END EDGE DOMINATION IN SUB DIVISION OF GRAPHS

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Abstract: Let \( S(G) \) be the subdivision graph of a graph \( G = (V, E) \). An edge dominating set \( D \) of a subdivision graph \( S(G) \) is an end edge dominating set if \( D \) contains all end edges of \( S(G) \). The end edge domination number \( \gamma^e_{\text{SG}}(G) \) of \( S(G) \) is the minimum cardinality of an end edge dominating set of \( S(G) \). In this paper, some bounds for \( \gamma^e_{\text{SG}}(S(G)) \) were obtained and exact values of \( \gamma^e_{\text{SG}}(S(G)) \) for some standard graphs were also obtained. Further its relationships with other different domination parameters were obtained. Also we relate split domination and end edge domination numbers in \( G \).

Keywords: Sub division graph; End edge dominating set; End edge domination numbers; Split domination number.

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1. Introduction

In this paper, we follow the notations of [1]. All the graphs considered here are simple, finite, non-trivial, undirected and connected. As usual \( p=|V| \) and \( q=|E| \) denote the number of vertices and edges of a graph \( G \), respectively.

In general, we use \( \langle X \rangle \) to denote the sub graph induced by the set of vertices \( X \) and \( N(v) \) and \( N[v] \) denote the open and closed neighborhoods of a vertex \( v \), respectively.

The notation \( \beta_0 G \ (\beta_1 G) \) is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of \( G \). Let \( \deg(v) \) is the degree of vertex \( v \) and as usual \( \delta G \ (\Delta G) \) is the minimum (maximum) degree. The degree of an edge \( e = uv \) of \( G \) is defined by \( \deg e = \deg u + \deg v - 2 \) and \( \delta' G \ (\Delta' G) \) is the minimum (maximum) degree among the edges of \( G \).

A vertex of degree one is called a pendent vertex and its neighbor is called a support vertex. A vertex \( v \) of \( V \) is called a cut vertex if removing it from \( G \) increases the number of components of \( G \).

The subdivision graph \( S(G) \) of a graph \( G \) is the graph obtained by inserting a vertex of degree two to every edge of \( G \).
A spider is a tree with the property that the removal of all end paths of length two of \( T \) results in an isolated vertex, called the head of a spider.

A dominating set \( D \subset V \) is said to be a split dominating set of \( G \), if the induced subgraph \( \langle V - D \rangle \) is disconnected. The minimum cardinality of vertices in such a set is called the split domination number of \( G \) and is denoted by \( \gamma_s G \). This concept was introduced by Kulli and Janakiram [3].

A \( 2 \)-packing in a graph \( G \) is a set of vertices of \( D \) that are pair wise at distance at least 3 apart i.e., \( D \) is \( 2 \)-packing of \( G \) if and only if \( d(u,v) \geq 3 \) for all distinct \( u,v \in D \).

A set \( S \subset E \) in a graph \( G \) is an edge dominating set if every edge in \( E - S \) is adjacent to at least one edge in \( S \). The minimum cardinality of edges in such a set is called the edge domination number of \( G \) and is denoted by \( \gamma' G \). Edge domination was introduced by S. Mitchell and S. T. Hedetniemi [4] and is now well studied in graph theory see [2].

An edge dominating set \( S \subset E \) is said to be an end edge dominating set of \( G \), if \( S \) contains all end edges of \( E(G) \). The minimum cardinality of edges in such a set is called the end edge domination number of \( G \) and is denoted by \( \gamma_{e} G \). This concept was introduced by Muddebihal and Sedamkar [5].

An edge dominating set \( D \) of a sub division graph \( S(G) \) is an end edge dominating set if \( D \) contains all end edges of \( S(G) \). The end edge domination number
\[ \gamma'_e S(G) \] of \( S(G) \) is the minimum cardinality of an end edge dominating set of \( S(G) \). In this paper, some bounds for \( \gamma'_e(S(G)) \) were obtained and exact values of \( \gamma'_e(S(G)) \) for some standard graphs were also obtained. Further its relationships with other different domination parameters were obtained. Also we relate split domination and end edge domination numbers in \( G \).

2. Results:

We need the following Theorems to prove our later results.

**Theorem A.4 [5]:** For any path \( P_p \) with \( p \geq 2 \) vertices,

\[ \gamma'_e P_p = p/3 + 1 , \text{ if } p \equiv 0 \mod 3 \]

\[ = \lfloor p/3 \rfloor , \text{ otherwise.} \]

**Corollary A [5]:** For any connected graph \( G \), let \( A = v_1, v_2, \ldots, v_m, m \geq 1 \), be the set of vertices of degree one. If \( A \not\subseteq V(G) \), then \( \gamma'_e G = \gamma'_e G \).

3. Main Results:

We list out end edge domination number for subdivision of some standard graphs.

**Theorem 3.1:**
1) \( \gamma'_e(S(C_p)) = \gamma'_e(C_{2p}) = \left\{ \begin{array}{ll} \frac{2p}{3}, & \text{if } p \equiv 0 \pmod{3} \\ \frac{2p}{3}, & \text{otherwise} \end{array} \right. \)

2) \( \gamma'_e(S(P_p)) = \gamma'_e(P_{2p-1}) = \left\{ \begin{array}{ll} \frac{2p}{3}, & \text{if } p \equiv 0 \pmod{3} \\ \frac{2p}{3}, & \text{otherwise} \end{array} \right. \)

3) \( \gamma'_e(S(K_p)) = p - 1 \)

4) \( \gamma'_e(S(K_{1,p})) = p - 1, \text{ for } p \geq 2 \)

**Remark 3.2:** Subdivision of star \( K_{1,p}, S(K_{1,p}), p \geq 3 \) is always a spider.

We give the following Lemma to prove our next result.

**Lemma 3.3:** For any tree \( T, \beta_1(S(T)) = q \).

**Proof:** To prove this result we use induction on \( q \). Let \( T = e, S(T) = 2e, \beta_1(S(T)) = 1 = q \).

Assume the result is true for any tree with \( q \) edges. Let \( T \) be a tree with \( q + 1 \) edges and \( e' \) be an end edge of \( T \). Then by induction hypothesis, \( \beta_1(S(T - \{e\})) = q - 1 \), further \( \beta_1(S(T)) = \beta_1(S(T - \{e\})) + 1 \) and hence \( \beta_1(S(T)) = q \).

In the following theorem, we obtain an upper bound for \( \gamma'_e(S(G)) \) in terms of the number of edges of \( G \).

**Theorem 3.4:** For any connected \((p,q)\)-graph \( G \) with \( p > 2 \), \( \gamma'_e(S(G)) \leq q \).
Proof: For $p = 2, \gamma'_e(S(G)) \leq q$. Let $T$ be a spanning tree of $G$. Then by Lemma 1, $\beta_1(S(T)) = q$ and any collection of $q$ - independent edges of $S(T)$ is an end edge dominating set of $S(G)$. Hence $\gamma'_e(S(G)) \leq q$.

Now we obtain one more upper bound for $\gamma'_e(S(T))$ in terms of number of vertices of $T$.

**Theorem 3.5:** For any tree $T$ with $p \geq 3$, $\gamma'_e(S(T)) \leq p - 1$. Equality holds if and only if $T$ is isomorphic to sub division of a spider or wounded spider or $P_4$ or $P_5$.

Proof: Let $F = \{e_1, e_2, ..., e_m\}$ be the set of all end edges in $S(T)$. Suppose $F' = \{e_1, e_2, ..., e_n\}$ denote the set of edges which are adjacent to the edges of $F$ and $E(S(T)) - F' = I$. Then $H \subseteq I$ is a minimal edge dominating set of $I$. Clearly, $F \cup H$ is an edge dominating set of $S(T)$ and $|F \cup H| \leq q$. Also by Theorem 2, $\gamma'_e(S(T)) \leq p - 1$.

Suppose $T$ is not a spider or wounded spider or $P_4$ or $P_5$. Since $F \cup H$ is a $\gamma'_e$ - set of $S(T)$, there exist at least one non end edge $e_x \in N(E - F \cup H)$ whose at most one end is adjacent to an edge of $F \cup H$. Clearly $|F \cup H| < q$, a contradiction.

Conversely, if $T$ is isomorphic to a spider or wounded spider or $P_4$ or $P_5$. Then by Lemma 1, $|F \cup H| = q$ and hence $\gamma'_e(S(T)) = p - 1$.

The following theorem relates $\gamma'_e(T)$ and $\gamma'_e(S(T))$ in terms of vertices of $T$. 
Theorem 3.6: For any tree $T$, $\gamma'_e(T) + \gamma'_e(S(T)) \geq p + 1$. Equality holds if $T$ is isomorphic to path $P_p$.

Proof: Let $S$ be the $\gamma'_e$ -set of $T$. After the sub division of $T$, let $S' = \{e_1, e_2, \ldots, e_i\}$ denote the end edge dominating set of $S(T)$. Since, there exists at least two end edges common to both $T$ and $S(T)$, also by the Lemma 1, $|S' \cup S' | \geq q + 2$. Hence $\gamma'_e(T) + \gamma'_e(S(T)) \geq p + 1$.

Suppose $T$ is isomorphic to path, then by Theorem [A.4], we have

$$\gamma'_e(P_p) = \frac{p}{3} + 1, \text{ if } p \equiv 0 \pmod{3}$$

$$= \left\lfloor \frac{p}{3} \right\rfloor, \text{ otherwise}$$

and by 2 of Theorem 1, we have

$$\gamma'_e\left(S\left(P_p\right)\right) = \frac{2p}{3}, \text{ if } p \equiv 0 \pmod{3}$$

$$= \left\lfloor \frac{2p}{3} \right\rfloor, \text{ otherwise.}$$

By adding these two, the equality holds.

In the following Theorem, we provide characterization of $\gamma'_e(S(G))$ for some standard graphs.

Theorem 3.7:
\[
1) \quad \gamma'_c\left(S\left(K_p\right)\right) = p - 1.
\]

\[
2) \quad \gamma'_c\left(S\left(W_p\right)\right) = p - 1.
\]

\[
3) \quad \gamma'_c\left(S\left(K_{m,n}\right)\right) = p - 1.
\]

**Proof:** In view of Theorem 2, it is enough to prove that \(\gamma'_c\left(S\left(G\right)\right) \geq p - 1\), where \(G\) is either \(K_p, W_p\) or \(K_{m,n}\) with \(m+n = p\).

**Case 1:** Suppose \(G\) is isomorphic to \(K_p\). Let \(V_1 = V\left(K_p\right)\) after the subdivision, let \(V_2 = V\left(S\left(K_p\right)\right) - V\left(K_p\right)\). Further, let \(S\) be any independent set of \(p - 2\) edges of \(S\left(K_p\right)\) and \(S'\) be the set of vertices of \(S\left(K_p\right)\) which are incident to the edges of \(S\).

Clearly, \(|S'| = 2(p - 2), |S' \cap V_1| = p - 2\) and \(|S' \cap V_2| = p - 2\). Hence there exist exactly two vertices \(u, v\) in \(V_1 - S'\). Now the edges \(uv, wv\), where \(w \in S\left(K_p\right)\) that sub divides the edge \(uv\) are not dominated by any edge of \(S\). Hence \(\gamma'\left(S\left(K_p\right)\right) \geq p - 1\). Since by Corollary [A], \(\gamma'_c = \gamma'\), it follows that \(\gamma'_c\left(S\left(K_p\right)\right) \geq p - 1\).

**Case 2:** Suppose \(G\) is isomorphic to \(W_p\). Let \(V_1 = V\left(W_p\right)\) and \(v_k\) be the centre of \(W_p\).

After the sub division of \(G\), let \(V_2 = V\left(S\left(W_p\right)\right) - V\left(W_p\right)\). Further, let \(S\) be any independent set of \(p - 2\) edges of \(S\left(W_p\right)\) and \(S'\) be the set of vertices of \(S\left(W_p\right)\) which are incident to the edges of \(S\).
Clearly, $|S'|=2(p-2), |S' \cap V_1|= p-2$ and $|S' \cap V_2|= p-2$. Hence there exists exactly two vertices $u, v$ in $V_1-S'$. If $uv$ is an edge in $W_p$, then the edges $uw$ and $wv$ where $w$ is the vertex of $S(W_p)$ that subdivides the edge $uv$ are not dominated by $S$. Suppose $uv$ is not an edge in $W_p$. Let $w_1, w_2$ be the vertices of $S(W_p)$ which sub divide the edge $v_iu, v_iv$ respectively. Since $S$ is independent, at least one of the edges $v_iw_1, v_iw_2$ does not belong to $S$. Suppose $v_iw_1 \not\in S$, so $w_iu$ is not dominated by $S$. Thus $\gamma'_e(S(W_p)) \leq p-1$.

**Case 3:** Suppose $G$ is isomorphic to $K_{m,n}$ with $m+n=p$. The proof of this case is similar to that of Case 2.

The following Theorem relates end edge domination and split domination in $G$.

**Theorem 3.8:** For any end edge dominating set $S$ of $G$, if there exists at least one end edge $e \in S$. Then $G$ has a split dominating set.

**Proof:** Let $e=uv \in S$ be an end edge in $G$. Suppose $v$ is an end vertex of $e$ in $G$. Then there exist a cut vertex $u \in N(v)$ in $G$. Let $D$ be a dominating set of $G$. Further, if $u \in D$, then $D$ is a split dominating set of $G$. Suppose $u$ is an end vertex, then $v \in D$ is a cut vertex. Hence $D^{-1} = (D-\{v\}) \cup \{u\}$ is a split dominating set of $G$. 
**Theorem 3.9:** If $G$ is not a tree and $S$ is a $\gamma'_e$-set of $G$. Then for some $e_i \in S$ which are non-end edges, dominates the edges of $E(G) - S$ are also dominated by some $S - e_i$ edges.

**Proof:** Let $S$ be the $\gamma'_e$-set of $G$. If possible, assume that there exists at least one non-end edge $e \in S$ such that $e$ does not satisfy the given condition. Then $S' = S - \{e\}$ is an end edge dominating set of $G$, a contradiction.

Hence there exist at least one non-end edge $e \in S$, which dominates at least one edge of $E(G) - S$ which is also dominated by some $S - \{e_i\}$ edges.

The following Theorem relates $\gamma'_e(S(T))$ and $\gamma'_e(T)$.

**Theorem 3.10:** For any tree $T$, $\gamma'_e(S(T)) \leq 2 \cdot \gamma'_e(T)$. Equity holds if $T$ is isomorphic to a spider.

**Proof:** Let $S$ be the $\gamma'_e$-set of $T$. Insert a vertex of degree two to each edge of $T$ to obtain $S(T)$. Let $F = \{e_1, e_2, \ldots, e_m\}$ be the set of edges whose edge degree is one, which are incident to the support vertices and $F' \in N(F)$ in $S(T)$. Suppose $H$ is a $\gamma'$-set of $S(T) - \{F \cup F'\}$, then $F \cup H$ is an end edge dominating set of $S(T)$. Since, each edge is subdivide, $q(S(T)) = 2 \cdot q(T)$ and number of end edges in both $T$ and $S(T)$ are same, it follows that, $|F \cup H| \leq 2 |S|$. Hence, $\gamma'_e(S(T)) \leq 2 \cdot \gamma'_e(T)$.

**Corollary 3.11:** For any tree $T$, $\gamma'_e(T) \leq \gamma'_e(S(T)) \leq 2 \cdot \gamma'_e(T)$.
The following Theorem relates $\gamma'_e(S(G))$ and independence number of $G$.

**Theorem 3.12:** For any connected $(p,q)$-graph $G$, $\gamma'_e(S(G)) \leq 2(p - \beta_i)$. Equity holds if $G$ is isomorphic to $K_2$.

**Proof:** Suppose $B = \{u_i, v_i / 1 \leq i \leq \beta_j\}$ be a maximum independent set of edges of $G$. Then $B$ is an edge dominating set of $G$. Let $w_i$ be the vertex of $S(G)$ which is adjacent to both $u_i$ and $v_i$. Let $M$ be the set of vertices of $G$ which are not incident with any edge of $B$.

If $M = \phi$, then $S \subseteq E(S(G))$ is an end edge dominating set of $S(G)$ such that $|S| \leq 2 \cdot \beta_i = 2(p - \beta_i)$. Hence $\gamma'_e(S(G)) \leq 2(p - \beta_i)$. Suppose $M \neq \phi$, let $M = \{x_1, x_2, \ldots, x_n\}$. Since $B$ is an edge dominating set of $G$, $\langle M \rangle$ is independent.

Furthermore, since $G$ is connected and $\langle M \rangle$ is independent, each vertex $x_i$ in $M$ is adjacent to some $z_j \left( z_j = u_j \text{ or } v_k \right)$ in $G$. Let $y_i$ be the vertex of $S(G)$ which is adjacent to both $x_i$ and $z_i$ in $S(G)$. Then another set $S_1 \subseteq E(S(G))$ forms an end edge dominating set of $S(G)$ such that, $|S_1| \leq 2\beta_i + 2(p - 2\beta_i)$. Hence $\gamma'_e(S(G)) \leq 2(p - \beta_i)$.

Suppose $G$ is isomorphic to $K_2$. In this case $|S| = p = 2$ and $|B| = 1$. Clearly $|S| = 2(p - \beta_i)$ and hence $\gamma'_e(S(G)) = 2(p - \beta_i)$. 


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