A NOTE ON OSCILLATION CRITERIA FOR SOME PERTURBED HALF-LINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, we show if the half-linear part of an equation is oscillatory, so would be some of its related perturbed equations. For one-dimentional cases, it can be resumed as the following: if the half-linear equation
\[ \phi'(t)y' + c(t)y + g(t) = 0 \]
is oscillatory then any of its perturbed equations
\[ \phi'(t)y' + Q(t)h(y, y') = 0 \]
will also be oscillatory whenever \( Q \in C^1(\mathbb{R}) \) and \( h \in C(\mathbb{R}^2, \mathbb{R}) \).

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1. Introduction

This work is somehow an addendum to our earlier result in [4]. For a \( T > 0 \), define accordingly
\[ \Omega_T := \{ x \in \mathbb{R}^n \mid ||x|| > T, \quad 1 < n \in \mathbb{N} \} \quad \text{or} \quad \Omega_T := (T, \infty) \subset \mathbb{R}. \]
We investigate some oscillation criteria for equations of the type
\[
\begin{cases}
(i) & \left\{ a(t)y' \right\}' + c(t)y + g(t)f(y, y') = 0, \quad t \in \Omega_T \quad \text{or} \\
(ii) & \nabla \cdot \left\{ A(x)\Phi(\nabla v) \right\} + C(x)v + H(x) \cdot F(v, \nabla v) = 0, \quad x \in \Omega_T,
\end{cases}
\]

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where \( a, c, g \in C(\Omega_T, \mathbb{R}); f \in C(\mathbb{R}^2, \mathbb{R}); H \in C(\Omega_T, \mathbb{R}^n), F \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \); the dot denotes the scalar product in \( \mathbb{R}^n \). For some \( \alpha > 0, \quad \phi(S) := |S|^{\alpha-1}S \) for \( S \in \mathbb{R} \) and \( \Phi(\zeta) := |\zeta|^{\alpha-1}\zeta, \quad \zeta \in \mathbb{R}^n \). They have the following properties: \( \forall t, s \in \mathbb{R} \) and \( \zeta \in \mathbb{R}^n \)

\[
\phi(t)\phi(s) = \phi(ts); \quad t\phi'(t) = \alpha\phi(t); \quad t\phi(t) = |t|^{\alpha+1};
\]

\[
\phi(s)\Phi(\zeta) = \Phi(s\zeta); \quad \zeta\Phi(\zeta) = |\zeta|^{\alpha+1}.
\]

**Definition 1.1.** Let \( u \in C(\mathbb{R}, \mathbb{R}) \) (or \( C(\mathbb{R}^n, \mathbb{R}) \)).

1. A nodal set of \( u \) is any open and connected \( D(u) \neq \emptyset \) such that \( u \neq 0 \) in \( D(u) \) and \( u|_{\partial D(u)} = 0 \).
2. \( u \) is said to be oscillatory (strongly oscillatory) if it has a zero in any \( \Omega_R, \quad R > 0 \) (in any nodal set \( D(u) \subset \Omega_R \)).
3. An equation will be said to be oscillatory if any of its non-trivial and bounded solutions is oscillatory.

In the sequel the general hypotheses are: for some \( T, m > 0, \)

\((H1): \quad a \in C^1(\Omega_T, (m, \infty)) \) is non-decreasing; \( A \in C^1(\Omega_T, (m, \infty)) \);

\[
g \in C(\Omega_T, \mathbb{R});
\]

\((H2): \quad c, C \in C(\Omega_T, (m, \infty)) \) eventually; \( H, f, F \) are as stated above;

\((H3): \quad \text{On any compact } E \subset \Omega_T, \quad \exists k > 0 \text{ such that}
\]

(i) \( |g(t)f(S,w)| \leq k|w|^\alpha + \phi(S) \) if \( |w| < 1 \) and \( S > 0 \) for the \((1.1)(i)\) case;

(ii) \( |H(x) \cdot F(S,\zeta)| \leq k|\zeta|^\alpha + \phi(S) \) if \( |\zeta| < 1 \) and \( S > 0 \) for the \((1.1)(ii)\) case.

( The condition \((H3)\) is to ensure that non-trivial solutions are not compact-supported (see [2]).

Oscillation criteria for the equations \((1.1)(i)\) will be obtained through some comparison methods, using some Picone-type identity. Some oscillation criteria for the half-linear equations

\[
(i) \quad \left\{ a(t)\phi(y') \right\}' + c(t)\phi(y) = 0, \quad t \in \Omega_T \quad \text{and}
\]

\[
(ii) \quad \nabla \cdot \left\{ A(x)\Phi(\nabla v) \right\} + C(x)\phi(v) = 0, \quad x \in \Omega_T
\]

are well known; see e.g. [1,3,4] and references therein. For any \( w \in C(\mathbb{R}, \mathbb{R}^+) \) define \( W^+(r) := r^{n-1}\sup_{|x|=r} w(x) \) and \( W^-(r) := r^{n-1}\inf_{|x|=r} w(x) \). The equations in \((1.2)\) are oscillatory if
(i) \( a \) satisfies (H1) and \( c \) satisfies (H2) or \( t \mapsto \int_t^T c(s) \, ds \) diverges to infinity for (1.2) (i);
(Theorem 1.5 of [3])

(ii) \( a := A^- \) and \( c := C^+ \) satisfy the conditions displayed in (i) above for (1.2) (ii).
(Theorem 5.1 of [4])

The criteria for (1.2) (ii) are obtained from those of (1.2) (i) using some rightaway transformations and some Picone-type identities; see [1] [3] and the references therein.

2. Picone-type formulae for the equations in (1.1)

If \( y \) is a non-trivial \( C^2 \)-solution, non zero in some \( D \subset \Omega_T \) of (1.1) (i) and \( z \) such a solution for (1.2) (i) then

\[
\begin{align*}
(a) & \quad \text{if } \exists G \in C^1(\Omega_T, \mathbb{R}) \text{ such that } G'(t) = g(t) \text{ in } \Omega_T, \\
(b) & \quad \left\{ a(t)z\phi(z') - a(t)z\phi\left(\frac{z}{y}\right) - z\phi\left(\frac{z}{y}\right)G(t)f(y, y') \right\}' \\
& \quad = a(t)\zeta_\gamma(z, y) - G(t)\left\{ z\phi\left(\frac{z}{y}\right)f(y, y') \right\}',
\end{align*}
\]

(2.1)

where, \( \forall \gamma > 0 \), the two-form function \( \zeta_\gamma \) is defined \( \forall u, v \in C^1(\mathbb{R}, \mathbb{R}) \) by

\[
(Z1) : \quad \zeta_\gamma(u, v) = |u'|^{\gamma+1} - (\gamma + 1)u'\phi_{\gamma}(\frac{u}{v}') + \gamma v'\frac{u}{v}\phi_{\gamma}(\frac{u}{v}')
\]

is strictly positive for non null \( u \neq v \) and is null only if \( u = \lambda v \) for some \( \lambda \in \mathbb{R} \). Similarly, if \( v \in C^2(\Omega_T, \mathbb{R}) \) is a non-trivial solution for (1.1) (ii) and \( u \) such a solution of (1.2) (ii) then
wherever \( v \neq 0 \)

\[
\begin{align*}
(a) & \quad \text{if } \exists h \in C^1(\Omega_T, \mathbb{R}) \text{ such that } \nabla h = H(x) \text{ in } \Omega_T, \\
(b) & \quad \nabla \cdot \left\{ A(x)\phi(\nabla u) - A(x)\phi\left(\frac{u}{v}\right) - u\phi\left(\frac{u}{v}\right)h(t)F(v, \nabla v) \right\} \\
& \quad = A(x)Z_\alpha(u, v) - h(t)\nabla \cdot \left\{ \frac{u}{v}F(v, \nabla v) \right\},
\end{align*}
\]

(2.2)

where \( \forall \gamma > 0, \forall u, v \in C^1(\mathbb{R}^n) \).

\( (Z2): \ Z_\gamma(u, v) := |\nabla u|^{\gamma+1} - (\gamma + 1)\Phi\left(\frac{\nabla u}{\nabla v}\right) \cdot \nabla v + \gamma \frac{\nabla u}{\nabla v}|^{\gamma+1} \\
= |\nabla u|^{\gamma+1} - (\gamma + 1)\left|\frac{\nabla u}{\nabla v}\right|^{\gamma+1} \frac{\nabla u}{\nabla v} \cdot \nabla v + \gamma \frac{\nabla u}{\nabla v}|^{\gamma+1}.
\)

We recall that \( \forall \gamma > 0 \) the two-form \( Z_\gamma(u, v) > 0 \) for distinct non null \( u, v \) and is null only if \( \exists k \in \mathbb{R}; u = kv; \) see [1].

3. Main results

**Theorem 3.1.** Assume that \( a, c, g \) and \( f \) satisfy (H1) to (H3). Then

\[
\begin{align*}
(i) & \quad \left\{ a(t)\phi(y') \right\}' + c(t)\phi(y) = 0, \ t > T \quad \text{is oscillatory}, \\
(ii) & \quad \text{so is } \left\{ a(t)\phi(y') \right\}' + c(t)\phi(y) + g(t)f(y, y') = 0, \ t \in \Omega_T
\end{align*}
\]

provided that \( \exists G \in C^1(\Omega_T); \ G'(t) = g(t). \)

**Theorem 3.2.** Assume that \( A, C, F, \) with \( a := A^-, \ c := C^+ \) and \( H \) satisfy (H1) to (H3). Then

\[
\nabla \cdot \left\{ A(x)\Phi(\nabla v) \right\} + C(x)\phi(v) + H(x) \cdot F(v, \nabla v) = 0, \ x \in \Omega_T
\]

is oscillatory provided that \( \exists h \in C^1(\Omega_T, \mathbb{R}); \ \nabla h(x) = H(x). \)

Since the proofs of the two theorems are similar, we prove only the first one.

**Proof of Theorem 3.1.** In equation (3.1) (ii) \( g(t) \) can be replaced by \( G'_\mu(t) := [G(t) + \mu]' \), \( \forall \mu \in \mathbb{R} \). With that replacement, if \( y \) is a non-trivial solution of (3.1)(ii) with \( y > 0 \) in
an \( \Omega_R \), then for any oscillatory solution \( z \) of (3.1) (i), for any nodal set \( D(z) \subset \Omega_R \)

\[
0 = \int_{D(z)} \left[ a(t) \zeta(z, y) \right] dt \\
- \int_{D(z)} \left( G(t) + \mu \right) \left\{ z \phi \left( \frac{z}{y} \right) f(y, y') \right\}' dt \quad \forall \mu \in \mathbb{R}.
\]

For \( \mu = 0 \) we get

\[
0 = \int_{D(z)} \left[ a(t) \zeta(z, y) \right] dt - \int_{D(z)} G(t) \left\{ z \phi \left( \frac{z}{y} \right) f(y, y') \right\}' dt
\]

whence \( \mu \left[ z \phi \left( \frac{z}{y} \right) f(y, y') \right] \equiv 0 \) and so is \( \zeta(z, y) \) in any such a \( D(z) \). Therefore no such a solution \( y \) can be non-zero in any \( \Omega_R \); it has to have a zero in any \( D(z) \subset \Omega_R \).

**Remark 3.3.** Following the processes similar to those in the proofs of Theorem 3.4 and Theorem 5.1 of [4], the hypotheses on \( G \) and \( H \) can be weakened to

\[
\exists k \in C(\Omega_T, \mathbb{R}) \quad \text{and} \quad K \in C(\Omega_T, \mathbb{R}^n)
\]

bounded in \( \Omega_T \) such that the functions \( G \) and \( h \) above satisfy

\[
G'(t) = g(t) + k(t) \quad \text{and} \quad \nabla h(x) = H(x) + K(x).
\]

But the penalty to pay is that the corresponding solutions \( y \) will be oscillatory unless \( \liminf_{t \to \infty} |y(t)| = 0 \) \( (\liminf_{|x| \to \infty} |y(x)| = 0) \).

**Conflict of Interests**

The author declares that there is no conflict of interests.

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**References**

