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HOPF BIFURCATION FOR DELAYED PREY-PREDATOR SYSTEM WITH ALLEE EFFECT

MOHAMED HAFDANE[1,*], JUANCHO A. COLLERA[3], IMANE AGMOUR[1], YOUSSEF EL FOUTAYENI[1,2]

[1]Analysis, Modeling and Simulation Laboratory, Hassan II University, Morocco

[2]Unit for Mathematical and Computer Modeling of Complex Systems, IRD, France

[3]Department of Mathematics and Computer Science, University of the Philippines Baguio, Philippines

Abstract. In this study, we take into account a predator-prey system with two delays, the prey is sea urchins and the predator is crabs. The focus is given to the Allee effect where the prey population undergoes, the poisoning of few predators, and a fishing effect on both species considered as selective for the prey. We aim to analyze the system's stability around interior equilibrium using the theory of bifurcations and determine stability intervals related to delays. The theory of normal form and the center manifold are used to determine the direction of the bifurcations. Finally, numerical simulations are given by numerical methods in DDE-Biftool Matlab package to illustrate the theoretical results.

Keywords: predator-prey; stability analysis; Hopf bifurcation; discrete delay; fishing effort.

2020 AMS Subject Classification: 91B05, 91A06, 91B02, 91B50.

INTRODUCTION

Across the seas, marine resources are at risk of extinction due to several factors: predation by strange species, affectation through chemicals or toxic species, and overexploitation. Therefore, marine biodiversity is threatened, requiring the intervention of relevant agencies to ensure the

*Corresponding author
E-mail address: med.hfdn@gmail.com
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preservation of these resources through several facets, these include limited access and selective fishing based on size and age to protect the juvenile population from early fishing.

Mathematical models stand as a tool allowing the study of biological phenomena and their influence on the dynamics of populations, they identify the interactions between marine species, the variation of their density as well as the effect of toxins.

The exponential growth established by Malthus remains among these mathematical models, it considers that the species density evolves exponentially with a constant growth rate. However, Verhulst criticizes this concept and claims that the environment has a maximum capacity of inhabitants that it cannot exceed when the population size approaches the carrying capacity, competing on the natural resources and causing a decrease in growth rate.

$$\dot{x}(t) = x(r - dx)$$

Among the threatened marine species, sea urchins are distinguished as essential invertebrate herbivores in the Mediterranean that undergo the Allee effect with an aggregate behavior. Sea urchins are characterized by external fertilization: reproduction decreases if the density is insufficient to create effective reproductive aggregates. Therefore, there is a positive dependence between the growth rate and size of the sea urchin population.

$$\dot{x}(t) = x\left(\frac{ax}{x+b} - c - dx\right)$$

Our model considers sea urchins as prey of crabs. With this prey-predator model of two exploitable species, we use the equations of Lotka-Volterra while adding the fishing effect. Some types of sea urchins are toxic; this toxicity affects crabs after a certain time of predation of these Sea urchins.

$$\begin{cases} \dot{x}(t) = x\left(\frac{ax}{x+b} - c - dx\right) - g_1 xy - E_1 x \\ \dot{y}(t) = -my + g_2 xy - g_3 x(t-\tau)y(t-\tau) - E_2 y \end{cases}$$

Sea urchins play a vital role in maintaining the balance of ecosystems. Given the decline in available stocks, action has been taken to regulate catches of this species. A minimum size designating an early age has been set for fishing to protect the juvenile population.

Finally, our model is expressed in the following form

$$(1) \qquad \begin{cases} \dot{x}(t) = x\left(\frac{ax}{x+b} - c - dx\right) - g_1xy - E_1x(t-\tau_1) \\ \dot{y}(t) = -my + g_2xy - g_3x(t-\tau_2)y(t-\tau_2) - E_2y \end{cases}$$

such that x sets for sea urchin biomass, and y represents crab biomass. The following table summarizes the different parameters and their explanations.

Parameter	Meaning
a	Per capita maximum filtering rate of population
b	Strength of Allee effect
c	Death rate for preys
d	Strength of intracompetition
g_1	Mortality rates due to predation effect
g_2	Reproductive rates of predators based on prey encountered
m	Predator death rate
g_3	Mortality rates by toxicity effect
τ_1	age selection for harvesting
τ_2	lag for affectation by toxicity

TABLE 1. The meaning of bioeconomical parameters

With initial conditions $x(\theta) = \phi_1(\theta) \geqslant 0$ and $y(\theta) = \phi_2(\theta) \geqslant 0$ for all $\theta \in [-\tau, 0]$, where $\tau = \max\{\tau_1, \tau_2\}$ are ϕ_i are continuous functions.

The remainder of this paper is organized as follows. After presenting the model in the introduction, section 1 focuses on the existence and boundedness of the system's solutions. The stability of the interior equilibrium point is given in section 2, in addition to the search of bifurcation points according to the delay parameters values τ_1 and τ_2. Section 3 discusses the stability and direction of Hopf bifurcation, which is followed in section 4 by the global stability. Finally, the numerical simulations of theoretical results are provided in section 5.

1. Existence and Boundedness of the Solution

1.1. Boundedness of solutions. The first equation of system (1) verifies the following inequality

$$\begin{aligned}\dot{x} &\leqslant x\left(\frac{ax}{x+b}-c-dx\right)\\ &\leqslant x(a-c-dx)\\ &\leqslant x\left(1-\frac{d}{a-c}x\right)(a-c)\end{aligned}$$

So $\exists M>0$ such that $x\leqslant M$.

We consider $w(t)=x(t)+y(t)$

$$\dot{\omega}(t)+p\omega(t)\leqslant x\left(\tfrac{ax}{x+b}-c-dx\right)+g_2xy-g_1xy+px+py-my$$

For $p<m$, we have

$$\dot{w}(t)+pw(t)\leqslant ax+px\leqslant(a+p)M$$

Then x and y are bounded.

1.2. Existence and uniqueness of solution. The system (1) can be represented in the following form

$$\dot{u}=f(u(t),u(t-\tau_1),u(t-\tau_2))$$

with $u=(x,y)$ and $f=(f_1,f_2)^T$ such that

$$\begin{aligned}f_1 &= x\left(\tfrac{ax}{x+b}-c-dx\right)-g_1xy-E_1x(t-\tau_1)\\ f_2 &= -my+g_2xy-g_3x(t-\tau_2)y(t-\tau_2)-E_2y\end{aligned}$$

The function f is continuous and the partial derivatives of f_i are continuous and bounded, then f is a Lipschitzian function. Consequently, the conditions of Cauchy Lipschitz are satisfied. According to the fundamental theorem of functional differential equations cited in [6], system (1) admits a unique solution.

2. Stability Analysis

2.1. Equilibrium points.

To find the positive equilibrium points, we solve the following system

$$\begin{cases} \frac{ax}{x+b} - c - dx - g_1 y - E_1 = 0 \\ -m + g_2 x - g_3 x - E_2 = 0 \end{cases}$$

Then the system (1) admits a unique strictly positive equilibrium point $P^*(x^*, y^*)$, where

$$x^* = \frac{E_2+m}{g_2-g_3}$$
$$y^* = \frac{1}{g_1}\left[\frac{ax^*}{x^*+b} - dx^* - c - E_1\right]$$

2.2. Characteristic equation.

To study the stability of the system (1), we must first calculate its characteristic equation which will be expressed as follows

$$P(\lambda) = \lambda^2 + A\lambda + B + (C\lambda + D)e^{-\lambda\tau_1} + (E\lambda + F)e^{-\lambda\tau_2} + Ge^{-\lambda(\tau_1+\tau_2)} = 0 \tag{2}$$

The coefficients of equation (2) are represented in the following table

Coefficient	Expression
A	$c + 2dx + g_1 y - \frac{a(x^2+2bx)}{(x+b)^2} + m + E_2 - g_2 x$
B	$\left(c + 2dx + g_1 y - \frac{a(x^2+2bx)}{(x+b)^2}\right)(m + E_2 - g_2 x) + g_1 g_2 xy$
C	E_1
D	$(m + E_2 - g_2 x)E_1$
E	$g_3 x$
F	$g_3 x\left(c + 2dx - \frac{a(x^2+2bx)}{(x+b)^2}\right)$
G	$g_3 x E_1$

TABLE 2. Expressions for the coefficients in 2

2.3. Study of local stability.

Case 1: Without delays. For $\tau_1 = \tau_2 = 0$, the characteristic equation becomes as follows

$$\lambda^2 + (A + C + E)\lambda + B + D + F + G = 0$$

According to the Routh Hurwitz criterion, if $A+C+E>0$ and $B+D+F+G>0$, then the system without delays is locally asymptotically stable around the equilibrium point P^*.

Case 2: One delay. For $\tau_1=0$ and $\tau_2>0$

$$\lambda^2+(A+C)\lambda+B+D+(E\lambda+F+G)e^{-\lambda\tau_2}=0 \tag{3}$$

We assume that iw $(w>0)$ is a root of (3) and we get the following couple of equations

$$\begin{cases} (F+G)\cos\omega\tau_2+E\omega\sin\omega\tau_2=\omega^2-(B+D) \\ E\omega\cos\omega\tau_2-(F+G)\sin\omega\tau_2=-(A+C)\omega \end{cases} \tag{4}$$

To arrive at the next system, we must square the previous equations and sum them

$$\omega^4+\left((A+C)^2-E^2-2(B+D)\right)\omega^2+(B+D)^2-(F+G)^2=0 \tag{5}$$

We note the following conditions

- $(\mathbf{H}_1)$ $A+C+E>0$ and $B+D+F+G>0$
- $(\mathbf{H}_2)$ $(A+C)^2-E^2-2(B+D)>0, \quad (B+D)^2-(F+G)^2>0$
- $(\mathbf{H}_3)$ $(B+D)^2-(F+G)^2<0$
- $(\mathbf{H}_4)$ $E^2-(A+C)^2+2(B+D)>0, \quad (B+D)^2-(F+G)^2>0$
  and $\left[E^2-(A+C)^2+2(B+D)\right]^2>4\left[(B+D)^2-(F+G)^2\right]$

If $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold, then Eq (5) has no positive roots. Hence, all roots of Eq (3) have negative real parts when $\tau_2\in[0,\infty)$.

If $(\mathbf{H}_1)$ and$(\mathbf{H}_3)$ hold, then (5) has a unique positive root ω_0^2. Substituting ω_0^2 into (4), we have

$$\tau_{2_n}=\frac{1}{\omega_0}\cos^{-1}\left[\frac{(F+G)\left(\omega_0^2-B-D\right)-(A+C)E\omega_0^2}{E^2\omega_0^2+(F+G)^2}\right]+\frac{2n\pi}{\omega_0}, \quad n=0,1,2,\dots$$

If $(\mathbf{H}_1)$ and $(\mathbf{H}_4)$ hold, then (5) has two positive roots ω_+^2 and ω_-^2. Substituting $\omega_\pm^2$ into (4) gives

$$\tau_{2_k}^{\pm}=\frac{1}{\omega_\pm}\cos^{-1}\left[\frac{(F+G)\left(\omega_\pm^2-B-D\right)-(A+C)E\omega_\pm^2}{E^2\omega_\pm^2+(F+G)^2}\right]+\frac{2k\pi}{\omega_\pm}, \quad k=0,1,2,\dots$$

Let $\lambda(\tau_2)$ be the root of (3) satisfying $\operatorname{Re}\lambda(\tau_{2_n})=0$ (rep. $\operatorname{Re}\lambda\left(\tau_{2_k}^{\pm}\right)=0$) and $\operatorname{Im}\lambda(\tau_{2_n})=\omega_0$ (rep. $\operatorname{Im}\lambda\left(\tau_{2_k}^{\pm}\right)=\omega_{\pm}\right)$. We can obtain that

$$\left[\frac{d}{d\tau_2}\operatorname{Re}(\lambda)\right]_{\tau_2=\tau_{2_0},\omega=\omega_0}>0,\quad \left[\frac{d}{d\tau_2}\operatorname{Re}(\lambda)\right]_{\tau_2=\tau_{2_k}^{+},\omega=\omega_+}>0,\quad \left[\frac{d}{d\tau_2}\operatorname{Re}(\lambda)\right]_{\tau_2=\tau_{2_k},\omega=\omega_-}<0.$$

Theorem 1. *For $\tau_1=0$, assume that $(\mathbf{H}_1)$ is satisfied. Then the following conclusions hold:*

- *If $(\mathbf{H}_2)$ holds, then equilibrium (x^*,y^*) is asymptotically stable for all $\tau_2\geqslant 0$.*
- *If $(\mathbf{H}_3)$ holds, then equilibrium (x^*,y^*) is asymptotically stable for $\tau_2<\tau_{2_0}$ and unstable for $\tau_2>\tau_{2_0}$. Furthermore, system (1.2) undergoes a Hopf bifurcation at (x^*,y^*) when $\tau_2=\tau_{20}$.*
- *If $(\mathbf{H}_4)$ holds, then there is a positive integer m such that the equilibrium is stable when $\tau_2\in\left[0,\tau_{2_0}^{+}\right)\cup\left(\tau_{2_0}^{-}\cup\tau_{2_1}^{+}\right)\cup\cdots\cup\left(\tau_{2_{m-1}}^{-},\tau_{2_m}^{+}\right)$, and the system (1) undergoes a Hopf bifurcation at (x^*,y^*) when $\tau_2=\tau_{2'}^{\pm},k=0,1,2,\ldots$*

Case 3: two delays. For two delays, it is assumed that conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_3)$ are checked. Moreover, the delay τ_2 is in its stability interval. Allow $i\omega(\omega>0)$ to stand as a solution of Eq (2), yet we can acquire

$$\omega^4+\widetilde{A}\omega^2+B^2+F^2-D^2-G^2+2\widetilde{B}\sin\omega\tau_2+2\widetilde{C}\cos\omega\tau_2=0, \tag{6}$$

where

Coefficient	Expression
$\widetilde{A}$	$A^2+E^2-2B-C^2$
$\widetilde{B}$	$\omega CG-\omega^3E-\omega AF+\omega BE$
$\widetilde{C}$	$-DG-\omega^2F+BF+\omega^2AE$

TABLE 3. Expressions for the coefficients in (6)

We define

$$F(\omega)=\omega^4+\widetilde{A}\omega^2+B^2+F^2-D^2-G^2+2\widetilde{B}\sin\omega\tau_2+2\widetilde{C}\cos\omega\tau_2.$$

If the condition $(\mathbf{H}_5)\quad (B+F)^2-(D+G)^2<0$ is checked, it's trusting to agree that $F(0)<0$ and $F(\infty)=\infty$. Then Eq (6) has finite positive roots $\omega_1,\omega_2,\ldots,\omega_k$. For every fixed $\omega_i, i=1,2,\ldots,k$, there exists a sequence $\left\{\tau_{1_i}^j \mid j=1,2,3,\ldots\right\}$, such that (6) holds.

$$\tau_{1_i}^j=\left(\frac{1}{\omega_i}\right)\cos^{-1}\left[\frac{L}{M}\right]+\frac{2j\pi}{\omega_i}, i=1,2,\ldots,k; j=1,2,\ldots \tag{7}$$

where,

Coefficient	Expression
L	$NS+PT+(QS+RT)\cos\omega_i\tau_2+(RS-QT)\sin\omega_i\tau_2$
M	S^2+T^2
N	$-\omega_i^2+B$
P	$A\omega_i$
Q	F
R	$E\omega_i$
S	$-(G\cos\omega_i\tau_2+D)$
T	$G\sin\omega_i\tau_2-C\omega_i$

TABLE 4. Expressions for the coefficients in (7)

Let $\tau_{1_0}=\min\left\{\tau_{1_i}^j \mid i=1,2,\ldots,k; j=1,2,3,\ldots\right\}$. When $\tau_1=\tau_{1_0}$, Eq. (2) has a pair of purely imaginary roots $\pm i\omega^0$ for $\tau_2\in[0,\tau_{2_0})$.
Finally, we accept that $(H_6)\quad \left[\frac{d}{d\tau_1}(\operatorname{Re}\lambda)\right]_{\lambda=i\omega_0}\neq 0$. Consequently, we retain the given result on the stability and bifurcation of our system.

Theorem 2. *For system (6), suppose parameters satisfy conditions of Theorem 1; H_3, H_5 and $\tau_2\in[0,\tau_{2_0})$. Then the equilibrium $E^*(x^*,y^*)$ is asymptotically stable when $\tau_1\in(0,\tau_{1_0})$, unstable when $\tau_1>\tau_{1_0}$ and a Hopf bifurcation occurs when $\tau_1=\tau_{1_0}$.*

3. Stability and direction of Hopf Bifurcation

The objective of this section is to determine the direction of Hopf bifurcation and analyze the stability of periodic solutions. To achieve this objective, we will apply the theory of normal form and the center manifold theorem to our system.

First, we start by linearizing the system (1) by changing variables $u = x - x^*$, $v = y - y^*$ and we get the following system

$$\begin{cases} \dot{u} = l_1 u + m_1 v + nu(t - \tau_1) + F_1 \\ \\ \dot{v} = l_2 u + m_2 v + pu(t - \tau_2) + qv(t - \tau_2) + F_2 \end{cases} \tag{8}$$

where

Coefficient	Expression
F_1	$a_1 u^2 + b_1 uv + c_1 u^3 + \dots$
F_2	$a_2 uv + b_2 u(t - \tau_2) v(t - \tau_2)$
l_1	$\frac{a\left(x^2+2bx\right)}{(x+b)^2} - c - 2dx - g_1 y$
m_1	$-g_1 x$
n	$-E_1$
a_1	$\frac{ab^2}{(x+b)^3} - 2d$
b_1	$-g_1)$
c_1	$-\frac{ab^2}{(x+b)^4}$
l_2	$g_2 y$
m_2	$-m_2 + g_2 x - E_2$
p	$-g_3 y$
q	$-g_3 x$
a_2	g_2
b_2	$-g_3$

TABLE 5. Expressions for the coefficients in (8)

Under the loss of generalities, we assume that: $\tau_2^* < \tau_1^0$.

We pose $\tau_2 = \tau_2^0 + \eta$ and $\varphi_t(\theta) = \varphi(t + \theta) \in C$.

We express the system (8) as the following functional differential system in $C([-\tau_1^*, 0], \mathbb{R}^2)$

$$\dot{\varphi}(t) = L_\eta(\varphi_t) + f(\eta, \varphi_t), \tag{9}$$

where

$$L_\eta: \quad C \to \mathbb{R}^2$$
$$\chi \to A_0\chi(0) + A_1\chi(-\tau_1^*) + A_2\chi(-\tau_2^0)$$

With

$$A_0 = \begin{pmatrix} l_1 & m_1 \\ l_2 & m_2 \end{pmatrix} \quad A_1 = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix}$$

and $f : \mathbb{R} \times C \to \mathbb{R}^2$ is expressed as follows

$$f(\eta,\chi) = \begin{pmatrix} a_1\chi_1^2(0) + b_1\chi_1(0)\chi_2(0) + c_1\chi_1^3(0) + \dots \\ a_2\chi_1(0)\chi_2(0) + b_2\chi_1(-\tau_1^*)\chi_2(-\tau_2^0) \end{pmatrix}.$$

Riesz's representation ensure the existence of a second-order matrix $g(\theta,\eta)$ of bounded variation for $\theta \in [-\tau_1^*, 0]$, such as

$$L_\eta\chi = \int_{-\tau_1^*}^{0} dg(\theta,\eta)\chi(\theta), \ \forall \chi \in C.$$

and

$$g(\theta,\eta) = A_0\delta(\theta) + A_1\delta(\theta+\tau_1^*) + A_2\delta(\theta+\tau_2^0)$$

where

$$\delta(\theta) = \begin{cases} 0, & \theta \neq 0 \\ 1, & \theta = 0 \end{cases}.$$

We can also express our system as follows

$$\dot{\varphi}_t = M(\eta)\varphi_t + R(\eta)\varphi_t.$$

where

$$M(\eta)\chi = \begin{cases} \frac{d\chi(\theta)}{d\theta}, & \theta \in [-\tau_1^*, 0) \\ \int_{-\tau_1^*}^{0} dg(\xi,\eta)\chi(\xi), \theta = 0 \end{cases} \qquad R(\eta)\chi = \begin{cases} 0, & \theta \in [-\tau_1^*, 0) \\ f(\eta,\chi), & \theta = 0 \end{cases}.$$

The adjoint operator of M is written as follows

$$M^*\chi = \begin{cases} -\frac{d\chi(s)}{ds}, s \in (0, \tau_1^*] \\ \int_{-\tau_1^*}^{0} dg^T(t,\theta)\chi(-t), s = 0 \end{cases}.$$

We use the following bilinear form in $C^1([-\tau_1^*,0],\mathbb{R}^2)\times C^1([0,1],(\mathbb{R}^2)^*)$

$$<\psi,\chi>=\bar{\psi}^T(0)\chi(0)-\int_{-\tau_1^*}^{0}\int_{\xi=0}^{0}\bar{\psi}^T(\xi-\theta)dg(\theta)\chi(\xi)d\xi$$

$\pm i\omega_0$ are eigenvalues of $M(0)$ and M^*, we easily check that $\rho(\theta)=\rho(0)e^{i\omega_0\theta}$ is an eigenvector of $M(0)$ associated with $i\omega_0$. Then $M(0)$ is written as $M(0)=i\omega_0\rho(\theta)$.

For $\theta=0$, we have

$$\left[i\omega_0 I-\int_{-\tau_1^*}^{0}dg(\theta)e^{i\omega_0}\right]\rho(0)=0,$$

we get $\rho(0)=(1,\alpha)$, where

$$\alpha=\frac{i\omega_0-l_1-ne^{-i\omega_0\tau_1}}{m_1}$$

Similarly, the eigenvector of M^* associated with $-i\omega_0$ is written as $\rho^*(s)=\overline{D}(1,\alpha^*)e^{i\omega_0 s}$, with

$$\alpha^*=\frac{-i\omega_0-l_1-ne^{i\omega_0\tau_1}}{l_2+pe^{i\omega_0\tau_2}}$$

Using the fact that $<\rho^*(s),\rho(s)>=1$, we have

$$\overline{D}=\frac{1}{1+\alpha\alpha^*+n\tau_1e^{-i\omega_0\tau_1}+p\alpha^*\tau_2e^{-i\omega_0\tau_2}+q\alpha\alpha^*\tau_2e^{-i\omega_0\tau_2}},$$

The following table gives the expression of K_{ij} that is important to determine the direction of Hopf bifurcation and the stability of periodic solutions. This result is obtained by Hassard's algorithm and the calculation steps used in [5].

Parameter	Expression
K_{11}	$a_1+b_1\alpha$
K_{21}	$a_2\alpha+b_2e^{-2i\omega_0\tau_2}\alpha$
K_{12}	$2a_1+b_1(\alpha+\bar{\alpha})$
K_{22}	$(a_2+b_2)(\alpha+\bar{\alpha})$
K_{13}	$a_1+b_1\bar{\alpha}$
K_{23}	$a_2\bar{\alpha}+b_2\bar{\alpha}e^{2i\omega_0\tau_2}$
K_{14}	$a_1\left(2w_{11}^{(1)}(0)+w_{20}^{(1)}(0)\right)+b_1\left(w_{11}^{(2)}(0)+\frac{w_{20}^{(2)}(0)}{2}+\bar{\alpha}\frac{w_{20}^{(1)}(0)}{2}+\alpha w_{11}^{(1)}(0)\right)+3c_1$
K_{24}	$a_2\left(w_{11}^{(2)}(0)+\frac{w_{20}^{(2)}(0)}{2}+\bar{\alpha}\frac{w_{20}^{(1)}(0)}{2}+p_1w_{11}^{(1)}(0)\right)$ $+b_2\left(e^{-i\omega_0\tau_2}\omega_{11}^{(2)}(-\tau_2)+e^{i\omega_0\tau_2}\frac{\omega_{20}^{(2)}(-\tau_2)}{2}+\bar{\alpha}\omega_{20}^{(1)}(-\tau_2)e^{i\omega_0\tau_2}+\alpha e^{-i\omega_0\tau_2}\omega_{11}^{(1)}(-\tau_2)\right)$

TABLE 6. Expressions for the parameters in (10)

The coefficients g_{ij} are given by the following formulas

$$(10)\qquad \begin{aligned} g_{20} =& \ 2\bar{D}(K_{11}+\bar{\alpha}^*K_{21}) \\ g_{11} =& \ \bar{D}(K_{12}+\bar{\alpha}^*K_{22}) \\ g_{02} =& \ 2\bar{D}(K_{13}+\bar{\alpha}^*K_{23}) \\ g_{21} =& \ 2\bar{D}(K_{14}+\bar{\alpha}^*K_{24}) \end{aligned}$$

However

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_0}\rho(\theta)e^{i\omega_0\theta}+\frac{i\overline{g}_{20}}{3\omega_0}\bar{\rho}(\theta)e^{-i\omega_0\theta}+\Lambda_1 e^{2i\omega_0\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}}{\omega_0}\bar{\rho}(\theta)e^{i\omega_0\theta}+\frac{i\overline{g}_{11}}{\omega_0}\bar{\rho}(\theta)e^{-i\omega_0\theta}+\Lambda_2, \end{aligned}$$

Where

$$\Lambda_1 = 2\begin{pmatrix} 2i\omega_0 - l_1 - ne^{-i\omega_0\tau_1} & -m_1 \\ -l_2 - pe^{-i\omega_0\tau_2} & 2i\omega_0 - m_2 - qe^{-i\omega_0\tau_2} \end{pmatrix}^{-1}\begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix}$$

and

$$\Lambda_2 = \begin{pmatrix} l_1+n & m_1 \\ l_2+p & m_2+q \end{pmatrix}^{-1}\begin{pmatrix} K_{12} \\ K_{22} \end{pmatrix}$$

Finally, we can compute the following results

$$\begin{aligned} &C_1(0) = \tfrac{i}{2\omega_0}(g_{11}g_{20}-2|g_{11}|^2-\tfrac{|g_{02}|^2}{3})+\tfrac{g_{21}}{2}, \\ &\eta_2 = -\tfrac{Re(C_1(0))}{Re(\lambda'(\tau_2^0))}, \\ &\beta_2 = 2Re(C_1(0)), \\ &T_2 = -\tfrac{Im(C_1(0))+\mu_2 Im(\lambda'(\tau_2^0))}{\omega_0} \end{aligned}$$

Theorem 3. *For system(8), under loss of generalities, we assume that $\tau_2^* < \tau_1^0$ and we get the following results:*

- *The direction of Hopf bifurcation is determined by the sign of η_2; if $\eta_2 > 0$ ($\eta_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the periodic solutions exist for $\tau_2 > \tau_2^0$ ($\tau_2 < \tau_2^0$).*
- *The stability of the periodic solution is determined by the sign of β_2: the bifurcations periodic solutions are orbitally asymptotically stable (unstable) if $\tau_2 > \tau_2^0$ ($\tau_2 < \tau_2^0$). The period of the periodic solutions is determined by the sign of T_2 : if $T_2 > 0$ ($T_2 < 0$), the periodic solutions increase (decrease).*

4. GLOBAL STABILITY

We choose the following Lyapunov function

$$V(t) = \alpha_1\left(x - x^* - \ln\left(\frac{x}{x^*}\right)\right) + \alpha_2\left(y - y^* - \ln\left(\frac{y}{y^*}\right)\right)$$

where α_1 and α_2 are positive constants. The derivation of this function will be expressed as follows

$$\begin{aligned}
\dot{V}(t) =& \alpha_1 \frac{x-x^*}{x}\dot{x} + \alpha_2 \frac{y-y^*}{y}\dot{y} \\
=& \alpha_1 (x-x^*)\left[\frac{ax}{x+b} - c - dx - g_1 y - E_1 \frac{x(t-\tau_1)}{x}\right] \\
&+ \alpha_2 (y-y^*)\left[-m + g_2 x - g_3 \frac{x(t-\tau_2)\, y(t-\tau_2)}{y} - E_2\right] \\
\leqslant& \alpha_1 (x-x^*)\left[\frac{ax}{x+b} - \frac{ax^*}{x^*+b} - d(x-x^*) - g_1 (y-y^*)\right] \\
&+ \alpha_2 (y-y^*)\left[g_2 (x-x^*) - g_3 (x-x^*)\right] \\
\leqslant& \alpha_1 ab (x-x^*)^2 \frac{1}{(x+b)(x^*+b)} - \alpha_1 d (x-x^*)^2 - \alpha_1 g_1 (x-x^*)(y-y^*) \\
&+ \alpha_2 (g_2 - g_3)(x-x^*)(y-y^*) \\
\leqslant& \alpha_1 \left(\frac{ab}{(x+b)(x^*+b)} - d\right)(x-x^*)^2 + \left(\alpha_2 (g_2-g_3) - \alpha_1 g_1\right)(x-x^*)(y-y^*)
\end{aligned}$$

we have $g_2 - g_3 > 0$, then we can choose α_1 and α_2 such as $\alpha_2(g_2 - g_3) = \alpha_1 g_1$. If the condition $ab < d(x+b)(x^*+b)$ is verified, then $\dot{V}(t) \leqslant 0$. Therefore, system (1) is globally asymptotically stable at the interior equilibrium point P^*.

5. DISCUSSION

In this section, numerical simulations will be performed to illustrate the theoretical results obtained in the previous sections. The simulations are performed by DDE-Biftool; a Matlab package designed for the numerical continuation and bifurcation analysis of the system. Its recent version DDE-Biftool V3.1.1 is conducted by J. Sieber in [2]. This numerical tool provides the course of the study with steady-state continuation, and periodic orbit solutions but also a bifurcation continuation in two parameters in which we are interested.

We start by giving the values of bioeconomic parameters; a = 2.8, b = 0.01, c = 0.3, d = 0.02, g_1 = 0.6, g_2 = 0.04, m = 0.1, g_3 = 0.01, E_1 = 0.3 and E_2 = 0.18. The system admits a single point of equilibrium strictly positive P*=(9.3333, 3.3505). In the following, we vary the two delays τ_1 and τ_2 between 0 and 15 to notice the impact of this variation on the equilibrium as well as its stability.

Then the 2 figures below show the variation of the equilibrium branch x according to the delay parameters τ_1 and τ_2, the two colors used to represent the nature of stability such that red shows the unstable part of the branch, green presents the stable part of the branch and asterisks are used to determine the points of Hopf bifurcation.

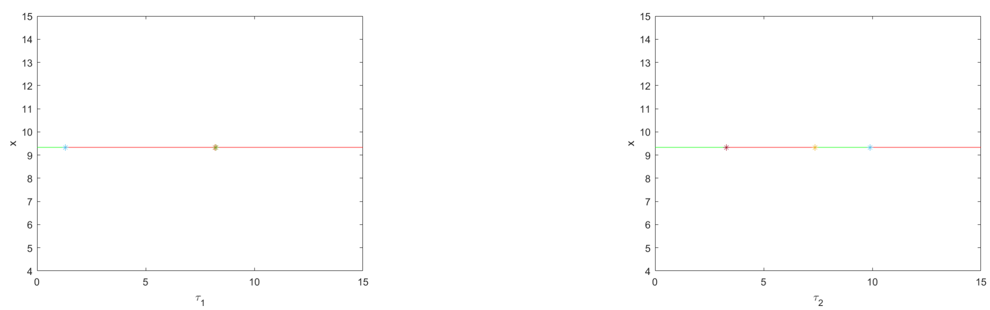


Fig 1. (Left) The equilibrium branch x with stability information for τ_1. (Right) The equilibrium branch x with stability information for τ_2

The following two figures represent the bifurcation diagrams which are obtained by the maximum and minimum amplitude of x, in which we notice that our system undergoes the Hopf bifurcation for τ_1=1.309 and τ_2 =3.284, moreover the nature of this bifurcation which is supercritical for both parameters.

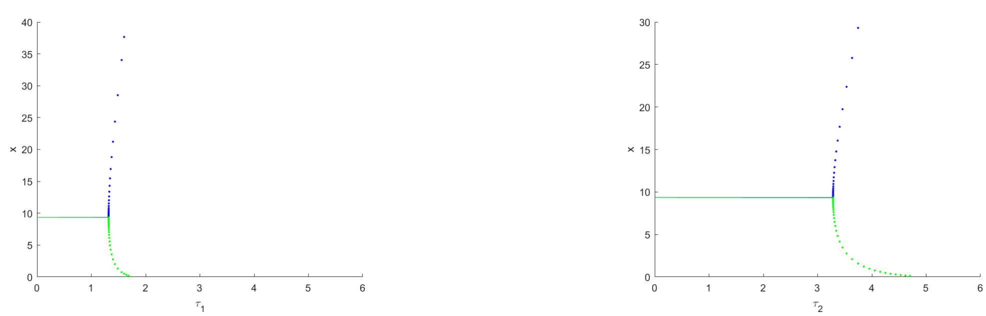
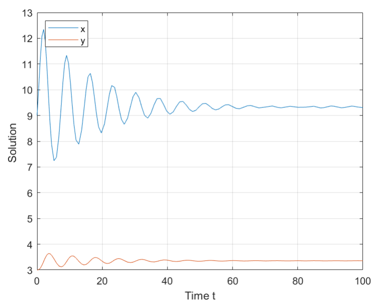
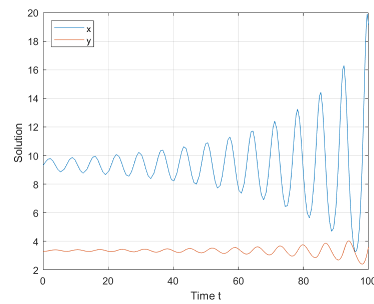


Fig 2. The Hopf bifurcation diagrams for τ_1 and τ_2

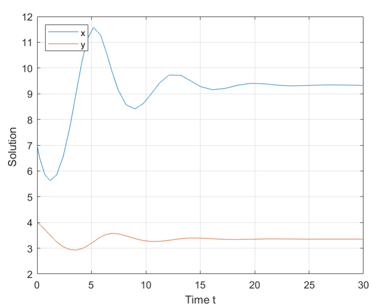
In the following, we will choose some values for the delay parameters and plot the variation of the system solution (1) in time around the equilibrium P*. The following figure contains six graphs. In the three graphs on the left, we have taken values of τ_1 and τ_2 located in the stability intervals of these two parameters and we notice that the solution converges toward the equilibrium point P^*, which is not the case in the three graphs on the right where chosen values of τ_1 and τ_2 are outside the stability interval.



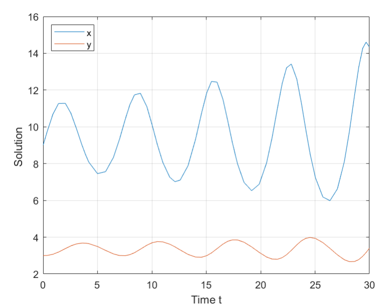
τ_1=1 and τ_2=0



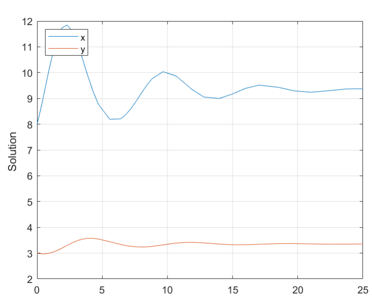
τ_1=1.5 and τ_2=0



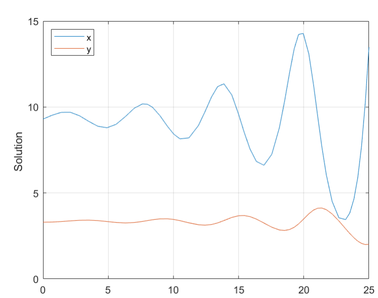
τ_1=0 and τ_2=1.2



τ_1=0 and τ_2=4.2



τ_1=0.4 and τ_2=0.5



τ_1=1.4 and τ_2=3.5

Fig 3. The temporal solution for different values of τ_1 and τ_2

Finally, we summarize our numerical study by the following figure in which we draw the line of Hopf in $\tau_1\tau_2$-plane. This line separates the stable region which is the area limited by τ_1-axis, τ_2-axis and the Hopf line. The other surface represents the unstable region.

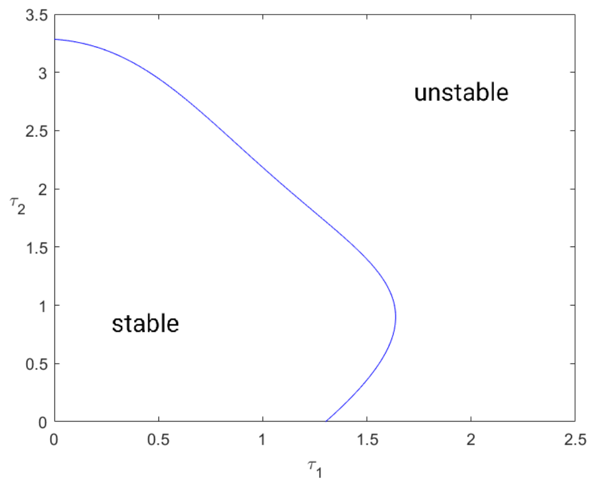


Fig 4. Stability region of P* and Hopf bifurcation curves in $\tau_1\tau_2$-plane

Conclusion

Our study focuses on the stability analysis of a prey-predator model consisting of sea urchins and crabs, taking into account the Allee effect in the sea urchin population, a mortality rate in crabs due to the sea urchins toxicity, and the fishing effect in both species. In addition, the fishery is considered selective for sea urchins, to conserve the juvenile population and preserve marine biodiversity. The stability analysis is established by the search for points of bifurcation and intervals of stability linked to delays. In the next work, we aim to add the diffusion effect for a more concretization of our research.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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