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POSITIVE PERIODIC SOLUTION OF A DISCRETE LOTKA-VOLTERRA COMMENSAL SYMBIOSIS MODEL

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Abstract. In this paper, sufficient conditions are obtained for the existence of positive periodic solution of the following discrete Lotka-Volterra commensal symbiosis model

 $x_1(k+1) = x_1(k) \exp\{a_1(k) - b_1(k)x_1(k) + c_1(k)x_2(k)\},\$

 $x_2(k+1) = x_2(k) \exp\{a_2(k) - b_2(k)x_2(k)\},\$

where $\{b_i(k)\}, i = 1, 2, \{c_1(k)\}$ are all positive ω -periodic sequences, ω is a fixed positive integer, $\{a_i(k)\}$ are ω -periodic sequences, which satisfies $\overline{a}_i = \frac{1}{\omega} \sum_{k=0}^{\omega-1} a_i(k) > 0, i = 1, 2$.

Keywords: Lotka-Volterra commensal symbiosis model; Positive periodic solution; Predator-prey.

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1. Introduction

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There are several relationship among the two species: competition, predator-prey, mutualism, etc. To this day, the dynamic behaviors of the mutualism model has been extensively investigated [1-12] and many excellent works concerned with the persistence, existence of positive periodic solution, and stability of the system were obtained. However, there are still few study on the commensal symbiosis model. The intraspecific commensal relationship means that a relationship which is only favorable to the one side and have no influence to the other side. Epiphyte and plants with epiphyte is one of the typical commensal relationship.

To describe the intraspecific commensal relationship, Sun and Wei [13] proposed the following model:

$$\frac{dx}{dt} = r_1 x \left(\frac{k_1 - x + ay}{k_1} \right),$$

$$\frac{dy}{dt} = r_2 y \left(\frac{k_2 - y}{k_2} \right).$$
(1.1)

The authors investigated the local stability of all equilibrium points. They showed that there is only one local stable equilibrium point in the system. However, they did not investigate the global stability property of the system.

As was pointed out by Fan and Wang [14], the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numeric simulations. This motivated us to propose the following discrete commensal symbiosis model

$$x_{1}(k+1) = x_{1}(k) \exp \left\{ a_{1}(k) - b_{1}(k)x_{1}(k) + c_{1}(k)x_{2}(k) \right\},$$

$$x_{2}(k+1) = x_{2}(k) \exp \left\{ a_{2}(k) - b_{2}(k)x_{2}(k) \right\},$$
(1.2)

where $\{b_i(k)\}, i = 1, 2, \{c_1(k)\}$ are all positive ω -periodic sequences, ω is a fixed positive integer, $\{a_i(k)\}$ are ω -periodic sequences, which satisfies $\overline{a}_i = \frac{1}{\omega} \sum_{k=0}^{\omega-1} a_i(k) > 0, i = 1, 2$. Here we assume that the coefficients of the system (1.2) are all periodic sequences which having a common integer period. Such an assumption seems reasonable in view of seasonal factors, e.g., mating habits, availability of food, weather conditions, harvesting, and hunting, etc. We only assume $\overline{a}_i = \frac{1}{\omega} \sum_{k=0}^{\omega-1} a_i(k) > 0, i = 1, 2$, the reason is that in bad conditions, the intrinsic growth rate $a_i(k)$ may be negative, hence, it is natural to use \overline{a}_i to describe the weight growth rate of

the species. As far as system (1.2) is concerned, one of the most important topic is to study the existence of positive periodic solution of the system, which plays a similar role played by the equilibrium of the autonomous models.

The aim of this paper is to obtain sufficient conditions to guarantee the existence of positive periodic solution of system (1.2).

2. Main results

In order to obtain the existence of positive periodic solutions of (1.2), for the reader's convenience, we shall summarize in the following a few concepts and results from [15] that will be basic for this paper.

Let X, Z be normed vector spaces, $L: DomL \subset X \to Z$ be a linear mapping, $N: X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = CodimImL < +\infty$ and ImL is closed in Z. If L is a Fredholm mapping of index zero there exist continuous projectors $P: X \to X$ and $Q: Z \to Z$ such that ImP = KerL, ImL =KerQ = Im(I - Q). It follows that $L|DomL \cap KerP: (I - P)X \to ImL$ is invertible. We denote the inverse of that map by K_P . If Ω be an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N: \overline{\Omega} \to X$ is compact. Since ImQis isomorphic to KerL, there exists an isomorphisms $J: ImQ \to KerL$.

In the proof of our existence theorem below, we will use the continuation theorem of Gaines and Mawhin([15, p40]).

Lemma 2.1. (Continuation Theorem) Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose

(a) For each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$;

(b) $QNx \neq 0$ for each $x \in \partial \Omega \cap KerL$ and

$$deg{JQN, \Omega \cap KerL, 0} \neq 0.$$

Then the equation Lx = Nx has at least one solution lying in $Dom L \cap \overline{\Omega}$.

Let Z, Z^+, R and R^+ denote the sets of all integers, nonnegative integers, real unumbers, and nonnegative real numbers, respectively. For convenience, in the following discussion, we will use the notation below throughout this paper:

$$I_{\omega} = \{0, 1, ..., \omega - 1\}, \ \overline{g} = \frac{1}{\omega} \sum_{k=0}^{\omega - 1} g(k), \ g^{u} = \max_{k \in I_{\omega}} g(k), \ g^{l} = \min_{k \in I_{\omega}} g(k),$$

where $\{g(k)\}$ is an ω -periodic sequence of real numbers defined for $k \in \mathbb{Z}$.

Lemma 2.2. [2] Let $g : Z \to R$ be ω -periodic, *i*. *e*., $g(k + \omega) = g(k)$. Then for any fixed $k_1, k_2 \in I_{\omega}$, and any $k \in Z$, one has

$$g(k) \le g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$

$$g(k) \ge g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

We now reach the position to establish our main result.

Theorem 2.1. System (1.2) admits at least one positive ω -periodic solution.

Proof. Let

$$x_i(k) = \exp\{u_i(k)\}, i = 1, 2,$$

so that system (1.2) becomes

$$u_1(k+1) - u_1(k) = a_1(k) - b_1(k) \exp\{u_1(k)\} + c_1(k) \exp\{u_2(k)\},$$

$$u_2(k+1) - u_2(k) = a_2(k) - b_2(k) \exp\{u_2(k)\}.$$
(2.1)

Define

$$l_2 = \left\{ y = \{ y(k) \}, y(k) = (y_1(k), y_2(k))^T \in \mathbb{R}^2 \right\}.$$

For $a = (a_1, a_2)^T \in \mathbb{R}^2$, define $|a| = \max\{|a_1|, |a_2|\}$. Let $l^{\omega} \subset l_2$ denote the subspace of all ω sequences equipped with the usual normal form $||y|| = \max_{k \in I_{\omega}} |y(k)|$. It is not difficult to show that l^{ω} is a finite-dimensional Banach space. Let

$$l_0^{\omega} = \{ y = \{ y(k) \} \in l^{\omega} : \sum_{k=0}^{\omega-1} y(k) = 0 \}, \ l_c^{\omega} = \{ y = \{ y(k) \} \in l^{\omega} : y(k) = h \in \mathbb{R}^2, k \in \mathbb{Z} \},$$

then l_0^{ω} and l_c^{ω} are both closed linear subspace of l^{ω} , and

$$l^{\omega} = l_0^{\omega} \oplus l_c^{\omega}, \ dim l_c^{\omega} = 2.$$

Now let us define $X = Y = l^{\omega}$, (Ly)(k) = y(k+1) - y(k). It is trivial to see that L is a bounded linear operator and

$$KerL = l_c^{\omega}$$
, $ImL = l_0^{\omega}$, $dimKerL = 2 = CodimImL$.

Then it follows that L is a Fredholm mapping of index zero. Let

$$N(u_1, u_2)^T = (N_1, N_2)^T := N(u, k),$$

where

$$\begin{cases} N_1 = a_1(k) - b_1(k) \exp\{u_1(k)\} + c_1(k) \exp\{u_2(k)\}, \\ N_2 = a_2(k) - b_2(k) \exp\{u_2(k)\}. \end{cases}$$
$$Px = \frac{1}{\omega} \sum_{s=0}^{\omega-1} x(s), x \in X, \ Qy = \frac{1}{\omega} \sum_{s=0}^{\omega-1} y(s), y \in Y. \end{cases}$$

It is not difficult to show that P and Q are two continuous projectors such that

$$ImP = KerL$$
 and $ImL = KerQ = Im(I - Q)$.

Furthermore, the generalized inverse (to L) K_p : ImL \rightarrow KerP \cap DomL exists and is given by

$$K_p(z) = \sum_{s=0}^{k-1} z(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) z(s).$$

Thus

$$QNx = \frac{1}{\omega} \sum_{k=0}^{\omega-1} N(x,k),$$
$$Kp(I-Q)Nx = \sum_{s=0}^{k-1} N(x,s) + \frac{1}{\omega} \sum_{s=0}^{\omega-1} sN(x,s) - \left(\frac{k}{\omega} + \frac{\omega-1}{2\omega}\right) \sum_{s=0}^{\omega-1} N(x,s)$$

Obviously, QN and $K_p(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, it is not difficult to show that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on any open bounded set $\Omega \subset X$. The isomorphism J of ImQ onto KerL can be the identity mapping, since ImQ=KerL.

Now we are at the point to search for an appropriate open, bounded subset Ω in *X* for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0, 1)$, we have

$$u_1(k+1) - u_1(k) = \lambda[a_1(k) - b_1(k)\exp\{u_1(k)\} + c_1(k)\exp\{u_2(k)\}],$$

$$u_2(k+1) - u_2(k) = \lambda[a_2(k) - b_2(k)\exp\{u_2(k)\}].$$
(2.2)

Suppose that $y = (y_1(k), y_2(k))^T \in X$ is an arbitrary solution of system (2.6) for a certain $\lambda \in (0, 1)$. Summing on both sides of (2.2) from 0 to $\omega - 1$ with respect to *k*, we reach

$$\sum_{k=0}^{\omega-1} [a_1(k) - b_1(k) \exp\{u_1(k)\} + c_1(k) \exp\{u_2(k)\}] = 0,$$

$$\sum_{k=0}^{\omega-1} [a_2(k) - b_2(k) \exp\{u_2(k)\}] = 0.$$

That is,

$$\sum_{k=0}^{\omega-1} b_1(k) \exp\{u_1(k)\} = \bar{a}_1 \omega + \sum_{k=0}^{\omega-1} c_1(k) \exp\{u_2(k)\},$$
(2.3)

$$\sum_{k=0}^{\omega-1} b_2(k) \exp\{u_2(k)\} = \bar{a}_2 \omega.$$
(2.4)

From (2.3) and (2.4), we have

$$\begin{split} &\sum_{k=0}^{\omega-1} |u_1(k+1) - u_1(k)| \\ &= \lambda \sum_{k=0}^{\omega-1} |a_1(k) - b_1(k) \exp\{u_1(k)\} + c_1(k) \exp\{u_2(k)\}| \\ &\leq \sum_{k=0}^{\omega-1} |a_1(k)| + \sum_{k=0}^{\omega-1} (b_1(k) \exp\{u_1(k)\} + c_1(k) \exp\{u_2(k)\}) \\ &= \sum_{k=0}^{\omega-1} |a_1(k)| + \bar{a}_1 \omega + 2 \sum_{k=0}^{\omega-1} c_1(k) \exp\{u_2(k)\} \\ &= (\bar{A}_1 + \bar{a}_1) \omega + 2 \sum_{k=0}^{\omega-1} c_1(k) \exp\{u_2(k)\}, \\ &\qquad \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| \\ &= \lambda \sum_{k=0}^{\omega-1} |a_2(k) - b_2(k) \exp\{u_2(k)\}| \\ &\leq (\bar{A}_2 + \bar{a}_2) \omega. \end{split}$$

$$(2.5)$$

where $\bar{A}_1 = \frac{1}{\omega} \sum_{k=0}^{\omega-1} |a_1(k)|, \ \bar{A}_2 = \frac{1}{\omega} \sum_{k=0}^{\omega-1} |a_2(k)|.$ Since $\{u(k)\} = \{(u_1(k), u_2(k))^T\} \in X$, there exist $\eta_i, \delta_i, i = 1, 2$ such that

$$u_i(\eta_i) = \min_{k \in I_{\omega}} u_i(k), \ u_i(\delta_i) = \max_{k \in I_{\omega}} u_i(k).$$
(2.6)

By (2.4), we have

$$\exp\{u_2(\eta_2)\}\sum_{k=0}^{\omega-1}b_2(k)\leq \bar{a}_2\omega.$$

So

$$u_2(\eta_2) \le \ln \frac{\bar{a}_2}{\bar{b}_2}.\tag{2.7}$$

It follows from Lemma 2.2, (2.5) and (2.7) that

$$u_2(k) \le u_2(\eta_2) + \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| \le \ln \frac{\bar{a}_2}{\bar{b}_2} + (\bar{A}_2 + \bar{a}_2)\omega.$$
(2.8)

From (2.4), we also have

$$\exp\{u_2(\delta_2)\}\sum_{k=0}^{\omega-1}b_2(k)\geq \bar{a}_2\omega,$$

and so

$$u_2(\delta_2) \ge \ln \frac{\bar{a}_2}{\bar{b}_2}.\tag{2.9}$$

It follows from Lemma 2.2, (2.5) and (2.9) that

$$u_2(k) \ge u_2(\delta_2) - \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| \ge \ln \frac{\bar{a}_2}{\bar{b}_2} - (\bar{A}_2 + \bar{a}_2)\omega, \qquad (2.10)$$

which together with (2.8) leads to

$$|u_2(k)| \le \max\left\{ |\ln\frac{\bar{a}_2}{\bar{b}_2} + (\bar{A}_2 + \bar{a}_2)\omega|, |\ln\frac{\bar{a}_2}{\bar{b}_2} - (\bar{A}_2 + \bar{a}_2)\omega| \right\} \stackrel{\text{def}}{=} H_2.$$
(2.11)

It follows from (2.3) and (2.8) that

$$\sum_{k=0}^{\omega-1} b_1(k) \exp\{u_1(\eta_1)\} \leq \bar{a}_1 \omega + \sum_{k=0}^{\omega-1} c_1(k) \exp\{(\bar{A}_2 + \bar{a}_2) \omega + \ln \frac{\bar{a}_2}{\bar{b}_2}\},\$$

and so,

$$u_1(\eta_1) \le \ln \frac{\Delta_1}{\overline{b}_1},\tag{2.12}$$

where

$$\Delta_1 = \bar{a}_1 + \frac{\bar{c}_1 \bar{a}_2}{\bar{b}_2} \exp\{(\bar{A}_2 + \bar{a}_2)\omega\}.$$

It follows from Lemma 2.2, (2.6) and (2.12) that

$$u_{1}(k) \leq u_{1}(\eta_{1}) + \sum_{k=0}^{\omega-1} |u_{1}(k+1) - u_{1}(k)|$$

$$\leq (\bar{A}_{1} + \bar{a}_{1})\omega + \ln \frac{\Delta_{1}}{\bar{b}_{1}} + 2 \frac{\bar{c}_{1}\bar{a}_{2}\omega}{\bar{b}_{2}} \exp\{(\bar{A}_{2} + \bar{a}_{2})\omega\} \stackrel{\text{def}}{=} M_{1}.$$
(2.13)

It follows from (2.3) and (2.10) that

$$\sum_{k=0}^{\omega-1} b_1(k) \exp\{u_1(\delta_1)\} \geq \bar{a}_1 \omega + \sum_{k=0}^{\omega-1} c_1(k) \exp\{\ln \frac{\bar{a}_2}{\bar{b}_2} - (\bar{A}_2 + \bar{a}_2)\omega\},\$$

and so,

$$u_1(\delta_1) \ge \ln \frac{\Delta_2}{\overline{b}_1},\tag{2.14}$$

where

$$\Delta_2 = \bar{a}_1 + \frac{\bar{c}_1 \bar{a}_2}{\bar{b}_2} \exp\{-(\bar{A}_2 + \bar{a}_2)\omega\}.$$

It follows from Lemma 2.2, (2.6) and (2.14) that

$$u_{1}(k) \geq u_{1}(\delta_{1}) - \sum_{k=0}^{\omega-1} |u_{1}(k+1) - u_{1}(k)|$$

$$\geq \ln \frac{\Delta_{2}}{\bar{b}_{1}} - (\bar{A}_{1} + \bar{a}_{1})\omega - 2\frac{\bar{c}_{1}\bar{a}_{2}\omega}{\bar{b}_{2}} \exp\{-(\bar{A}_{2} + \bar{a}_{2})\omega\} \stackrel{\text{def}}{=} M_{2}.$$
(2.15)

It follows from (2.13) and (2.15) that

$$|u_1(k)| \le \max\left\{|M_1|, |M_2|\right\} \stackrel{\text{def}}{=} H_1.$$
 (2.16)

Clearly, H_1 and H_2 are independent on the choice of λ . Obviously, the system of algebraic equations

$$\bar{a}_1 - \bar{b}_1 x_1 + \bar{c}_1 x_2 = 0, \ \bar{a}_2 - \bar{b}_2 x_2 = 0$$
 (2.17)

has a unique positive solution $(x_1^*, x_2^*) \in \mathbb{R}_2^+$, where

$$x_1^* = \frac{\bar{a}_1 + \bar{c}_1 x_2^*}{\bar{b}_1}, \ x_2^* = \frac{\bar{a}_2}{\bar{b}_2}.$$

Let $H = H_1 + H_2 + H_3$, where $H_3 > 0$ is taken sufficiently enough large such that $||(\ln\{x_1^*\}, \ln\{x_2^*\})^T|| = |\ln\{x_1^*\}| + |\ln\{x_2^*\}| < H_3$. Let $H = H_1 + H_2 + H_3$, and define

$$\Omega = \left\{ u(t) = (u_1(k), u_2(k))^T \in X : ||u|| < H \right\}$$

It is clear that Ω verifies requirement (a) in Lemma 2.1. When $u \in \partial \Omega \cap KerL = \partial \Omega \cap R^2$, *u* is constant vector in R^2 with ||u|| = B. Then

$$QNu = \begin{pmatrix} \bar{a}_1 - \bar{b}_1 \exp\{u_1\} + \bar{c}_1 \exp\{u_2\} \\ \bar{a}_2 - \bar{b}_2 \exp\{u_2\} \end{pmatrix} \neq 0.$$

Moreover, direct calculation shows that

$$deg\{JQN, \Omega \cap KerL, 0\} = \operatorname{sgn}\left(\bar{b}_1\bar{b}_2\exp\{x_1^*\}\exp\{x_2^*\}\right) = 1 \neq 0.$$

where deg(.) is the Brouwer degree and the J is the identity mapping since ImQ = KerL.

By now we have proved that Ω verifies all the requirements in Lemma 2.1. Hence (2.1) has at

least one solution $(u_1^*(k), u_2^*(k))^T$, in $DomL \cap \overline{\Omega}$. And so, system (1.2) admits a positive periodic solution $(x_1^*(k), x_2^*(k))^T$, where $x_i^*(k) = \exp\{u_i^*(k)\}, i = 1, 2$, This completes the proof of the claim.

Conflict of Interests

The author declares that there is no conflict of interests.

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