MATHEMATICAL ANALYSIS OF THE SMALL OSCILLATION OF A
HEAVY HETEROGENEOUS VISCOUS LIQUID IN AN OPEN IMMOVABLE
CONTAINER

H. ESSAOUNI1,*, J. ELBAHAOU1, L. ELBAKKALI1 AND P. CAPODANNO2

1University Abdelmalek Essaâdi, Faculty of Science, M2SM ER28/FS/05, 93000, Tetuan, Morocco

2Université de Franche-Comté, 2B Rue des jardins, 2500 Besançon, France

Copyright © 2014 H. Essaouini et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract: The aim of this paper is the study of the influence of the viscosity on the oscillations of a heterogeneous liquid in a container. Above all, it is proved that the presence of viscosity removes the essential spectrum which appears in the case of a heterogeneous inviscid liquid. From the equations of the system container-liquid, we deduce the variational equation of the problem, and then an operatorial equation in a suitable Hilbert space. The study of the normal oscillations is reduced to the study of an operator bundle whose kind is well known. We obtain an infinity of a periodic damped motions and, for sufficiently small viscosity, a finite number of oscillatory damped motions. The existence and uniqueness of the associated evolution problem are then proved using the weak formulation.

Key words: heterogeneous and viscous liquid; small oscillations; variational method; operator bundle; spectral and evolution problems.

2010 AMS Subject Classifications: 76D03, 76D05, 49R05, 47A75.
1. Introduction

Studying small oscillations of a container, partially filled by a heavy liquid is a subject of a great interest in engineering for example: construction of tanks, of trucks for the companies of transport of liquids, double-skin construction of ships, etc.

The theoretical results are very important for numerical and experimental calculations of hydroelastic properties and dynamic characteristics of such structures.

The case of an immovable container with homogeneous and viscous heavy liquid has been studied by many authors: [1], [2], [9].

The problem of a heterogeneous ideal heavy liquid was studied at first by Rayleigh by considering the density of the liquid in equilibrium under the form \( \rho_0(x) = ke^{-\beta x} \) [10].

It seems that the problem has been considered next by a limited number of authors [10], [7], [3], [4].

Moreover, Capodanno and his collaborators have studied the problem when the liquid is inviscid and "almost homogeneous", (i.e its density in the equilibrium position is a linear function of the depth, which a little from a constant) [5], [6]. This problem is more complicated because, in this case, an essential spectrum appears, in contrast to the case of a homogeneous liquid, where the spectrum is entirely discrete.

The aim of this work is to prove that the presence of the viscosity removes the essential spectrum.

After writing the general equations of motion of the liquid, we linearize the problem assuming small displacements from an equilibrium position. We reformulate the equations as a variational problem and finally as an operatorial problem involving non bounded linear operators on suitable function spaces.

In this way, we reduce the problem to the study of a well-known operator bundle. There are an infinity of aperiodic damped motions and, for a sufficiently small viscosity, an at most finite number of oscillatory damped motions.

Finally, using the weak formulation, we give an existence and uniqueness theorem for the solution of the evolution problem.
2. Position of the problem

Let assume that a heavy heterogeneous liquid fills partially an immovable container and in the equilibrium state occupies a domain $\Omega$ that is bounded by the solid boundary $S$ and the free surface $\Gamma$. $\Gamma$ is a plane that is orthogonal to the acceleration $\bar{g}$ of the gravitation field. As usual, we choose the system of coordinates $Ox_1x_2x_3$ such that $\bar{g} = -g\bar{x}_3$ and its center $O$ is located on the equilibrium surface $\Gamma$ (Figure 1).

We are going to study the small oscillations of the liquid about its equilibrium position, obviously in linear theory.

As usual, we are considering that the linearized velocities and accelerations are “true” velocities and accelerations, in order to avoid writing needless formulas in the following calculations.

3. Equations of the motion

We denote with $\mathbf{u}(x,t)$ the small displacement of a particle of the liquid which occupies the position $x$ at the instant $t$ from its equilibrium position. $\rho^*(x,t)$, $P^*(x,t)$ are respectively the density and pressure.

The coefficient of viscosity of the liquid at constant temperature $\mu^*$ is function of the density $\rho^*$, we have $\mu^* = \mu^*(\rho^*)$.

The equations of the motion of the liquid can be written in the form:
\begin{align*}
\rho^* \ddot{u}_i &= -\frac{\partial P^*}{\partial x_j} - \rho^* g \delta_{i3} \quad \text{for } i = 1, 2, 3 \\
\text{div}(\dot{u}) &= 0 \quad \text{in } \Omega \text{ (incompressibility)},
\end{align*}

where \( \Sigma_i(\dot{u}, P^*) \) are the components of the stress tensor and \( \delta_i \) indicates the Kronecker’s symbol.

We add the laws between the stresses and the velocities of deformation:

\[ \Sigma_i(\dot{u}, P^*) = -P^* \delta_{ij} + 2\mu^*(\rho^*) \varepsilon_{ij}(\dot{u}). \]  

\( \varepsilon_{ij}(\dot{u}) = \frac{1}{2} \left( \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) \) are the components of the tensor of the velocities of deformation.

We write the \textbf{linearized Navier-Stokes equation of the motion of the liquid} in the following form:

\begin{align*}
\rho^* \ddot{u}_i &= -\frac{\partial P^*}{\partial x_i} + \frac{\partial}{\partial x_i} \left( 2\mu^*(\rho^*) \varepsilon_{ij}(\dot{u}) \right) \quad \text{for } i = 1, 2, 3 \\
\text{div}(\dot{u}) &= 0 \quad \text{in } \Omega \text{ (incompressibility)},
\end{align*}

We must add the continuity equation

\[ \frac{\partial \rho^*}{\partial t} + \text{div}(\rho^* \dot{u}) = 0 \quad \text{in } \Omega \]  

Since \( \text{div}(\dot{u}) = 0 \), the equation (3.4) becomes

\[ \frac{\partial \rho^*}{\partial t} + \dot{u} \cdot \text{grad}\rho^* = 0 \quad \text{in } \Omega \]  

On the other hand, in linear theory, we can integrate the equation \( \text{div}(\dot{u}) = 0 \) from the date of the equilibrium position to the instant \( t \), and we obtain

\[ \text{div}(\dot{u}) = 0 \quad \text{in } \Omega \]  

- Now we consider the \textbf{boundary conditions}:
i) The no-slip condition at the rigid wall $S$ is:

$$\mathbf{u}_S = 0;$$

in the same way, we can replace it by

$$\mathbf{u}_S = 0; \quad (3.7)$$

In the following we denote by $\mathbf{n}$ the unit vector of the external normal to $\partial \Omega$.

ii) In order to write the dynamic conditions, we consider the linearized equation $x_3 = u_{n,\tau}$ of the free moving surface $\Gamma_r$ (Figure 2), then we must have

$$\Sigma_{ij} \left( \mathbf{\dot{u}}, P^* \right) n_{ij} = -p_a n_{ij} \quad \text{on} \quad \Gamma_r$$

where $p_a$ is the atmospheric pressure, which is assumed to be constant.

Taking into account the equation (2.3), the equation (3.8) becomes

$$\left[ -P_{\Gamma_r} \delta_{ij} + 2\mu^* \varepsilon_{ij} \left( \mathbf{\dot{u}} \right) \right] n_{ij} = -p_a n_{ij} \quad \text{on} \quad \Gamma_r$$

This equation can be be written, since the coefficients of the $n_{ij}$ are small:

$$- \left( P_{\Gamma_r}^* - p_a \right) n_i + 2\mu^* \varepsilon_{ij} \left( \mathbf{\dot{u}} \right) n_j = 0 \quad \text{on} \quad \Gamma$$

On $\Gamma$, we have $n_1 = n_2 = 0$, $n_3 = 1$, and from (3.10), we can deduce

$$\begin{cases}
\varepsilon_{13} \left( \mathbf{\dot{u}} \right) = 0, & \varepsilon_{23} \left( \mathbf{\dot{u}} \right) = 0 \\
- \left( P_{\Gamma_r}^* - p_a \right) + 2\mu^* \left( \rho^* \right) \varepsilon_{33} \left( \mathbf{\dot{u}} \right) = 0
\end{cases} \quad \text{on} \quad \Gamma$$

We are going to study the equilibrium of the system:

If $\rho_0$ and $P_a$ are the density and the pressure of the liquid in the equilibrium position, we have

$$\text{grad} P_a = -\rho_0 g \, \mathbf{x}_3,$$

so that $P_a$ and $\rho_0$ are functions of $x_3$, with

$$\frac{d P_a \left( x_3 \right)}{d x_3} = -\rho_0 \left( x_3 \right) g,$$  \quad (3.12)
We have
\[ P_{st} = -g \int_{0}^{w} \rho_0(w) \, dw + p_a \]
Setting
\[ R_0(x_3) = \int_{0}^{w} \rho_0(w) \, dw, \]
we can write
\[ P_{st} = -gR_0(x_3) + p_a, \]
and
\[ R_0(0) = 0, \quad R'_0(x_3) = -\rho_0(x_3) \]

Finally the **Navier-Stokes equations** take the form:
\[ \rho^* \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} - \left( \rho^* - \rho_0(x_3) \right) g \delta_{ij} \quad \text{in} \quad \Omega \]
where \( p(x,t) = P^*(x,t) - P_{st} \) is the dynamic pressure, and \( \sigma_{ij} = -p \delta_{ij} + 2\mu^* \varepsilon_{ij}(\ddot{u}) \).

The equations (3.15), (3.5), (3.6), (3.7), (3.11) are the equations of the small motions of the system.
From these equations we want to deduce the variational equation of the problem.

### 4. Formal variational formulation of the problem

i) For a formal calculation, we introduce the space of the admissible displacements:
\[ W = \left\{ \mathbf{v} / \text{div}(\mathbf{v}) = 0, \quad \mathbf{v}|_{\partial \Omega} = 0 \right\}, \]
with \( \mathbf{v} \) sufficiently smooth. This space will be précised later.

By multiplying the i-th Navier-Stokes equation (3.15) for \( \bar{v}_i \), by adding with respect to i, and by integrating on \( \Omega \), we obtain
\[ \int_{\Omega} \rho^* \ddot{\mathbf{u}} \cdot \bar{v} \, d\Omega = \int_{\Omega} \frac{\partial \sigma_{ij}}{\partial x_j} \cdot \bar{v}_i \, d\Omega - \int_{\Omega} \left[ \rho^* - \rho_0(x_3) \right] g \bar{v}_i \, d\Omega \]
But
\[ \int_{\Omega} \frac{\partial \sigma_{ij}}{\partial x_j} \cdot \nabla_i d\Omega = \int_{\Omega} \frac{\partial}{\partial x_j} \left( \sigma_{ij} \cdot \nabla_i \right) d\Omega - \int_{\Omega} \sigma_{ij} \frac{\partial v_i}{\partial x_j} d\Omega, \]

and the Green formula gives

\[ \int_{\Omega} \frac{\partial}{\partial x_j} \left( \sigma_{ij} \cdot \nabla_i \right) d\Omega = \int_{\Gamma} \sigma_{ij} n_j \cdot \nabla_i d\sigma \]

From (3.11), we deduce \( \sigma_{13} = \sigma_{23} = 0 \) on \( \Gamma \), on the other hand, we have

\[ \sigma_{33} = -p_{1r} + 2\mu^* e_{33}, \]

\[ = -p_{1r} + p_{j}^* - p_a \]

\[ = P_{ujr} - p_a \]

\[ = -gR_0 \left( u_{i\Gamma} \right) \]

Taking into account that \( v_{jr} = 0 \), we get easily

\[ \int_{\Omega} \frac{\partial}{\partial x_j} \left( \sigma_{ij} \cdot \nabla_i \right) d\Omega = \int_{\Gamma} \sigma_{ij} n_j \cdot \nabla_i d\Gamma = -\int_{\Gamma} gR_0 \left( u_{i\Gamma} \right) \nabla_3 d\Gamma \]

As \( \sigma_{ij} \) are symmetric,

\[ \int_{\Omega} \sigma_{ij} \frac{\partial v_i}{\partial x_j} d\Omega = \frac{1}{2} \int_{\Omega} \left( \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \sigma_{ji} \frac{\partial v_j}{\partial x_i} \right) d\Omega = \int_{\Omega} \sigma_{ij} e_{ij} (\bar{v}) d\Omega \]

therefore

\[ \int_{\Omega} \sigma_{ij} \frac{\partial v_i}{\partial x_j} d\Omega = \int_{\Omega} \left( -p e_{ij} + 2\mu^* e_{ij} (\hat{u}) \right) e_{ij} (\bar{v}) d\Omega \]

But

\[ \delta_{ij} e_{ij} (\bar{v}) = e_{ij} (\bar{v}) = \text{div} (\bar{v}) = 0, \]

Finally, we have

\[ \int_{\Omega} \sigma_{ij} \frac{\partial v_i}{\partial x_j} d\Omega = \int_{\Omega} 2\mu^* (\rho^*) e_{ij} (\hat{u}) e_{ij} (\bar{v}) d\Omega \]

Therefore, we obtain the variational equation
Reciprocally, we are proving that, from the equation (4.1), we can deduce the equations of motion and the dynamic boundary conditions of the problem.

We take \( \bar{v}(x) \) sufficiently smooth in \( \Omega \), such that \( \bar{v}_{\partial \Omega} = 0 \), but not verifying \( \text{div} \bar{v} = 0 \).

Then, introducing a multiplier \( \lambda_0 \) associated to the constraint \( \text{div} \bar{v} = 0 \), we replace the equation (4.1) by

\[
\left\{ \begin{array}{l}
\int_\Omega \rho^* \dddot{u} \cdot \dot{\bar{v}} \, d\Omega + \int_\Omega 2\mu^* (\rho^*) \varepsilon_{ij} \left( \dddot{u} \right) \varepsilon_{ij} \left( \bar{v} \right) \, d\Omega + \int_\Gamma gR_0 \left( u_{3p} \right) \bar{v}_{3p} \, d\Gamma \\
+ \int_\Omega \left[ \rho^* - \rho_0 (x_3) \right] g \bar{v}_j \, d\Omega = 0
\end{array} \right.
\]

(4.2)

Setting

\[
\hat{\sigma}_j = -p \delta_j + 2\mu^* \varepsilon_{ij} \left( \dddot{u} \right)
\]

Taking into account that \( \hat{\sigma}_j \) are symmetric, we have

\[
\int_\Omega \sigma_{ij} \frac{\partial \bar{v}_j}{\partial x_i} \, d\Omega = \int_\Omega \sigma_{ij} \varepsilon_{ij} \left( \bar{v} \right) \, d\Omega
\]

\[
= \int_\Omega \left( \lambda_0 \delta_j + 2\mu^* \varepsilon_{ij} \left( \bar{u} \right) \right) \varepsilon_{ij} \left( \bar{v} \right) \, d\Omega
\]

and therefore

\[
\int_\Omega \hat{\sigma}_{ij} \frac{\partial \bar{v}_j}{\partial x_i} \, d\Omega = \int_\Omega \lambda_0 \text{div} \bar{v} \, d\Omega + \int_\Omega 2\mu^* \varepsilon_{ij} \left( \bar{u} \right) \varepsilon_{ij} \left( \bar{v} \right) \, d\Omega
\]

Then the equation (4.2) becomes

\[
\left\{ \begin{array}{l}
\int_\Omega \rho^* \dddot{u} \cdot \dot{\bar{v}} \, d\Omega + \int_\Omega \hat{\sigma}_{ij} \frac{\partial \bar{v}_j}{\partial x_i} \, d\Omega + \int_\Gamma gR_0 \left( u_{3p} \right) \delta_{ij} \bar{v}_i \, d\Gamma \\
+ \int_\Omega \left[ \rho^* - \rho_0 (x_3) \right] g \delta_{i3} \bar{v}_i \, d\Omega = 0
\end{array} \right.
\]

(4.3)

On the other hand, and by applying the Green formula

\[
\int_\Omega \frac{\partial \sigma_{ij}}{\partial x_i} \cdot \bar{v}_j \, d\Omega = \int_\Omega \frac{\partial}{\partial x_i} \left( \sigma_{ij} \bar{v}_j \right) - \sigma_{ij} \frac{\partial \bar{v}_j}{\partial x_i} \, d\Omega
\]
\[
= \int_{S + \Gamma} \sigma \frac{\partial \hat{\sigma}}{\partial x_j} n_j \, d\sigma - \int_{\Omega} \sigma \frac{\partial \hat{v}}{\partial x_j} \, d\Omega
\]

Taking into account that \( \hat{v} = 0 \), the equation (4.3) becomes

\[
\int_{\Omega} \left[ \rho^* \hat{\dot{u}}_i - \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} + \left( \rho^* - \rho_0(x_3) \right) g \delta_{ij} \right] \hat{v}_i \, d\Omega + \int_{\Gamma} \left[ \hat{\sigma}_{ij} n_j + gR_0(u_{3r}) \delta_{ij} \right] \hat{v}_{ij} \, d\Gamma = 0 \tag{4.4}
\]

We take \( \hat{v} \in [\partial (\Omega)]^3 \) (according to \( \hat{v} = 0 \)), and we have

\[
\int_{\Omega} \left[ \rho^* \hat{\dot{u}}_i - \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} + \left( \rho^* - \rho_0(x_3) \right) g \delta_{ij} \right] \hat{v}_i \, d\Omega, \quad \forall \hat{v} \in [\partial (\Omega)]^3
\]

therefore

\[
\rho^* \hat{\dot{u}}_i - \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} + \left( \rho^* - \rho_0(x_3) \right) g \delta_{ij} = 0, \quad \text{in } \partial'(\Omega); \ (i,j=1,2,3)
\]

Also, we deduce from (4.4)

\[
\int_{\Gamma} \left[ \hat{\sigma}_{ij} n_j + gR_0(u_{3r}) \delta_{ij} \right] \hat{v}_{ij} \, d\Gamma, \text{ for every admissible } \hat{v},
\]

then for arbitrary \( \hat{v}_r \).

We obtain

\[
\hat{\sigma}_{ij} n_j + gR_0(u_{3r}) \delta_{ij} = 0 \quad \text{on } \Gamma.
\]

Taking into account the definition of \( \hat{\sigma}_{ij} \), we have

\[
\lambda_0 n_i + 2\mu^* \epsilon_{ij} \left( \hat{u} \right) n_j + gR_0 \left( u_{3r} \right) \delta_{ij} = 0 \quad \text{on } \Gamma
\]

Now for \( i=1, i=2 \), we have respectively

\[
\epsilon_{13} \left( \hat{u} \right) = 0; \quad \epsilon_{23} \left( \hat{u} \right) = 0 \quad \text{on } \Gamma
\]

On the other hand, for \( i=3 \), we obtain

\[
\lambda_{3r} + 2\mu^* \epsilon_{32} \left( \hat{u} \right) + gR_0 \left( u_{3r} \right) = 0.
\]

Setting \( \lambda_{3r} = -p \) [15], we find the equations of motion and the dynamic boundary conditions.

iii) In order to give the variational equation in the case of the small motion, we
set:

$$\rho^* = \rho_0(x_3) + \tilde{\rho}(x,t) + \ldots$$

$\tilde{\rho}$ is the first order with respect to the amplitude of the oscillations and the dots indicate terms of higher order.

The continuity equation is, at the first order

$$\frac{\partial \tilde{\rho}}{\partial t} + \text{div} \left( \rho_0(x_3) \tilde{u} \right) = 0$$

or taking into account of $\text{div} \left( \tilde{u} \right) = 0$:

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{u} \cdot \text{grad} \rho_0(x_3) = 0 \quad \text{in } \Omega$$

Integrating from the instant of the equilibrium to the instant $t$, we obtain the linearized continuity equation

$$\tilde{\rho} = -u_3 \rho_0'(x_3)$$

(4.5)

Since $\varepsilon_{ij}(\tilde{u})$ is the first order, we can replace, in linear theory, the term $\mu^*(\rho^*)$ by $\mu^*(\rho_0(x_3))$.

On the other hand, we have the following equations

$$R \left( u_{3^n} \right) = R(0) + u_{3^n} R'(0) + \ldots = u_{3^n} \rho_0(0) + \ldots;$$

$$\rho^* - \rho_0(x_3) = \tilde{\rho}(x,t) + \ldots = -u_3 \rho_0'(x_3) + \ldots$$

Since $u_{3^n} = u_{3^n}$, we deduce the final variational equation of the problem

$$\begin{cases}
\int_\Omega \rho_0(x_3) \tilde{u} \cdot \tilde{v} \, d\Omega + \int_\Omega 2 \mu^*(\rho_0(x_3)) \varepsilon_{ij}(\tilde{u}) \varepsilon_{ij}(\tilde{v}) \, d\Omega \\
+ \rho_0(0) g \int_{\Gamma} u_{3^n} \nabla u_{3^n} \, d\Gamma + g \int_\Omega -\rho_0'(x_3) u_3 \nabla_3 \, d\Omega = 0
\end{cases}$$

(4.6)

$$\forall \tilde{v} \in W.$$

5. Variational formulation of the problem

Theorem 5.1. The exact variational formulation of the problem is: to find $\tilde{u} \in V$ such that:

$$\left( \tilde{u}, \tilde{v} \right)_H + \left( \tilde{u}, \tilde{v} \right)_V + \rho_0(0) g \int_{\Gamma} u_{3^n} \nabla u_{3^n} \, d\Gamma + g \int_\Omega -\rho_0'(x_3) u_3 \nabla_3 \, d\Omega = 0, \quad \forall \tilde{v} \in V$$

(5.1)

Proof.
In the following, we shall assume that \( \rho_0(x_3), \ \rho'_0(x_3) \) and \( \mu'(\rho_0(x_3)) \) are sufficiently smooth functions.

In this paragraph we are giving the exact variational formulation of the problem. For this reason we introduce the space (of the admissible displacements of the liquid):

\[
V = \left\{ \bar{u} \in \mathbb{H}^1(\Omega) = \left[H^1(\Omega)\right]^3; \ \text{div} \bar{u} = 0, \ \bar{u}_p = 0 \right\},
\]

equipped with the scalar product

\[
(\bar{u}, \bar{v})_V = \left( \int_{\Omega} 2\mu'(\rho_0(x_3)) \varepsilon_3(\bar{u}) \varepsilon_3(\bar{v}) d\Omega \right)^{1/2}
\]

Its associated norm \( \| \cdot \|_V \) is equivalent to the classical norm \( \| \cdot \| \) of \( \mathbb{H}^1(\Omega) \) by virtue of Korn inequality and under the hypothesis about \( \mu'(\rho_0(x_3)) \).

We denote with \( H \) the functional completion of \( V \) for the norm associated with the scalar product

\[
(\bar{u}, \bar{v})_H = \int_{\Omega} \rho_0(x_3) \bar{u} \cdot \bar{v} d\Omega
\]

It is easy to see that the norm \( \| \bar{u} \|_H \) is equivalent to the classical norm \( \| \bar{u} \|_{L^2(\Omega)} \) of \( L^2(\Omega) = \left[L^2(\Omega)\right]^3 \).

The embedding of \( V \) into \( H \) is, obviously, dense, continuous and it is compact (by virtue of Rellich theorem).

Then, we can deduce from (4.6) the variational equation (5.1).

6. Operatorial equation

**Theorem 6.1.** The operatorial equation is:

to find \( \hat{U}(\cdot) \in H \) such that

\[
A^{-1} \hat{U} + \hat{U} + \rho_0(0) g A^{1/2} T \gamma_n A^{-1/2} \hat{U} + g A^{-1/2} K A^{-1/2} \hat{U} = 0; \quad (6.3)
\]

where all the operators are bounded.

**Proof.**

In order to deduce an operatorial equation of the problem with bounded operators, we want to substitute the equation (5.1) with an equation on the space \( H \).
Classically [11], if $A$ is the unbounded operator of $H$ which is associated to the sesquilinear form $(\tilde{u}, \tilde{v})_H$, and to the pair $(V, H)$, we have

$$ (\tilde{u}, \tilde{v})_H = (A\tilde{u}, \tilde{v})_H, \quad \forall \tilde{u} \in D(A) \subset V \subset H, \quad \forall \tilde{v} \in V \quad \text{(a.6)} $$

For every $X \in L^2(\Gamma) = \{ w \in L^2(\Gamma), \int_{\Gamma} w \, d\Gamma = 0 \}$, we can write

$$ \left| \int_{\Gamma} X \bar{v}_{n|\Gamma} \, d\Gamma \right| \leq \|X\|_{L^2(\Gamma)} \|\bar{v}_{n|\Gamma}\|_{L^2(\Gamma)}, \quad \forall \bar{v} \in V $$

By virtue of a trace theorem, we obtain

$$ \left| \int_{\Gamma} X \bar{v}_{n|\Gamma} \, d\Gamma \right| \leq c_i \|X\|_{L^2(\Gamma)} \|\bar{v}\|_{V}, \quad (c_i > 0) $$

so that, by using a Riesz theorem, there exists a bounded operator $T$ from $L^2(\Gamma)$ into $V$ such that:

$$ \int_{\Gamma} X \bar{v}_{n|\Gamma} \, d\Gamma = (TX, \bar{v})_V, \quad \forall \bar{v} \in V $$

then, we have

$$ \int_{\Gamma} u_{n|\Gamma} \bar{v}_{n|\Gamma} \, d\Gamma = (Tu_{n|\Gamma}, \bar{v})_V = (TY_n\tilde{u}, \bar{v})_V, \quad \forall \tilde{u}, \bar{v} \in V \quad \text{(b.6)} $$

where $Y_n$ is the restriction of the application normal trace $(\Xi^1(\Omega) \to L^2(\Gamma))$ to $V$.

In the same manner, we can write

$$ \int_{\Omega} -\rho_0(x_j) u_j v_j \, d\Omega = (K\tilde{u}, \tilde{v})_H, \quad \forall \tilde{u}, \tilde{v} \in H \quad \text{(c.6)} $$

where $K$ is a non negative self-adjoint bounded operator from $H$ into $H$.

Finally, from (a.6), (b.6), (c.6), we can write the variational equation of the problem:

to find $\tilde{u}(\cdot) \in V$

$$ (\tilde{u}, \tilde{v})_H + (A\tilde{u}, \tilde{v})_H + \rho_0(0) g (ATY_n\tilde{u}, \tilde{v})_H + g (K\tilde{u}, \tilde{v})_H = 0, \quad \forall \tilde{v} \in V \quad \text{(6.1)} $$

This variational equation is equivalent to the operatorial equation

$$ \tilde{u} + A(\tilde{u} + \rho_0(0) g TY_n\tilde{u}) + g K\tilde{u} = 0, \quad \tilde{u} \in V. \quad \text{(6.2)} $$

In order to eliminate the unbounded operator, setting $A^{1/2} = \tilde{U} \in H$ and applying the operator $A^{-1/2}$, we obtain a final operatorial equation (6.3) of the problem.
Now, we are going to study the different operators which appears in the precedent operatorial equation (6.3).

i) it is well known that the operator $A^{-1}$ is compact of $H$ in $H$, self adjoint and positive definite;

ii) We consider the two operators: $A^{1/2}$ compact bounded from $H$ into $L^2(V)$, $A^{1/2}$ bounded from $H$ into $L^2(V)$, we have, for every $X \in L^2(\Gamma)$ and $\tilde{U} \in H$:

$$
\left( A^{1/2}_n T X, \tilde{U} \right)_H = \left( A^{1/2}_n T X, A^{1/2} \tilde{u} \right)_H = \left( T X, \tilde{u} \right)_H
= \left( X, u_n \right)_{E(\Gamma)}
= \left( X, \gamma_n \tilde{u} \right)_{E(\Gamma)}
= \left( X, \gamma_n A^{-1/2} \tilde{U} \right)_{E(\Gamma)}
$$

Then, the operators $\gamma_n A^{-1/2}$ and $A^{1/2} T$ are mutually adjoint, $\gamma_n A^{-1/2}$ is compact from $H$ in $L^2(\Gamma)$. By virtue of a Schauder’s theorem, $A^{1/2} T$ is compact too from $L^2(\Gamma)$ into $H$.

iii) Setting $B = A^{1/2} T \gamma_n A^{-1/2}$, it is easy to see that $B$ is a self-adjoint and compact operator from $L^2(\Gamma)$ in $L^2(\Gamma)$.

We have

$$
\left( B \tilde{U}, \tilde{U} \right)_H = \left\| \gamma_n A^{-1/2} \tilde{U} \right\|^2_{E(\Gamma)} \geq 0
$$

From $\left( B \tilde{U}, \tilde{U} \right)_H = 0$, we deduce $\gamma_n \tilde{u} = u_{0 \Gamma} = 0$, and then $\tilde{u}$ belongs to a space which contains

$$
J^1_0(\Omega) = \left\{ \tilde{u} \in \Xi^1(\Omega); \ \text{div} \ \tilde{v} = 0, \ \tilde{u} \big|_{\partial\Omega} = 0 \right\},
$$

therefore, $B$ is non negative.

iv) We have, $H \overset{\text{compact}}{\longrightarrow} H \overset{K}{\longrightarrow} H \overset{\text{compact}}{\longrightarrow} H$, therefore

$A^{-1/2} K A^{-1/2}$ is self-adjoint, non negative and compact operator from $H$ into $H$. 
7. Operator bundle of the problem and existence of the eigenvalues

**Theorem 7.1.** The problem has a countably eigenvalues $\lambda$ with real part positive admitting $\lambda = 0$ and $\lambda = \infty$ for points of accumulation.
There are countably aperiodic motions arbitrary strongly damped ($\lambda^+_{\infty}$ real $\to \infty$) and countably aperiodic motions arbitrary weakly damped ($\lambda^-_{\infty}$ real $\to 0$).

1) If $4\|A^{-1}\|\|B_0\| < 1$, all eigenvalues are real and there are no oscillatory motions (it is the case in which $\mu$ is sufficiently great).

2) If $4\|A^{-1}\|\|B_0\| \geq 1$, there is at most a finite number of complex eigenvalues, in the circular ring $\frac{1}{2\|A^{-1}\|} \leq |\lambda| \leq 2\|B_0\|$: oscillatory damped motions correspond to these eigenvalues.

**Proof.**

We are finding solutions of the equation (6.3) under the form:

$$\hat{U}(x,t) = e^{-\lambda t}U(x), \quad \lambda \in \mathbb{C};$$

we obtain

$$\lambda^2 A^{-1} \hat{U} - \lambda \hat{U} + \left[ \rho_0(0) g B + g A^{-1/2} K A^{-1/2} \right] \hat{U} = 0 \quad (7.1)$$

From the equation $\left[ \rho_0(0) g B + g A^{-1/2} K A^{-1/2} \right] \hat{U} = 0$ and taking into account of the properties of the operators $B$ and $K$, we deduce

$$\hat{u} \in V, \quad u_{n|\Gamma} = 0, \quad u_3 = 0,$$

so that $\hat{u}$ belongs to a space containing $J_{\Omega}^1(\Omega)$. Consequently $\lambda = 0$ is an eigenvalue with infinite multiplicity.

Now, we eliminate this case, and setting

$$B_0 = g \left[ \rho_0(0) B + A^{-1/2} K A^{-1/2} \right]$$

We obtain

$$L(\lambda) \hat{U} \equiv \left( I_{n|\Gamma} - \lambda A^{-1} - \lambda^{-1} B_0 \right) \hat{U} = 0, \quad \hat{U} \in H \quad (7.2)$$
The equation (7.2) is of the kind considered by Askerov, Krein, Laptev [1], i.e

\[ f = \lambda Pf + \frac{1}{\lambda} Qf , \quad f \in H \]

with

\[ P = A^{-1} \quad \text{and} \quad Q = B_0 \]

It is proved in paragraph 5 that \( P \) is self adjoint, compact, positive definite operator and \( Q \) is a self adjoint, compact and non negative operator.

We have the theorem 7.1.

8. Conclusions

i) The presence of viscosity removes the continuous spectrum which appears in the case of a heterogeneous inviscid liquid.

ii) The problem is reduced to the study of a classical Askerov, Krein, Laptev pencil. The small motions of the system depend on the viscosity coefficient. There are always damped motions, but damped oscillatory motions can appear only for weak viscosity coefficient.

9. Existence and uniqueness theorem

**Theorem 9.1.** If the initial data verify: \( \vec{w}^0 \in V ; \vec{\dot{w}}^0 \in H \), the problem:

\[ \text{to find} \ \vec{w}(t) \in V \quad \text{such that} \]

\[ \frac{d}{dt} c(\vec{\dot{w}}, \vec{v}) + b(\vec{\dot{w}}, \vec{v}) + a(\vec{w}, \vec{v}) = 0, \quad \forall \vec{v} \in V \]

has one and only one solution such that

\[ \vec{w} \in L^2(0,T;V) ; \quad \vec{\dot{w}} \in L^2(0,T;V) ; \]

where \( T \) is a positive constant.

**Proof.**

Setting \( \vec{w} = e^{t} \vec{w} \), the variational problem (5.1) is equivalent to:
To find \( \bar{w}(t) \in V \) such that
\[
\begin{aligned}
\left\{(\bar{w}, \bar{v})_H + (\bar{w}, \bar{v})_H + 2(\bar{w}, \bar{v})_H \right.
\left.+ (\bar{w}, \bar{v})_H, (\bar{w}, \bar{v})_H + \rho_0(0) g (T_{\gamma_n} \bar{w}, \bar{v}), + g (K \bar{w}, \bar{v})_H = 0 \right.
\forall \bar{v} \in V
\end{aligned}
\tag{9.1}
\]
We define the sesquilinear forms in \( V \times V \) [8,664-665]
\[
\begin{align*}
c(\bar{w}, \bar{v}) &= (\bar{w}, \bar{v})_H, \\
b(\bar{w}, \bar{v}) &= (\bar{w}, \bar{v})_H + 2(\bar{w}, \bar{v})_H, \\
a(\bar{w}, \bar{v}) &= (\bar{w}, \bar{v})_H + (\bar{w}, \bar{v})_H + \rho_0(0) g (T_{\gamma_n} \bar{w}, \bar{v}) + g (K \bar{w}, \bar{v})_H
\end{align*}
\]
The imbedding \( V \subset H \) is continuous, we have:
\[
\exists c_0 > 0 ; \text{ such that } \|\bar{v}\|_H \leq c_0 \|\bar{v}\|_V . \text{ As the trace application is continuous from } V \text{ into } L^2(\Gamma) , \text{ we get}
\]
\[
\left| \left( T_{\gamma_n} \bar{w}, \bar{v} \right)_V \right| \leq \left| \left( w_{n|r}, v_{n|r} \right)_{E(\Gamma)} \right| \leq \left\| w_{n|r} \right\|_{E(\Gamma)} \cdot \left\| v_{n|r} \right\|_{E(\Gamma)} \leq c_1 \|\bar{v}\|_V \cdot \|\bar{v}\|_V
\]
\[c_1 \text{ is a constant } > 0 , \]
then the last sesquilinear forms are continuous in \( V \times V \)

On the other hand, we have: \( \forall (\bar{w}, \bar{v}) \in V \times V \)
\[
\begin{itemize}
\item \( t \mapsto a(\bar{w}, \bar{v}) \) is a \( C^1([0,T]) \), \( a(\bar{w}, \bar{v}) = a(\bar{v}, \bar{w}) \) and \( a(\bar{v}, \bar{v}) \geq \|\bar{v}\|^2 \),
\item \( t \mapsto b(\bar{w}, \bar{v}) \) is a \( C^1([0,T]) \), \( b(\bar{w}, \bar{v}) = b(\bar{v}, \bar{w}) \) and \( b(\bar{v}, \bar{v}) \geq \|\bar{v}\|^2 \),
\item \( t \mapsto c(\bar{w}, \bar{v}) \) is a \( C^1([0,T]) \), \( c(\bar{w}, \bar{v}) = c(\bar{v}, \bar{w}) \) and \( c(\bar{v}, \bar{v}) = \|\bar{v}\|^2 \).
\end{itemize}

From (9.1), we deduce the following problem:

to find \( \bar{w}(t) \in V \) such that
\[
\frac{d}{dt} c(\bar{w}, \bar{v}) + b(\bar{w}, \bar{v}) + d(\bar{w}, \bar{v}) = 0 , \quad \forall \bar{v} \in V \tag{9.2}
\]

Therefore, by virtue of a known theorem [8, pp 664-670], we have the proof of theorem 9.1.
REFERENCES


