COMMON FIXED POINT THEOREM IN FUZZY SYMMETRIC SPACES

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Abstract. The main aim of this paper is to prove a unique common fixed point theorem for six self mappings in fuzzy symmetric spaces for occasionally weakly compatible mappings.

Keywords: Fuzzy symmetric space; Occasionally weakly compatible mappings; Weakly compatible mappings; Coincidence point; Fixed point.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

Wilson introduced the concept of semi-metric spaces in 1931, many fixed point theorems have been proved in this space. Jungck and Rhoads [1] initiated the the concept of weakly compatible mappings which are weaker than compatible mappings. Recently Jungck and Rhoads [9] introduced the concept of occasionally weakly compatible mappings which are more general among compatible mappings. The purpose of this paper is to obtain a common fixed point theorem for six self maps in fuzzy symmetric space.

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Received November 11, 2014
2. Preliminaries

**Definition 2.1.** A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous $t$-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative

(ii) $*$ is continuous

(iii) $a * 1 = a$

(iv) $a * b \leq c * d$, where $a \leq c, b \leq d$ and $a, b, c, d \in [0, 1]$.

**Definition 2.2.** The 3-tupple $(X, M, *)$ is called a fuzzy metric space if $X$ is an arbitrary non empty set, $*$ is continuous $t$-norm and $M$ is a fuzzy set in $X^2 \times (0, \infty)$ which satisfies the following conditions:

(i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$.

**Definition 2.3.** The pair $(X, M)$ is called fuzzy symmetric space if $X$ is an arbitrary non empty set and $M$ is fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$.

If $(X, M)$ is a symmetric space, then $M$ is called fuzzy symmetric space for $X$.

Every fuzzy metric space is a fuzzy symmetric space but not conversely.

**Example 2.4.** Consider $X = [0, 2)$ and $M(x, y, t) = \frac{t}{x+y+1}$. Let $x = 1, y = 1/2, z = 0, t = 1, s = 0$ then (iv) of definition is not satisfied and hence $(X, M)$ is fuzzy semi-metric space but a not fuzzy metric space.
Definition 2.5. Let $A$ and $B$ be two self mappings of a fuzzy symmetric space $(X, M)$. Then $A$ and $B$ are said to be compatible if $\lim_{n \to \infty} (ABx_n, BAx_n, t) = 1$ whenever a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} (Ax_n, x, t) = \lim_{n \to \infty} (Bx_n, x, t) = 1$ for some $t \in X$.

Definition 2.6. Let $X$ be a set $A$ and $B$ be self mappings of $X$. A point $x$ in $X$ is called a coincidence point of $A$ and $B$ if and only if $Ax = Bx$. We denote $w = Ax = Bx$ a point of coincidence of $A$ and $B$.

Definition 2.7. Let $A$ and $B$ be two self mappings of a fuzzy symmetric space $(X, M)$ then, $A$ and $B$ are said to be weakly compatible if they commute at their coincidence point.

Definition 2.8. Let $A$ and $B$ be two self maps of a fuzzy symmetric space $(X, M)$ then $A$ and $B$ are to be occasionally weekly compatible if there is a point $x \in X$ which is coincidence point of $A$ and $B$ at which $A$ and $B$ commute.

Lemma 2.9. Let $X$ be a set. Let $A$ and $B$ be occasionally weakly compatible self maps of $X$. If $A$ and $B$ have a unique point of coincidence $w = Ax = Bx$ then $w$ is the unique common fixed point of $A$ and $B$.

Lemma 2.10. If for all $x, y \in X, t > 0$ and for a number $k \in (0, 1)$, then $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Proof. Suppose that there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$, for all $x, y$ in $X$ and $t > 0$. Then $M(x, y, t) \geq M(x, y, t/k)$ and after $n$-th iteration $M(x, y, t) \geq M(x, y, t/k_n)$ for some positive integer taking limit as $n \to \infty$ we have $M(x, y, t) \geq 1$. Hence $x = y$.

3. Main results

Theorem 3.1. Let $(X, M)$ be a fuzzy symmetric space. $A, B, S, T, P$ and $Q$ be self maps of $X$ such that

I: $(AP, S), (BQ, T)$ are occasionally weakly compatible

II: $M(APx, BQy, qt) \geq \min \{M(Sx, Ty, t), M(APx, Sx, t), M(BQy, Ty, t), M(APx, Ty, t)\}$ for all $x, y \in X, q \in (0, 1)$. 
Then $AP, BQ, S,$ and $T$ have unique common fixed point. Further if $(A, P)$ and $(B, Q)$ are commuting pair of mapping $s$ then $A, B, S, T, P$ and $Q$ have a unique common fixed point.

**Proof.** Since $(AP, S)$ and $(BQ, T)$ are occasionally weekly compatible then there exits $x,y \in X$ Such that $APx = x$ and $BQy = Ty$. We claim $APx = BQy$.

$$M(APx, BQy, qt) \geq \min\{M(Sx, Ty, t), M(APx, Sx, t), M(BQy, Ty, t), M(APx, Ty, t)\}.$$ \[ (3.1.1) \]

Also from (3.1.1) and (3.1.2) $APx = APz \Rightarrow t = x$. Hence $w = APx = Sx$ for $w \in X$ is the unique point of coincidence of $AP$ and $S$. By the lemma, $w$ is a unique common fixed point of $AP$ and $S$ Hence $APw = Sw = w$. Similarly there is a unique common fixed point of $BQ$ and $T$. Hence $BQw = Tw = u$ Suppose $u \neq w$ which contradicts the inequality (3.I-I). Hence $w$ is unique common fixed point of $AP, BQ, S$ and $T$. We show that $w$ is only common fixed point of $A, B, S, T, P$ and $Q$. If the pairs $(A, Q)$ $(B, Q)$ are commuting pairs then for this we have $$A(APw) = A(PAw) = AP(Aw) = Aw.$$

if $x = Aw, y = w$ in (3.II). \[ M(AP(Aw)BQw, qt) \geq \min\{M(S(Aw), Tw, t), M(AP(Aw), S(Aw), t), M(BQw, Tw, t), M(AP(Aw), Tw, t)\}. \]

Then $AP, BQ, S, T, P, Q$ have a unique common fixed point.

**Example 3.2.** consider $X = [0,2]$ with the fuzzy semi metric space $(X, M)$ defined by $M(x, y, t) = \frac{t}{1 + e^{-k|x-y|}}$ for $x, y$ all in $X$. Define self mappings $A, B, S, T, P$ and $Q$ as

$$A(x) = Q(x) = \begin{cases} x & \text{if } x \in (0,2); \\ \end{cases}$$
\[ B(x) = BQ(x) = \begin{cases} 
 3/4 & \text{if } x \in [0,1); \\
 1 & \text{if } x \in [1,2).
\end{cases} \]

\[ T(x) = \begin{cases} 
 3x/2 & \text{if } x \in [0,1); \\
 1 & \text{if } x \in [1,2).
\end{cases} \]

\[ P(x) = AP(x) = \begin{cases} 
 x/2 & \text{if } x \in [0,1); \\
 1 & \text{if } x = 1; \\
 0.95 & \text{if } x \in (1,2). 
\end{cases} \]

\[ S(x) = \begin{cases} 
 1/4 & \text{if } x \in [0,1); \\
 1/x & \text{if } x = 1; \\
 1/x^2 & \text{if } x \in (1,2). 
\end{cases} \]

It is easy to verify that the pairs \((AP, S)\) and \((BQ, T)\) are occasionally weakly compatible mappings and 1 is common fixed point.

The above example reveals that occasionally weakly compatible mappings are not weakly compatible. Since it has two coincidence points 1/2 and 1 \((AP, S)\) and \((BQ, T)\) are not commuting at \(x=1/2\). We observed that the self mappings \((A, P)\) and \((B, Q)\) are commuting and the mappings \(A, B, S, T, P\) and \(Q\) have unique common fixed point.

**Corollary 3.3.** Let \((X, M)\) be a fuzzy symmetric space. \(A, B, S, T, P\) and \(D\) be self maps of \(X\) such that

**I:** \((AP, S)\) and \((BQ, T)\) are occasionally weakly compatible

**II:** \[ [M(APx, BQy, kt)]^2 \cdot [M(APx, BQy, kt) \cdot M(Sx, Ty, t)] \geq k_1 [M(BQy, Sx, 1.25kt) \cdot M(APx, Ty, 1.25kt)] + k_2 [M(APx, Sx, 2.5kt) \cdot M(BQy, Ty, 2.5kt)] M(Sx, Ty, t). \]

for all \(x, y\) in \(X\) and \(k_1, k_2 \geq 0, k_1 + k_2 \geq 1\). Then \(AP, BQ, S,\) and \(T\) have unique common fixed point. If \((A, P)\) and \((B, Q)\) are commuting pair of mappings then \(A, B, S, T, P\) and \(Q\) have a unique common fixed point theorem.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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