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A NEW NON-STANDARD FINITE DIFFERENCE METHOD FOR AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. This paper presents a new approach to constructing Non-Standard Finite Difference Method (NSFDM) for the solution of autonomous Ordinary Differential Equations (ODEs). The need for this method came up due to the shortcomings of the standard methods; in which the qualitative properties of the exact solutions are not usually transferred to the numerical solutions. These shortcomings affect the stability of the standard approach. The new nonstandard finite difference method have the property that its solution do not exhibit numerical instabilities.

Keywords: Autonomous function; Denominator function; Numerator function; NSFDM.

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1. Introduction

Many real world problems are modelled by differential equations, for which analytical solutions are not always easy to find. One of the most difficult problems is how to solve these differential equations efficiently. Several researchers have tried to do this in various different ways using finite element methods, standard finite difference methods, spline approximation

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methods, block methods, among others. In recent years, to get reliable results with less effort, researchers have applied the NSFDMs and obtained competitive results, among these authors are Anguelov and Lubuma [1], Anguelov and Lubuma [2], Anguelov *et al.* [3], Mickens[4], Mickens[5], Mickens[6], Mickens[7], Mickens[8], Mickens and Smith [9], Ibijola *et al.* [10], Sunday [11], Sunday [12] and Sunday *et al.* [13].

In this paper, a new NSFDM is constructed for the solution of autonomous first-order differential equations of the form,

$$\frac{dy}{dt} = f(y). \tag{1}$$

We shall assume that

$$f(y) = 0 \tag{2}$$

has only simple zeros.

The construction of the NSFDM is not always straight forward and there are no general criteria for doing so, Patidar [14]. However, several important rules have been discovered by Mickens [6], which we state shortly. The design of the NSFDMs starts mostly with the concept of exact schemes, see Sunday [12]. A major advantage of having an exact difference scheme for a differential equation is that questions related to the usual considerations of consistency, stability and convergence need not to arise, Hilderbrand [15]. For equation (1), numerical instabilities will occur whenever the linear stability properties of any of the fixed-points for the difference scheme differs from that of the differential equation, Mickens[4].

Definition 1.1 (Borowski and Borwein [16]) A differential equation is autonomous if it does not depend on the variable of differentiation (often time), that is, such that there is no explicit occurence of the independent variable in the equation.

Definition 1.2 (Anguelov and Lubuma [2]) The fixed point of the differential equation (1) is any zero y of the function $f : f(y) = 0$.

2. Theory of the NSFDM

The general form of NSFDM can be written as

$$y_{n+1} = F(h, y_n). \tag{3}$$

Definition 2.1 (Anguelov and Lubuma [1]) A finite difference scheme is called NSFDM, if at least one of the following conditions is met

i) In the discrete derivative, the traditional denominator is replaced by a non-negative function ϕ such that,

$$\phi(h) = h + o(h^2), \quad h \to 0. \tag{4}$$

ii) Non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable functions of several points of the mesh. For example,

$$y^2 \approx y_n y_{n+1},\ y_{n-1} y_n, \quad y^3 \approx y_{n-1} y_n y_{n+1},\ y_n^2 y_{n+1}$$

The NSFDMs were developed using the collection of rules set by Mickens [6] as follows.

Rule 1: The orders of the discrete derivatives must be exactly equal to the order of the corresponding derivatives of the differential equation. If this rule is violated, this can lead to numerical instability in the form of oscillations which may be bounded or unbounded. The mathematical reason for the above occurrence is that discrete equations have large class of solutions than differential equations.

Rule 2: Denominator function for the discrete derivatives must, in general, be expressed in terms of more complicated function of the step-sizes than those conventionally used. These denominator functions, generally, are functions, that are related to particular solutions or properties of the general solution to the differential equation.This rule allows the introduction of complex analytic function of h in the denominator.

Rule 3: The non-linear terms must, in general, be modeled (approximated) non-locally on the computational grid or lattice in many different ways.

Rule 4: Special solutions of the differential equations should also be accompanied by special discrete solutions of the finite-difference models.

Rule 5: The finite-difference equation should not have solutions that do not correspond exactly to the solution of the differential equations.

Rule 6: For differential equations having $N(\geq 3)$ terms, it is generally useful to construct finite difference schemes for various sub-equations composed of M terms, where $M < N$, and then combine all the schemes together in an overall consistent finite difference model.

It is to be noted that any method, which is not standard can be considered as nonstandard. However, when we talk about the NSFDMs, we mean by those which consider at least one of the nonstandard modelling rules proposed by Mickens as above.

3. Methodology

Our objective is to construct a new NSFDM for equation (1) that do not exhibit elementary numerical instabilities (i.e. solutions to the finite difference equation that do not correspond to any of the solutions to the differential equation). Another objective is to prove that, for equation (1), it is possible to construct NSFDM that has the correct linear stability properties for all finite step-sizes, Mickens and Smith[9].

Derivation technique of the new NSFDM

We shall assume that

$$\frac{dy}{dt} \approx \frac{y_{n+1} - \psi(h) y_n}{\phi(h)}, \tag{5}$$

where ψ and ϕ functions of the step-size, $h = \Delta t$, and are known as the numerator and the denominator functions respectively; $t_n = t_0 + hn$ and $y(t) = y_n$. The ψ and ϕ have the following properties

$$\left\{ \begin{array}{c} \psi = 1 + o(h) \\ \phi = h + o(h^2). \end{array} \right\} \tag{6}$$

Recall that the conventional discrete representation for the first derivative takes $\psi = 1$ and $\phi = h$ (Hilderbrand [15]), for the Standard Finite Difference Methods (SFDMs) and that is why this method can be regarded as an extension of the classical finite difference meethod.

This implies that this method approaches the SFDM as $h \to 0$. For systems of coupled first-order ODEs, there exists a systematic method for constructing the denominator function. Also, unless the "system" has dissipation, the numerator function is usually equal to one, Mickens [8].

Let the fixed-points of equation (1) be given by

$$\{\bar{y}_i;\ i = 1,2,3,...,I\},$$

where I may be bounded. The fixed-points are the real n solutions to the equation,

$$f(\bar{y}) = 0. \tag{7}$$

Let P_i be defined by

$$P_i = \frac{df}{dy}_{y=\bar{y}_i} \tag{8}$$

and P^* as

$$P^* \equiv Max\ \{|P_i|;\ i = 1,2,3,...,I\}. \tag{9}$$

The new NSFDM for equation of the form (1) is given by

$$\frac{y_{n+1} - \psi y_n}{\phi} = f(y_n), \tag{10}$$

where ϕ the denominator function is defined by the new function

$$\phi = \frac{\phi(h,P^*)}{P^*}. \tag{11}$$

This form replaces the simple "h" function found in the SFDM. That is,

$$\frac{dy}{dt} \to \frac{y_{n+1} - y_n}{h}. \tag{12}$$

Note that $\phi(h,P^*)$ in equation (11) has the properties

$$\phi(h,P^*) = h + o(P^*h^2),\ 0 < \phi < \frac{1}{P^*}. \tag{13}$$

If we consider a dynamical system where the independent variable t is time, it follows that P_i have units of inverse time and a set of time scales can be defined by means of the relations

$$T_i = \frac{1}{P_i},\ \ i = 1,2,3,...,n;\ T^* = \frac{1}{P^*}. \tag{14}$$

Thus, T^* corresponds to the smallest time scale and this shows that the denominator function is in the range $0 < \phi(h,T^*) < T^*$ .Therefore, from equations (10) and (11), we have our new scheme as

$$\frac{y_{n+1} - \psi y_n}{\left[\frac{\phi(h,P^*)}{P^*}\right]} = f(y_n) \tag{15}$$

That is,

$$y_{n+1} = \psi y_n + \frac{\phi(h,P^*)}{P^*} f(y_n). \tag{16}$$

Equation (16) is the new NSFDM capable of solving equations of the form (1).

4. Analysis of basic properties of the NSFDM

We shall consider basic properties that are relevant to NSFDM. Let us assume that the function $F(h,y)$ in equation (3) has a continuous derivatives with respect to both variables for $h > 0, y \in \mathbb{R}$ and satisfies,

$$F(0,y) = y \text{ and } \frac{\partial F(0,y)}{\partial h} = f(y) \tag{17}$$

Another assumption made is that the difference scheme (3) is consistent with the differential equation (1). We note that consistency implies that (17) is satisfied when y is the solution of (1).

4.1. Monotone dependence on initial value

Theorem 4.1.1. (Anguelov and Lubuma [2]) *A set $G(\Omega)$ which is a real valued function defined on a subset $[t_0,\infty)$ is said to monotonically depend on the initial value at t_0 if for any two functions $y,z \in \Omega$, we have*

$$y(t_0) \leq z(t_0) \Rightarrow y(t) \leq z(t), \;\; t \in \Omega. \tag{18}$$

It is necessary to state at this stage that since equation (1) is assumed to satisfy Lipchitz condition (i.e. the theorem that guarantees the existence and uniqueness of solutions of differential equation), the set of solutions for (1) is monotonically dependent on the initial value at t_0.

Theorem 4.1.2. (Anguelov and Lubuma [2]) *The NSFDM (3) is stable with respect to monotone dependence on initial value if*

$$\frac{\partial F(h, y_n)}{\partial y} \geq 0, \quad y \in \mathbb{R}, \; h > 0. \tag{19}$$

It is important to note that our new NSFDM (16) satisfies Theorems 4.1.1 and 4.1.2. Let us now prove that the new NSFDM (16) satisfies Theorem 4.1.2.

Proof. Let the numerator function ψ and the denominator function $\frac{\phi(h,P^*)}{P^*}$ in equation (16) be defined by

$$\left\{ \begin{array}{c} \psi = 1, \\ \frac{\phi(h,P^*)}{P^*} = \phi. \end{array} \right\} \tag{20}$$

Substituting (20) in (16), we have

$$\frac{y_{n+1} - y_n}{\phi} = f(y_n), \tag{21}$$

$$y_{n+1} = y_n + \phi f(y_n). \tag{22}$$

Note that (22) is of the form (3). Therefore,

$$F(h, y_n) = y_n + \phi f(y_n). \tag{23}$$

We now differentiate equation (23) partially with respect to y,

$$\frac{\partial F}{\partial y} = \frac{\partial (y)}{\partial y} + \frac{\partial (\phi)}{\partial y} f(y) + \frac{\phi \partial [f(y)]}{\partial y} \tag{24}$$

For all $y \in \mathbb{R}$ and $h > 0$, equation (24) satisfies (19). This shows that our new NSFDM is stable with respect to monotone dependence on initial value.

4.2. Monotonicity of solutions

Anguelov and Lubuma [2] said that due to the autonomous nature of the differential equation (1), its solution have a relatively simple structure with regard to their monotonicity. Every solution is either increasing or decreasing on the whole interval $[t_0, \infty)$. The increasing and decreasing solutions are separated by fixed points.

Definition 4.2.1 (Anguelov and Lubuma [2]) The NSFDM (3) is stable with respect to the property of monotonicity of solutions if for every $y_0 \in \mathbb{R}$, the solution y_n of (3) is an increasing or decreasing sequence according as the solution $y(t)$ of equation (1) is increasing or decreasing.

Theorem 4.2.1 (Anguelov and Lubuma [2]) Assume that the NSFDM (3) is stable with respect to monotone dependence on initial value. Assume also that for every $h > 0$, the equations

$$y = F(h,y) \;\text{ and }\; f(y) = 0 \tag{25}$$

hence, the same root considered with their multiplicity, then (3) is stable with respect to monotonicity of solutions. It must be mentioned here that if the condition in (25) is satisfied, then (3) is elementary stable.

4.3. Linear stability

Consider the scalar ODE (1) and assume that (2) has m simple zeros, where m may be bounded. The solution of equation (2) are fixed-points of the differential equation and correspond to constant solutions. The linear stability properties of the fixed-points are determined by investigating the behaviour of small perturbations about a given fixed-point. Consider the i-th fixed point, $\bar{y}_{(i)}$.

The perturbed trajectory takes the form,

$$y(t) = \bar{y}_{(i)} + \varepsilon(t), \tag{26}$$

where

$$|\varepsilon(t)| \leq \left|\bar{y}_{(i)}\right|. \tag{27}$$

Substituting equation (26) into equation (1) gives,

$$\frac{d\varepsilon}{dt} = f\left[\bar{y}_{(i)}\right] + P_i\varepsilon + o(\varepsilon^2), \tag{28}$$

where

$$P_i = \frac{df}{dy}_{y=\bar{y}_{(i)}}. \tag{29}$$

The linear stability equation is given by the linear terms of equation (28), i.e.,

$$\frac{d\varepsilon}{dt} = P_i \varepsilon. \tag{30}$$

The solution of this equation is,

$$\varepsilon(t) = \varepsilon_0 e^{P_i t}. \tag{31}$$

The fixed point, $y(t) = \bar{y}_{(i)}$, is said to be linearly stable if $P_i < 0$ and linearly unstable if $P_i > 0$.

5. Numerical implementations

We shall apply the newly developed NSFDM (16) on some sampled autonomous differential equations. We shall consider ODEs with one and two fixed points.

5.1. Numerical examples

Problem 1: ODE with One Fixed-Point (the Decay Equation)

Consider the ODE

$$\frac{dy}{dt} = \lambda y,\ y(0) = 100,\ \lambda = -0.0026,\ t \in [0,1] \tag{32}$$

with the exact solution

$$y(t) = 100e^{-0.0026t} \tag{33}$$

Comparing (32) with (1), we see that

$$f(y) = -0.0026y \tag{34}$$

A single globally stable fixed point exists at $\bar{y}_{(1)} = 0$ from (34). The application of equation (8) gives,

$$P_1 = \frac{df}{dy}_{y=\bar{y}_{(1)}} = -0.0026 \tag{35}$$

From equation (9),

$$P^* = 0.0026 \tag{36}$$

Define the numerator function ψ and the denominator function ϕ by

$$\left\{ \begin{array}{c} \psi = 1 \\ \phi = \frac{\phi(h,P^*)}{P^*} = \frac{1-e^{-0.0026h}}{0.0026} \end{array} \right\} \tag{37}$$

Substituting (37) into (10), we have

$$y_{n+1} = y_n + \left[\frac{1-e^{-0.0026h}}{0.0026}\right](-0.0026y_n) \tag{38}$$

This is the new NSFDM for (32).

Problem 2: ODE with Two Fixed-Points (the Logistic Equation)

Consider the ODE

$$\frac{dy}{dt} = y(1-y),\ y(0) = 0.5,\ t \in [0,1] \tag{39}$$

with the analytical solution

$$y(t) = \frac{0.5}{0.5(1+e^{-t})} \tag{40}$$

Comparing (39) with (1), we obtain

$$f(y) = y(1-y) \tag{41}$$

The equation has two fixed points at,

$$\left\{ \begin{array}{c} \bar{y}_{(1)} = 0 \\ \bar{y}_{(2)} = 1 \end{array} \right\} \tag{42}$$

From equation (8), we see that

$$P_1 = \frac{df}{dy}_{y=\bar{y}_{(1)}} = 1 \tag{43}$$

$$P_2 = \frac{df}{dy}_{y=\bar{y}_{(2)}} = -1 \tag{44}$$

and the application of equation (9) gives

$$P^* = 1. \tag{45}$$

Define the numerator and the denominator functions as

$$\left\{ \begin{array}{c} \psi = 1 \\ \phi = \frac{\phi(h,P^*)}{P^*} = \frac{1-e^{-(1)h}}{1} = 1 - e^{-h}. \end{array} \right\} \tag{46}$$

Substituting (46) into (10), we have

$$y_{n+1} = y_n \left[1 + (1 - e^{-h})(1 - y_n) \right]. \tag{47}$$

This is the new NSFDM for (39).

Table 1: Showing the Results for Problem 1 at $h = 0.01$

t	Exact Solution	New NSFDM Solution	Error	Eval. Time/s
0.010	99.9974000337997070	99.9974000337997070	0.000000e+000	2.8071
0.020	99.9948001351976640	99.9948001351976640	0.000000e+000	2.8459
0.030	99.9922003041920960	99.9922003041920960	0.000000e+000	2.8484
0.040	99.9896005407812540	99.9896005407812540	0.000000e+000	2.8509
0.050	99.9870008449633760	99.9870008449633760	0.000000e+000	2.8532
0.060	99.9844012167367280	99.9844012167367280	0.000000e+000	2.8541
0.070	99.9818016560995350	99.9818016560995350	0.000000e+000	2.8545
0.080	99.9792021630500330	99.9792021630500330	0.000000e+000	2.8547
0.090	99.9766027375864610	99.9766027375864610	0.000000e+000	2.8549
0.100	99.9740033797070850	99.9740033797070850	0.000000e+000	2.8551

Table 2: Showing the Results for Problem 1 at $h = 0.1$

t	Exact Solution	New NSFDM Solution	Error	Eval. Time/s
0.100	99.9740033797070850	99.9740033797070850	0.000000e+000	0.0411
0.200	99.9480135176568470	99.9480135176568470	0.000000e+000	0.0473
0.300	99.9220304120923400	99.9220304120923400	0.000000e+000	0.0497
0.400	99.8960540612571460	99.8960540612571460	0.000000e+000	0.0519
0.500	99.8700844633952300	99.8700844633952300	0.000000e+000	0.0545
0.600	99.8441216167510670	99.8441216167510670	0.000000e+000	0.0557
0.700	99.8181655195695610	99.8181655195695610	0.000000e+000	0.0560
0.800	99.7922161700960970	99.7922161700960970	0.000000e+000	0.0562
0.900	99.7662735665764730	99.7662735665764730	0.000000e+000	0.0564
1.000	99.7403377072569700	99.7403377072569700	0.000000e+000	0.0566

Table 3: Showing the Results for Problem 2 at $h = 0.01$

t	Exact Solution	New NSFDM Solution	Error	Eval. Time/s
0.010	0.5024999791668749	0.5024999791668749	0.000000e+000	0.1834
0.020	0.5049998333399998	0.5049998333399998	0.000000e+000	0.1896
0.030	0.5074994375506203	0.5074994375506203	0.000000e+000	0.1920
0.040	0.5099986668799655	0.5099986668799655	0.000000e+000	0.1943
0.050	0.5124973964842103	0.5124973964842103	0.000000e+000	0.1975
0.060	0.5149955016194100	0.5149955016194100	0.000000e+000	0.1990
0.070	0.5174928576663898	0.5174928576663898	0.000000e+000	0.1993
0.080	0.5199893401555819	0.5199893401555819	0.000000e+000	0.1995
0.090	0.5224848247918001	0.5224848247918001	0.000000e+000	0.1998
0.100	0.5249791874789400	0.5249791874789400	0.000000e+000	0.2000

Table 4: Showing the Results for Problem 2 at $h = 0.1$

t	Exact Solution	New NSFDM Solution	Error	Eval. Time/s
0.100	0.5249791874789400	0.5249791874789400	0.000000e+000	0.0577
0.200	0.5498339973124780	0.5498339973124780	0.000000e+000	0.0691
0.300	0.5744425168116590	0.5744425168116590	0.000000e+000	0.0821
0.400	0.5986876601124520	0.5986876601124520	0.000000e+000	0.0946
0.500	0.6224593312018546	0.6224593312018546	0.000000e+000	0.1070
0.600	0.6456563062257954	0.6456563062257954	0.000000e+000	0.1163
0.700	0.6681877721681662	0.6681877721681662	0.000000e+000	0.1272
0.800	0.6899744811276125	0.6899744811276125	0.000000e+000	0.1460
0.900	0.7109495026250039	0.7109495026250039	0.000000e+000	0.1629
1.000	0.7310585786300049	0.7310585786300049	0.000000e+000	0.1790

5.2. Discussion of results

From the results presented in the tables above, it is obvious that the new NSFDM does not exhibit numeriacl instabilities because the exact solution and the numerical solution are the same at the step-sizes $h = 0.01$ and $h = 0.1$. It is also observed that evaluation time per seconds at each point is also very small, thus showing that the method is computationally reliable.

6. Conclusion

From the presentation above, we have been able to develop a new NSFDM for the solution of autonomous first order ODEs. The performance of the method on some sampled problems shows that it is computationally reliable. This method is therefore recommended for the numerical solution of first order autonomous differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests.

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