac{m}{r}[N_h - N_0 + K] - \alpha X - C \tag{37}$$

Above condition demonstrates that the maximal benefit is related with the two choice factors: Optimum Dosage X and Economic Threshold value ET. To find both variables we differentiate above Equation with X and ET.

$$P_X = -\frac{\beta m}{r}\left[\frac{dN_h}{dK}K_X - 0 + K_X\right] - \alpha \tag{38}$$

Let

$$A = e^{r(t_h - \tau_1)} = \frac{N_0}{ET}e^{rt_h}$$

So

$$N_h = (ET - K)A$$

Hence $\frac{dN_h}{dK} = -A$

$$P_X = \frac{\beta m}{r}[A-1]K_X - \alpha \tag{39}$$

$$P_{ET} = -\frac{\beta m}{r}\left[K_{ET} - N_0 e^{rt_h}\left\{\frac{ET K_{ET} - K}{ET^2}\right\}\right] \tag{40}$$

Let $P_X = 0$ and $P_{ET} = 0$. Solving with respect to K_X and K_{ET} above equation yields, $K_{ET} = \frac{AK}{[A-1]ET}$ and $K_X = \frac{\alpha r}{\beta m[A-1]}$

We know,

$K(X,ET) = (ET)^\gamma F(X) = (ET)^\gamma(1 - e^{-\lambda X})$

Hence we get $K_{ET} = \frac{\gamma K}{ET}$ and $K_X = \lambda(ET)^\gamma e^{-\lambda X}$

Equating both K_{ET} gives

$$\gamma = \frac{A}{A-1}$$

which shows that $\gamma > 1$

On solving it gives, $\tau_1^* = t_h - \frac{\ln(\frac{\gamma}{\gamma-1})}{r}$

Resulting in

$$ET^* = \frac{(\gamma-1)e^{rt_h}N_0}{\gamma} \tag{41}$$

Equating both K_X gives

$$X^* = \frac{-1}{\lambda} \ln\left[\frac{\alpha r(\gamma-1)}{(ET^*)^{\gamma}\beta\lambda m}\right]$$

and $X^* > 0$ when $0 < \left[\frac{\alpha r(\gamma-1)}{(ET^*)^{\gamma}\beta\lambda m}\right] < 1$

Biologically $ET - K > 0$. In fact $ET - K = ET - (ET)^{\gamma}F(X)$. So $ET - K > 0$ is one of the following two conditions hold

$(c_1) ET \leq 1,\ \gamma > 1$

$(c_2) ET > 1,\ 1 \leq \gamma \leq 1 - \frac{\ln(F(X)}{\ln(ET)}$.

$$P_{ET^*ET^*} = \frac{1}{r}\left[\beta\gamma m(\gamma-1)(e^{(-X\lambda)}-1)((N_0 e^{(rt_h)}(\gamma-1))/\gamma)^{(\gamma-2)}\right] < 0$$

as all terms are positive and $(e^{(-X\lambda)}-1)$ is negative.

$$P_{X^*X^*} = \alpha\lambda(\gamma-1) > 0$$

hence dosage is minimum.

3.2.2. *When pesticides are applied n times when pest population reaches economic threshold value.* Now we assume that the pest population initiating from N_0 will reach ET n times before final harvest at time t_h ,which indicates that n treatment of pesticides must be applied. We assume that pest population reaches ET at time $\tau_1, \tau_2, ..., \tau_n$. Therefore our system (4) becomes

$$N(t) = \begin{cases} \frac{dN(t)}{dt} = rN(t), & \tau_{(i-1)} \leq t \leq \tau_i \\ N(\tau_i^+) = N(\tau_i) - K(X, ET), & t = \tau_i \\ N(0^+) = N_0 \end{cases} \tag{42}$$

gives

$$N(t) = \begin{cases} N_0 e^{rt}, & 0 \leq t \leq \tau_1 \\ (ET-K)e^{r(t-\tau_i)}, & \tau_i \leq t \leq \tau_{i+1}, \quad i = 1, 2, .. \end{cases} \tag{43}$$

hence

$$\tau_1 = \frac{1}{r}\ln\frac{ET}{N_0}, \qquad T = \frac{1}{r}\ln\left[\frac{ET}{ET-K}\right] \tag{44}$$

thus damage in time t_h is distributed in (n+1) parts

$$D_0 = \int_0^{\tau_1} mN(t)dt = \frac{m(ET - N_0)}{r},$$

$$D_1 = \int_{\tau_1}^{\tau_2} mN(t)dt = \frac{mK}{r},$$

$$D_n = \int_{\tau_n}^{\tau_{n+1}} mN(t)dt = \frac{m[N_h - (ET - K)]}{r}$$

Let $B = N_0 e^{rt_h}$, we have $\tau_n = \tau_1 + (n-1)T = \frac{1}{r}\ln\left[\frac{ET^n}{N_0(ET-K)^{(n-1)}}\right]$

So, $N_h = (ET - K)e^{r(t_h - \tau_n)} = B\left(1 - \frac{K}{ET}\right)^n$

Implies,

$$P = \beta H - \frac{\beta m}{r}[(ET - N_0) + (n-1)K + [N_h - (ET - K)]] - n\alpha X - nC$$

For kill function

(45) $$K(X, ET) = ET^{\gamma}(1 - e^{-\lambda X})$$

We get $K_X = (ET)^{\gamma}\lambda e^{-\lambda X}$, $K_{ET} = \frac{\gamma K}{ET}$,

$N_{hX} = -nB\left[1 - ET^{\gamma-1}\left(1 - e^{-\lambda X}\right)\right]^{n-1} \lambda ET^{\gamma-1}e^{-\lambda X}$ and

$N_{hET} = -nB\left[1 - ET^{\gamma-1}\left(1 - e^{-\lambda X}\right)\right]^{n-1} (\gamma - 1)ET^{\gamma-2}\left(1 - e^{-\lambda X}\right)$

Differentiating P with respect to ET and X, gives

(46) $$P_X = -\frac{\beta m[nK_X + N_{hX}]}{r} - n\alpha$$

(47) $$P_{ET} = -\frac{\beta m[nK_{ET} + N_{hET}]}{r}$$

Using $K_X, K_{ET}, N_{hX}, N_{hET}$ and letting P_X and $P_{ET} = 0$ gives,

(48) $$\begin{aligned} \gamma ET - B(\gamma-1)\left[1 - ET^{\gamma-1}\left(1 - e^{-\lambda X}\right)\right]^{n-1} &= 0 \\ \beta m\left[B\lambda ET^{\gamma-1}e^{-\lambda X}\left[1 - ET^{\gamma-1}\left(1 - e^{-\lambda X}\right)\right]^{n-1} - \lambda ET^{\gamma}e^{-\lambda X}\right] - r\alpha &= 0 \end{aligned}$$

To get analytic solution for ET we discuss the 2 cases:

Case 1: n=1

The solution verifies the result of section 3.2.1

Case 2 n=2,$\gamma = 2$

$$ET^* = \frac{B\lambda \pm \sqrt{B^2\lambda^2 + \frac{4\lambda(B+2)\alpha Br}{\beta m}}}{2\lambda(B+2)}$$
$$X^* = -\frac{1}{\lambda}\ln\left[1+\frac{2}{B}-\frac{1}{ET^*}\right]$$

Here X^* is positive if $ET^* < \frac{B}{B(1-e)+2}$

4. Biological Control of Exponentially Growing Pest Population

In the previous section we have discussed how spraying of chemical pesticides once or multiple times on pest population reduces its count. We obtained optimal timing of spray and optimal dosage of pesticides in each scenario to maximize the profit. This section will however focus exclusively on the scenario as how biological control can be one of the way to reduce pest population in the absence of chemical pesticides. This technique can be of use specially in the regions where farmers cannot afford these expensive pesticides. Hence, in this section we would try to focus on how inclusion of the natural enemy in a crop field can bring the pest population to the threshold level where it doesn't remain a threat for the crop population. A simple suggestive mathematical model is as follows:

$$\begin{aligned} \frac{dN}{dt} &= rN - nNP \\ \frac{dP}{dt} &= nNP - m_1P \end{aligned} \tag{49}$$

where N, P are pest population and natural enemy. n is the interaction rate between the pest and natural enemy and m_1 is the natural death rate of P. The equilibrium points corresponding to the above system 49 are $P_1 = (0,0)$, $P_2 = (\frac{m_1}{n}, \frac{r}{n})$

Jacobian of system (49) with equillibrium points is $P_2 = (\frac{m_1}{n}, \frac{r}{n})$ is $\begin{bmatrix} 0 & -m_1 \\ r & 0 \end{bmatrix}$

Now, we have the same system (49) with control and the system becomes:

$$\begin{aligned} \frac{dN}{dt} &= rN - nNP \\ \frac{dP}{dt} &= nNP - m_1P + U \end{aligned} \tag{50}$$

Here U is the control to reduce the pest population to desired economic threshold value N_d. The equations satisfied by the desired positive steady state with control are given below as:

$$\begin{aligned} rN^* - nN^*P^* &= 0 \\ nN^*P^* - m_1P^* + u^* &= 0 \end{aligned} \tag{51}$$

From the above equations (51), we obtain the control variable u^*,

$$u^* = -P^*[nN^* - m_1] \tag{52}$$

$$P^* = \frac{r}{n} \tag{53}$$

to maintain $N^* = N_d$. The equilibrium point $P_3(N_d, P^*)$ can be unstable for the control u^* and therefore, linear feedback control u may be applied to make it asymptotically stable.

Below we define new variables as:

$$y = \begin{bmatrix} N - N^* \\ P - P^* \end{bmatrix}, u = U - u^* \tag{54}$$

The new variables are substituted in (10)and admitting (51) we get,

$$\begin{aligned} \frac{dy_1}{dt} &= r(y_1 + N^*) - n(y_1 + x_1^*)(y_2 + P^*) \\ \frac{dy_2}{dt} &= n(y_1 + N^*)(y_2 + P^*) - m_1(y_2 + P^*) + u + u^* \end{aligned} \tag{55}$$

which becomes

$$\begin{aligned} \frac{dy_1}{dt} &= ry_1 - ny_1y_2 - nP^*y_1 - nN^*y_2 \\ \frac{dy_2}{dt} &= ny_1y_2 + nP^*y_1 + nN^*y_2 - ny_2 + u + u^* \end{aligned} \tag{56}$$

Hence we obtain the error system below:

$$\dot{y} = Ay + h(y) + Bu \tag{57}$$

Matrices A and B are given as

$A = \begin{bmatrix} r - nP^* & -nN^* \\ nP^* & nN^* - m_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

with the following form of the vector $h(y)$:

$h(y) = \begin{bmatrix} -y_1 y_2 \\ y_1 y_2 \end{bmatrix}$

We shall use the following theorem to find the feedback control u.

Theorem 1. *If there exist constant matrices Q and R, positive definite, Q being symmetric, such that the function*

$$l(y) = y^T Q y - h^T(y) P y - y^T P h(y) \tag{58}$$

is positive definite then the linear feedback control

$$u = -R^{-1} B^T P(t) y \tag{59}$$

is optimal, in order to transfer the nonlinear system (57) from an initial to a final state

$$y(\infty) = 0 \tag{60}$$

minimizing the functional

$$J = \int_0^\infty [l(y) + u^T R u] dt \tag{61}$$

where P the symmetric, positive definite matrix is the solution of the matrix algebraic Riccati equation

$$PA + A^T P - PBR^{-1} B^T P + Q = 0 \tag{62}$$

In addition, with the feedback control (59), there exists a neighbourhood $\Gamma_0 \subset \Gamma$, $\Gamma \subset R^n$ of the origin such that if $y_0 \in \Gamma_0$, the solution $y(t) = 0$, $t \geq 0$, of the controlled system (57) is locally asymptotically stable, and $J_{min} = y^T(0) P(0) y(0)$. Finally, if $\Gamma = R^n$ then the solution $y(t) = 0, t \geq 0$, of the controlled system (54) is globally asymptotically stable.
The next theorem determines the positive definiteness of the function $l(y)$ at the neighbourhood Γ_0 of the origin for the system (57).

Theorem 2. *For any matrix P and*

$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$, $h(y) = \begin{bmatrix} -y_1 y_2 \\ y_1 y_2 \end{bmatrix}$

function $l(y)$ defined in Equation (58) is positive definite at the neighbourhood Γ_0 of the origin $(0,0)$.

Proof Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ be symmetric. From (58), for all y we have,

$$l(y) = q_{11}y_1^2 + q_{22}y_2^2 + 2p_{11}y_1^2y_2 + 2p_{12}y_1y_2^2 - 2p_{12}y_1^2y_2 - 2p_{22}y_1y_2^2$$

and its first-order partial derivatives are

$$\frac{\partial l}{\partial y_1} = 2q_{11}y_1 + 4p_{11}y_1y_2 + 2p_{12}y_2^2 - 4p_{12}y_1y_2 - 2p_{22}y_2^2$$
$$\frac{\partial l}{\partial y_2} = 2q_{22}y_2 + 2p_{11}y_1^2 + 4p_{12}y_1y_2 - 2p_{12}y_1^2 - 4p_{22}y_1y_2$$

It is obvious that at $y_1 = y_2 = 0$

$$\frac{\partial l}{\partial y_1}(0) = \frac{\partial l}{\partial y_2}(0) = 0$$

the Hessian of $l(y)$(at the origin) we get

$$H(0) = \begin{bmatrix} 2q_{11} & 0 \\ 0 & 2q_{22} \end{bmatrix}$$

We see that it is positive definite, which gives us that the origin of function $l(y)$ is a strict local minimum point. And this function at the neighbourhood Γ_0 of the origin is positive definite. Hence, we can say that the error dynamical system (57) is locally asymptotically stable under linear feedback control u and thus, system (10) approaches to (N_d, P^*) under the control $U = u + u^*$. The next section, would discuss the comparative results of all the cases discussed above numerically with the data taken from [3] along with the system with control.

5. NUMERICAL SIMULATIONS

In this section, we will discuss a numerical example in support of the analytic results of our system. The value of $H = 500$ and $C = 3$ and the rest range of the parameters are considered as per table 4.

TABLE 4. Range of parameters and their distribution

Parameters	**Range**	**Distribution**
γ	$(0.65, 1.45)$	Uniform
β	$(2.5, 3.3)$	Normal
α	$(2.3, 3.1)$	Uniform
t_h	$(116, 124)$	Normal
m	$(0.0006, 0.0014)$	Normal
N_0	$(6, 14)$	Normal
r	$(0.036, 0.044)$	Normal

5.1. Comparative study of the cases. Comparative study of Case 3.1.1 and Case 3.2.1

The comparative results of when the pesticides are applied once at a fixed time and when it is applied as the pest population reaches the economic threshold are shown in table 5.

Remark 1 It shows from table 5 that applying pesticides after reaching economic thresholds requires less dosage in comparison to when pesticides are applied once before harvest which is both economically and environmentally beneficial and it further it does not have a significant difference on the profit.

TABLE 5. Optimal dosage, threshold, profit when pesticides are applied only once before harvest

Cases when pesticides are applied only once before harvest		
3.1.1		
OptimalX_i	Optimal(τ)	Profit
3.1847	43.8869	1457
3.2.1		
OptimalX_i	Optimal(ET)	Profit
3.0687	12.0307	1437.7

TABLE 6. Optimal dosage, survival and profit when pesticides are applied n times before harvest

Cases when pesticides are applied n times before harvest			
3.1.2			
n	OptimalX_i	Optimal(p_1)	Profit
6	1.1083	0.2645	1418.3
10	0.8360	0.3667	1406.3
15	0.6878	0.4381	1391.2
20	0.6103	0.4808	1376.1

Comparative study of Case 3.1.1, 3.2.1 with 3.1.2

Table 6 shows the optimal dosage, optimal survival and profit for the case 3.1.2 for the same set of data.

Remark 2 Table 6 concludes that when pesticides are applied n fixed times, the optimal dosage drastically reduces from 3.1847(Case 3.1.1), 3.0687(Case 3.2.1) to 1.1083 and gradually decreases further with increase in n. Although, it results in a slight decrease in profit but not major and hence, this strategy is beneficial for those pest population like eggplant fruit-and-shoot borer (EFSB) which destroys egg-plant in a very short span of time. This would also prevent pests to develop resistance against pesticides due to low dosage.

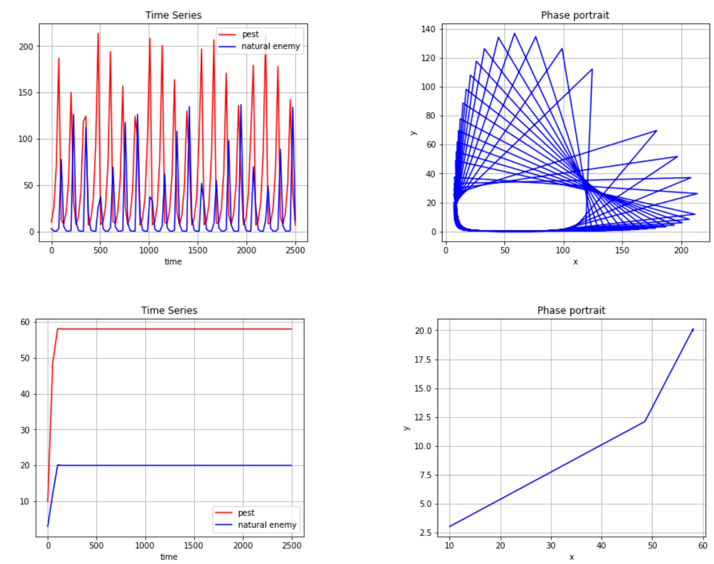


FIGURE 4. (a) Pest population without Control (b) Pest population with Control

5.2. System with Biological control. In this section, our main aim is to analyze how much biological control would be required if We wish to stabilize the system **(7.1)** at desired steady state with $N^* = N_d = 57.8621(58)$. The value of N_d is taken in reference to the case when pesticides are sprayed once as pest population reaches an economic threshold value. This will give us an idea of how much optimal control is required to reach the same threshold value in the presence of only biological control. The parameters assumed are $n = 0.002$ and $m_1 = 0.12$ keeping $r = 0.04$ as assumed in the previous case. We obtained the value of $P^* = 20$ from (53) and we get $u^* = 0.08$ from (52). Matrix obtained for the system is as follows:

$$A = \begin{bmatrix} 0 & -0.116 \\ 0.04 & -0.004 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assuming

$$Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \ R = \begin{bmatrix} 1 \end{bmatrix}$$

Using LQR commands in Matlab, from the Riccati equation we obtain

$$P(t) = \begin{bmatrix} 0.1475 & -0.0677 \\ -0.0677 & 0.1564 \end{bmatrix}$$

For $l(y)$ we get the following form:

$l(y) = 0.01(y_1^2 + y_2^2 + 29.5y_1^2y_2 + 13.54y_1^2y_2 - 31.28y_2^2y_1)$

Finally we have found the optimal strategy form as:

$U = u + u^* = 0.08 + 0.0677y_1 - 0.1564y_2$

Therefore, the eigen values which were $\lambda = i0.04.\lambda = -i0.04$ without control got stabilized by linear feedback control to $\lambda_1 = -0.1604, \lambda_2 = -1159.9599$ and also reduced the pest population. The optimal dosages of chemicals calculated in all the above cases are also sensitive to the parameters. Hence, in the next section global sensitivity analysis of X^* is done on the bases of the dependent parameters.

5.3. Uncertainty analysis. In this section, uncertainty and sensitivity analysis is discussed using the method of Latin hypercube sampling (LHS) scheme [19, 20] to see the variability in the optimal dosage and optimal time due to uncertainty in the input parameters in each case. Each input parameter is sampled 1000 times. The PRCC between X^*, τ_1^*, p_1^* and each of the parameters is calculated as per the cases discussed analytically. If the absolute value of PRCC is near $+1$ or -1, then there is an important relationship between the variable and the parameter. We also consider absolute values of PRCC > 0.4 as indicating an important correlation between an input parameter and output variables, values between 0.1 and 0.4 as moderate correlations, and values between 0 and 0.1 as not significantly different from zero [21]. The sensitivity of the parameters is done with a 0.05 level of significance and for the rest of the parameters below 0.05, our test is unable to test the significance. We shall understand the sensitivity of our output variable with the various parameters through the study of the combination of uncertainty analysis and PRCC. The explanation in detail for each case along with the graphs are mentioned in the Appendix. It shows the following results as per each case:

- From the Figure (5(a)), it is visible that as the absolute value of PRCC for $\gamma = 0.65$ is the highest of all the corresponding values of other parameters, thus it is highly correlated with X^*. γ is also positively correlated with X^* which means that with an increase in γ, the killing rate of pests decreases with that pest population decreases slowly which in turn would require a higher dosage of pesticides so the dosage amount should also increase accordingly.

The next highest positively correlated is r which also goes in lines as r is the exponential growth rate of the pest population. N_0 and t_h are positively correlated as an increase in initial pest population and longer harvest time would require more pesticides dosage .

α is negatively correlated as the pesticides cost increases, we will have to reduce the dosage to maintain the profit. The rest of the parameters λ, α, β, and m are negatively correlated.

- For τ_1^*, we can interpret from Figure (5(b)) that again r and γ are positively correlated to τ_1^* which is due to the reason that as γ increases, the dosage automatically rises as a result of which length of the optimal timing for pesticide spray increases. It also states that optimal dosage and optimal time are positively correlated to each other.

  t_h is also positively correlated through Figure (5(b)) which means an increase in harvest time would lead to an increase in optimal time of dosage.
- From Figure (6(a)), m_1 is highly correlated with p_1^* which means that the survival rate of pests is totally dependent on the rate at which the crops are damaged.
- Figure (6(b)) concludes that the rate at which the crops are damaged has a significant positive correlation to the survival rate of the pest population. As the rate of crop damage increases, the pest population would start increasing.
- Figure (7(a)) shows that γ is also positively correlated with X^* which means that with an increase in γ, pest population rises in comparison to the killing rate of pest decreases which results in the decline of pest population slowly, hence high dosage of pesticides would be required. This kind of behavior is also due to the reason that the pesticides are sprayed after the population reaches to a threshold level. The next highest positively correlated is r which also goes in lines as r is the exponential growth rate of the pest population. N_0 is positively correlated as an increase in the initial pest population would require more pesticide dosage. This also validates our assumption of exponential growth rate as in the initial phase the pest population follows an exponential growth rate. So, as N_0 increases, the dosage should increase. α and m are negatively correlated which means that with an increase in pesticide cost, the dosage has to be reduced to

maintain the profit. Therefore, if pesticides are sprayed once as pest population reaches an economic threshold, the crucial parameters should be taken into consideration so that aim of controlling pests could be achieved by maximizing profit.

- Figure (7(b)) shows that γ is positively correlated with X^* which means that with an increase in γ, pest population rises in comparison to the killing rate of pest decreases which results in the decline of pest population slowly, hence high dosage of pesticides would be required.

6. DISCUSSION

The pest control programs are fundamentally different from implementation approaches and handling pests to tolerating the effects of the pest. These all factors become critical when focusing on pest management tactics. To understand the current and potential roles of pesticide implementation strategy in reducing the harmful impact of pests on crops, it is first necessary to examine the components of the demographic and environmental mechanisms on the dynamics of pest and crop yield. The pest control action is only justified once the population of a pest reaches a certain level. However, determining the critical level of pest activity where a control action is needed can be challenging. Management tactics for a particular pest include a threshold population density, often termed the "action or economic threshold," that is used to determine if a control tactic is justified. As long as the pest density remains below this threshold no action is needed, but if the pest population exceeds this level, a control action is recommended. The level of this action depends on how much damage can the crop tolerate, which in turn varies depending on the situation and scenarios. The economic threshold of pest size has been the most problematic in incorporating and evaluating its impact on dynamics because it depends on predictions of pest population growth rates. Focusing research efforts on these aspects of the dynamics of pest control offers the prospect of improved responsiveness to pesticide implementation and reduction in pest resistance. In this paper, a mathematical model is developed by assuming the pest population grows exponentially with impulsive pesticide release. The pesticide spraying is carried out in different types of patterns. Initially, it is applied once at a fixed time, then multiple times, and later after reaching the economic threshold on the size of the pest population. The dynamics of each of the system is studied analytically and the optimal dosage, optimal timing of dosage, optimal pest survival rate, and optimal profit are obtained. The aim of both the model analysis is to bring the pest population to a threshold level. Finally, the analytic results are validated by a numerical scenario which reveals that *spraying pesticides after the pest population reaches the economic threshold is more beneficial in comparison to applying it once initially.* This is because the former strategy has reduced dosage and hence it is more economical. *Further, spraying pesticides at regular intervals has also a major advantage because as the value of n (the number of times pesticides are implemented) increases, the dosage decreases. Although, the profit only reduces slightly. This strategy would be significantly useful for staple crops.* Hence, the government should start with programs to educate farmers about how strategically pesticides should be used so that maximum profit could be made with fewer expenses. Finally, another mathematical model is proposed with biological control(natural enemy) with the dynamics studied through linear feedback control. Instead of applying chemical pesticides, LQR is applied to bring the pest population to the same threshold level where it doesn't remain dangerous to the crop population. The measures should be taken according to the growth rate of the pest population. If it's growing exponentially as generally in most of the cases due to favorable conditions, the frequency of pesticide spray should be increased to prevent pest resistance against pesticides as well as crop destruction. Further, in the absence of pesticides, biological control can also be used as an alternative to save crop damage. In addition to numerical simulations, we have also obtained the sensitive parameters which would be crucial for the optimal dosage, optimal time, and optimal survival rate. Global sensitivity for each of the cases has been done keeping in mind as sometimes in the rural areas, the availability of pesticides is a major concern. Hence, they are left up with only a choice of spraying pesticides at a fixed time. Therefore, for each of the cases, the sensitive parameters would play a vital role in the number of pesticides and the timing of spray. We also have obtained a significant result that how the growth rate of the pest population has come up as a sensitive parameter in the majority of the cases. The analysis and interpretation would be extremely beneficial for the government to deal with the situations where the crops are drastically affected due to the pest population.

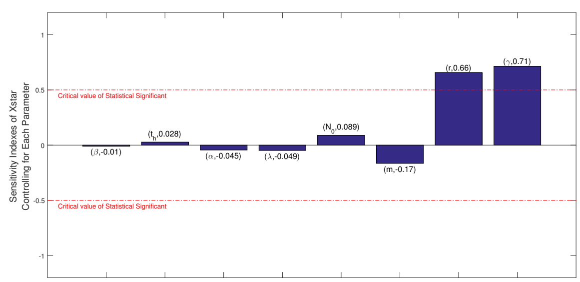
APPENDIX A. SUPPLEMENTARY DOCUMENT

A.1. Uncertainty analysis. The detailed elaboration of each of the cases is as follows:

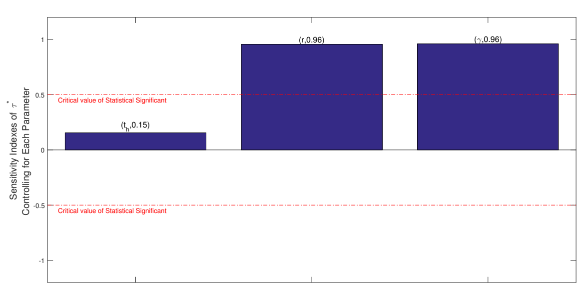
Case 3.1.1 In this case, the parameters α, γ have been considered with uniform distribution whereas $\beta, a, t_h, r, m, N_0, \lambda$ with normal distribution with means set as $\beta = 2.9, a = 0.1, t_h = 120, r = 0.04, m = 0.001, N_0 = 10, \lambda = 1.2$. We set the standard deviation for parameters a, β, r, m, λ to be very small(i.e 0.01) whereas for parameters N_0, t_h are varied in a larger range(i.e 0.2). We shall see the sensitivity analysis for X^* and τ_1^*:

Case 3.1.2 Next, we discuss the case when pesticides are applied n fixed times in harvest. We shall consider both cases to see the sensitivity of optimal survival p_1^*.

Case1 We consider the case that suppose pesticides are sprayed five times i.e n=5, we see that from Figure 6(a) that as the absolute value of PRCC for $m = 0.51$ is highest of all the corresponding values of other parameters.



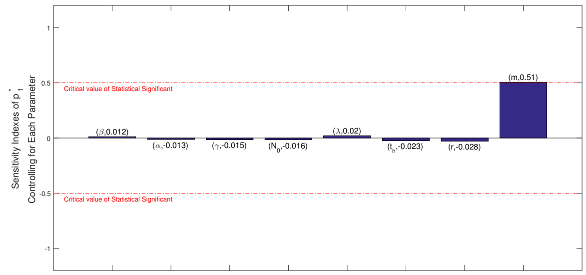
(a)



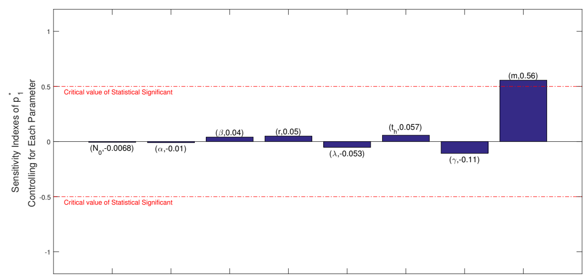
(b)

FIGURE 5. (a) Sensitivity of X^* with respect to parameters for case 3.1.1, (b)Sensitivity of τ_1^* with respect to parameters for case 3.1.1

3.1.2 (Case 2) Again as per Figure 6(b), p_1^* is highly correlated to m with PRCC 0.56.



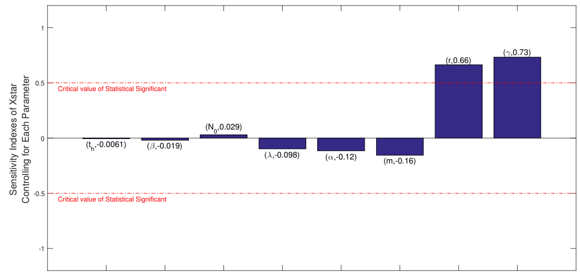
(a)



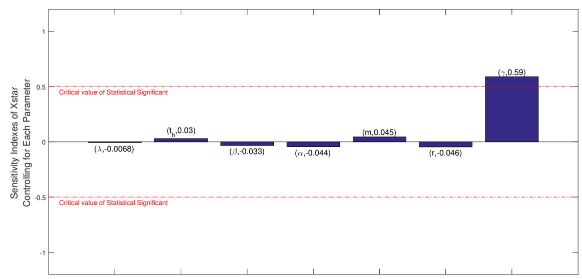
(b)

FIGURE 6. (a)Sensitivity of p_1^* with respect to parameters for case 3.1.2(Case 1), (b)Sensitivity of p_1^* with respect to parameters for case 3.1.2(Case 2)

Case 3.2.1 Next, we discuss the case when pesticides are applied once when pest population reaches Economic Threshold. We shall see the sensitivity analysis for X^*. From the graphs Figure 6(a) it is visible that as the absolute value of PRCC for $\gamma = 0.73$ is highest of all the corresponding values of other parameters, thus it is highly correlated with X^*.



(a)



(b)

FIGURE 7. (a) Sensitivity of X^* with respect to parameters for case 3.2.1, (b) Sensitivity of X^* with respect to parameters for case 3.2.2

Case 3.2.2 This case is when pesticides are applied n times when pest population reaches Economic Threshold value. From Figure 6(b) it is visible that as the absolute value of PRCC for $\gamma = 0.59$ is highest of all the corresponding values of other parameters, thus it is highly correlated with X^*. Other parameters $\lambda, t_h, \beta, \alpha, m, r$ do not show a significant correlation with X^*.

TABLE 7. Highlights of brief difference between the analytical and numerical results of exponential and logistic growth rate of pest population

Our Model with Exponential Growth	**Model with Logistic Growth in** [3]
Case 1: when pesticides are applied once at fixed time before harvest	
Optimal Dosage	
$X^* = \frac{-1}{\lambda}\ln\left[\frac{r\alpha(\gamma-1)}{\lambda(N_1^*)^\gamma m\beta}\right]$	$X^* = -\frac{1}{\lambda}\ln\left[\frac{r\alpha\left(N_1^*-\left(N_1^*\right)^\gamma+(\gamma-1)\left(s-N_1^*\right)\right)}{\left(N_1^*\right)^\gamma(sm\beta\lambda-r\alpha)}\right]$
Optimal pesticide spraying time	
$\tau_1^* = t_h - \frac{1}{r}\ln\frac{\gamma}{(\gamma-1)}$	$\tau_1^* = \frac{1}{r}\ln J$ where $J = \frac{-W+\sqrt{W^2+4N_0V}}{2N_0}$, $W = \gamma(s-N_0)$ and $V = (\gamma-1)(s-N_0)e^{rt_h}$
Case 2: When pesticides are applied once when pest population reaches economic threshold value.	
Economic Threshold	
$ET^* = \frac{(\gamma-1)e^{rt_h}N_0}{\gamma}$	$ET^* = \frac{[\gamma+2B(\gamma-1)]s-s\sqrt{\gamma^2+4B(\gamma-1)}}{2(\gamma-1)(B+1)}$ where $B = \frac{N_0e^{rt_h}}{s-N_0}$
Optimal Dosage	
$X^* = \frac{-1}{\lambda}\ln\left[\frac{\alpha r(\gamma-1)}{(ET^*)^\gamma\beta\lambda m}\right]$	$X^* = \frac{-1}{\lambda}\ln\left[\frac{r\alpha\left(ET^*-(ET^*)^\gamma+(\gamma-1)(s-ET^*)\right)}{(ET^*)^\gamma(sm\beta\lambda-r\alpha)}\right]$
Case 3: When pesticides are applied n times at fixed time before harvest.	
$X_1^* = -\frac{1}{\lambda}\left[\ln\left(p_1^*-1+N_1^{(\gamma-1)}\right)\right] - \frac{1}{\lambda}\left[(1-\gamma)\ln N_1\right]$ $X_2^* = \frac{-1}{\lambda}\ln\left[\frac{\frac{1}{A_1}-1+N_2^{(\gamma-1)}}{N_2^{\gamma-1}}\right]$	$X_1^* = \frac{-1}{\lambda}\left[\ln\left(p_1^*-1+N_1^{(\gamma-1)}\right)+(1-\gamma)\ln N_1\right]$ $X_2^* = -\frac{1}{\lambda}\ln \left[\frac{\left(p_1^*N_1\right)^{\gamma-1}(sA_1)^\gamma-(A_1-1)\left(s-p_1^*N_1\right)\left(p_1^*N_1(A_1-1)+s\right)^{\gamma-1}}{\left(p_1^*N_1\right)^{\gamma-1}(sA_1)^\gamma}\right]$

TABLE 8. Highlights of brief difference between the analytical and numerical results of exponential and logistic growth rate of pest population

Our Model with Exponential Growth	**Model with Logistic Growth in** [3]
Case 4: Case when pesticides are applied n times when pest population reaches economic threshold value.	
Economic Threshold	
$ET^* = \frac{B\lambda \pm \sqrt{B^2\lambda^2 + \frac{4\lambda(B+2)\alpha Br}{\beta m}}}{2\lambda(B+2)}$	ET^* = Not calculated due to mathematical complexity
Optimal Dosage	
$X^* = -\frac{1}{\lambda}\ln\left[1 + \frac{2}{B} - \frac{1}{ET^*}\right]$	$X^* = \frac{-1}{\lambda}\ln\left(\frac{r\alpha\left(s-(ET^*)^2\right)}{(sm\beta\lambda - r\alpha)(ET^*)^2}\right)$ Calculated as per assumed ET^*
Table 6 and Table 5, Global Uncertainty Analysis	Not done

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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