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A NEW PROXIMITY DEFINITION FOR 2-D AND 3-D CURVES

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Abstract. Let C_1 and C_2 be two continuous (coplanar or not) Jordan curves defined over a closed (finite) interval. We introduce a new approach concerning the distance between C_1 and C_2 and we examine some ramifications of this definition in the frame of metric spaces.

Keywords: Jordan curve; distance; metric space.

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PART I. Symbols and Definitions

For the case of coplanar curves, without loss of generality, we will consider our plane to be the usual XOY-coordinates plane. Let $\mathcal{I}$ be the collection of all closed (finite) intervals of the x-axis. We will denote by $\mathcal{C}$ the collection of all continuous rectifiable Jordan curves C over all intervals in I .In case the interval I=[a, b] is trivial , i.e. for a=b, the corresponding C will be considered shrunk to the point P(a, a). We denote by $d(,)$ the usual Euclidean distance between the points of the plane. In elementary calculus it is well known that the average value of a continuous function y=f(x) over an interval [a, b], a $\neq$b, denoted as $\overline{f(x)}$, is defined by

$$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)dx \quad (1)$$

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Let P be any point on the plane and C$\in\mathcal{C}$ over an interval [c, d].We will define as the distance of P from C the minimum value of the function *d*(P,Q) as Q traces C. With some abuse of notation we will denote this value simply by *d*(P, C). For C_1 and C_2 in $\mathcal{C}$ over nontrivial intervals I=[a,b] and J=[c, d], respectively, we will define the distance from C_1 to C_2, which in order to avoid any further abuse of notation and confusion we will denote by $D^*(C_1, C_2)$, as follows:

Definition 1

$$D^*(C_1, C_2) = \overline{d(P, C_2)} \text{ over I. (2)}$$

Remark1

Similarly we define the distance from C_2 to C_1 via the formula $D^*(C_2, C_1) = \overline{d(Q, C_1)}$ over J.

Remark 2

It is evident that in order to have $D^*(C_1, C_2) = D^*(C_2, C_1)$ we require I=J, which is a necessary but not a sufficient condition. We can easily see e.g. that for C_1 ={y=0, 0≤x≤1} and C_2={y= x/√3, 0≤x≤1}, $D^*(C_1, C_2)$ =1/4 while $D^*(C_2, C_1)$ =1/2√3. Note that if we do not have $D^*(C_1, C_2) = D^*(C_2, C_1)$ but the rest of the metric conditions hold, then we talk about a quasimetric. Still, in this case, a metric *D* can be defined as follows:

Definition 2

$$D(C_1, C_2) = [\, D^*(C_1, C_2) + D^*(C_2, C_1)] / 2 \quad (3)$$

Then, via (3), we can simply refer to this number as the distance between the two curves, provided that our *D** is a quasimetric, which will be a routine to check after we have imposed an additional condition on $\mathcal{C}$ presented in Remark 3.

Remark 3

If we consider as C_1 the x-axis interval I=[0, 1] and as $C_2 = C_1 \cup OA$ .where O=(0,0) and A=(0,1) .then $D(C_1, C_2)$=0 but the two curves are not identical. We impose now on $\mathcal{C}$, additionally, the condition our curves C to have at most a countable number of mutual intersections. Then it is rather straightforward that when we restrict ourselves to this (still very large class!) $\mathcal{C}$, defined over the same $I \in \mathit{I}$, a set of curves that will denote by ($\mathcal{C}$, I) , *D* is a metric on ($\mathcal{C}$, I)x($\mathcal{C}$, I) , for each $I \in \mathit{I}$.

For a trivial I the metric space ($\mathcal{C}$, I, *D*) is merely the classical ($\measuredangle^2$, *d*) and so we will consider only nontrivial intervals.

Remark 4

Al though we have used the Euclidean *d*, our *D* resembles but is not identical neither to the intrinsic metric endowed by *d* on the plane nor to the Housdorff metric even in the case that *D** is itself a metric and our curves are restricted to graphs of functions (for details of the theory of metric space see e. g [1] or [2]).

Remark 5

Since ($\mathcal{C}$, I, *D*), that from now on we shall denote it simply by V, is a metric space, we can define for any $C_0 \in V$, a (local) basis of open neighborhoods of C_0, N(C_0,ε), for the endowed topology, i.e the set of all C ∈ ($\mathcal{C}$, I) such that $D(C_0, C) < \varepsilon$, for ε>0.Due to the "exotic" nature of V, in order to be able to "visualize" the above basis, it is preferable to use I itself for C_0,in the sense that C_0 is the graph of the function y=0, a≤x≤ b. We will see (in Example 7) that, as expected, we cannot always calculate the exact distance between two curves in ($\mathcal{C}$, I) and then we make use of upper and lower bounds of *D* in which case we can talk about an ε-proximity of two curves in ($\mathcal{C}$, I).

Remark 6

Our approach could be extended, in a similar way, to define the distance between two (parametrically described over the same parameter interval) rectifiable curves in space (Example 7). In part (III), where we refer to the classical analysis concept of a Cauchy sequence It will emerge as a possibly usable tool of computer mathematics for the image analysis theory in reference to the geometry of data points curves: more specifically, among others, it will require the ability to determine convergence of a sequence of curves to a “curve-limit” that will rather look totally different in shape from the generating sequence; nevertheless in this work we focus our attention only to coplanar cases.

Remark 7

In Part III we also pose some (open so far) questions we consider challenging, concerning the type of the new metric space **V** and provide some pertinent heuristic claims.

PART II.EXAMPLES

We present now various examples of an increasing degree of calculative difficulty as far as the numerical calculations is concerned. The first four examoles can actually be worked out “by hand” and provide exact distance results while the rest like the fifth and the sixth, involving 2-D curves, along with the seventh which is our only 3-D example, make use of Mathematica and thus we will produce only ε-proximity results.

Example II.1

Let C_1 be the x-axis interval I=[0, 1] and C_2 the line segment joining the A (0,1) to B(1,1). Evident ly $D(C_1, C_2)$=1. Suppose now that at the points $A_n = (1-2^{-n}, 1)$ of C_2 , for n=0,1,2,... , we draw above AB vertical line segments of length 2^{-n} .This new curve , C_3 is defined over I and evidently has length 3, but when traced in the ccw sense in a continuous way the vertical lines at A_n , for n=1,2,..., are travelled twice and thus C_3 is not a Jordan path. Once again, $D^*(C_1, C_3)$ =1 but $D^*(C_3, C_1)$ is not well defined and thus $D(C_1, C_3)$ is not well defined either , even though C_3 is rectifiable; this remark explains the initial condition that our curves have to be Jordan paths.□

Example II.2

If C_1 is the graph of y=x+2 and C_2 the graph of y=2x+2, with domain [-2,2] , we can easily check that the minimum of the distance d(P,Q), for any fixed point P on C_1 as Q traces C_2 is $\frac{3}{5}|x|$ and thus $D^*(C_1, C_2) = \frac{3}{20}\int_{-2}^{2}|x|dx$ =1√5. In a similar way we have $D^*(C_2, C_1) = \frac{1}{4\sqrt{2}}\int_{-2}^{2}|x|dx$ =1/√2 and so $D(C_1, C_2) = \frac{1}{20}(12 + 5\sqrt{2})$.□

Example II.3

Let C_1 be the graph of y=x with domain [0,1] and C_2 the quarter of the unit circle centered at O (0,0) that lies in the first quadrant:

(i)We can easily check that the minimum of the distance d(P,Q), for any fixed point P on the line segment OA ,with A= (1,1), from any point on the arc is $\left|1 - x\sqrt{2}\right|$ where $x = x_P$.

Then $D^*(C_1, C_2) = \int_{0}^{1}\left|1 - x\sqrt{2}\right|dx$ = √2-1.

(ii)Similarly, for any fixed point Q of the arc , $d(Q, C_1) = \frac{1}{\sqrt{2}}\left|x - \sqrt{1-x^2}\right|$ (where $x = x_Q$) and so

$$D^*(C_2, C_1) = \frac{1}{\sqrt{2}}\left[\int_{0}^{1/\sqrt{2}}(\sqrt{1-x^2} - x)dx + \int_{1/\sqrt{2}}^{1}(x - \sqrt{1-x^2})dx\right] = 1/2\sqrt{2}.$$

We conclude that $D(C_1, C_2) = \frac{1}{8}(5\sqrt{2} - 4)$.□

Example II.4

Let C_1 be the graph of $y = \frac{3}{4} - x$, and C_2 the graph of $y = x^2$ over $[\frac{1}{4}, \frac{3}{4}]$.

(i)For a fixed P on the line segment AB , where A=($\frac{1}{4}$,$\frac{1}{2}$) and B=($\frac{3}{4}$,0) we first calculate the distance d(P,Q) between P and any point Q on the above parabolic arc, then we calculate $d(\mathrm{P}, \mathrm{C}_2)$ and finally we will calculate $D^*(\mathrm{C}_1, \mathrm{C}_2) = 2\int_{1/4}^{3/4} d(\mathrm{P}, \mathrm{C}_2)\mathrm{dx}$.

To facilitate our calculations we set, Q(t) for the point Q=(t,t²), 1/4≤ t≤ 3/4 and squaring distances we can check directly that the derivative (with respect to t) of $d(\mathrm{P},\mathrm{Q(t)})^2$ leads to the cubic equation

$2t^3 + (2x-\frac{1}{2})t - x = 0 \Leftrightarrow (t-\frac{1}{2})(2t^2 + t = 2x) = 0$ with the only acceptable root - critical point t=$\frac{1}{2}$ and by construction there is no need to compare d(P,Q($\frac{1}{2}$)) to d(P,Q(1/4)) or d(P,Q(3/4)) Note also that the second derivative of $d(\mathrm{P},\mathrm{Q(t)})^2$ is $12t^2 + 4x - 1 > 0$ and $d(\mathrm{P},\mathrm{Q}(\frac{1}{2}))^2 = 2(x-\frac{1}{2})^2$ .We conclude that $d(\mathrm{P}, \mathrm{C}_2) = \sqrt{2}\left|x - \frac{1}{2}\right|$ and thus $D^*(\mathrm{C}_2, \mathrm{C}_1) = 2\sqrt{2}\left[\int_{1/4}^{1/2} (\frac{1}{2} - x)\mathrm{dx} + \int_{1/2}^{3/4} (x - \frac{1}{2})\mathrm{dx}\right] = \frac{1}{4\sqrt{2}}$.

(ii)In a similar way for a fixed point Q= Q(x) on the parabolic arc , it is straightforward to see that

$d(\mathrm{Q}, \mathrm{C}_1) = \frac{\sqrt{(-3+4x+4x^2)^2}}{4\sqrt{2}}$, Thus $D^*(\mathrm{C}_2, \mathrm{C}_1) = 2\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{\sqrt{(-3+4x+4x^2)^2}}{4\sqrt{2}} dx = \frac{1}{4\sqrt{2}} \Rightarrow D^*(\mathrm{C}_1, \mathrm{C}_2) = \frac{1}{4\sqrt{2}}$.

We conclude that $D(\mathrm{C}_1, \mathrm{C}_2) = \frac{1}{4\sqrt{2}}$.□

Example II.5

Let C_1 and C_2 be, respectively, the graphs of $y = \frac{1}{4}x^2 - \frac{1}{2}$ and $y = x^2$, $0 \le x \le 1$.

(i)For a fixed point P on the first parabolic arc, in order to facilitate once more our calculations, ,we set

Q=Q(t) for the point Q tracing the second parabola. Following the same steps and symbols as in **Ex. 4**, we directly see that the derivative of $d(P,Q(t))^2$ leads to the cubic $2t^3 + (2 - \frac{1}{2}x^2)t - x = 0 \Leftrightarrow (t - \frac{x}{2})(2t^2 + xt + 2) = 0$, and since $0 \le x \le 1$ the only acceptable root-critical point is $t = \frac{x}{2}$. Naturally the second derivative of $d(P,Q(\frac{x}{2}))^2$ is $11x^2 + 4 > 0$ and since $d(P,Q(\frac{x}{2}))^2 = \frac{x^2+1}{4}$ we conclude that $d(P, C_2) = \frac{\sqrt{x^2+1}}{2} \Rightarrow D^*(C_1, C_2) = \frac{1}{2}\int_0^1 \sqrt{x^2+1}dx$ .Since $\int \sqrt{x^2+1}dx = \frac{1}{2}[x\sqrt{x^2+1} + \ln(x + \sqrt{x^2+1})]$ (modulo a constant) we obtain $D^*(C_1, C_2) = \frac{1}{4}(\sqrt{2} + \text{ArcSinh}[1]) \approx 0.573$.

Thus we can take $D^*(C_1, C_2)^- = 0.573$ and $D^*(C_1, C_2)^+ = 0.574$

(ii)Now, for a fixed point Q=Q(x) on the second parabolic arc and P=P(t) tracing the first one, we require calculations will not be as simple as in (i): when $0<t<1$, in order to find the critical points (if any), $t = \rho = \rho(x)$, we have to solve the cubic $t^3 + (1-2x^2)t - 2x = 0$ and then compare all three , $d(Q, P(\rho))$, $d(Q,P(0))$ and $d(Q,P(1))$ in order to produce $d(Q, C_1)$ as a function of x in [0,1].

Following the classical cubic equation theory (see e.g.[5]) we find as the only acceptable minimizing root $\rho(x) = [x + \sqrt{x^2 + (\frac{1-2x^2}{3})^3}]^{1/3} + [x - \sqrt{x^2 + (\frac{1-2x^2}{3})^3}]^{1/3}$ and $d(Q, C_1) = d(Q, P(\rho(x)))$.We conclude, via Mathematica, that $D^*(C_2, C_1) =$

$$\int_0^1 \sqrt{(\rho(x)-x)^2 + (\frac{1}{4}\rho(x)^2 - \frac{1}{2} - x^2)^2}\,dx =$$

$$\int_0^1 \sqrt{\left| x - \frac{\sqrt[3]{6}\left(18x + \sqrt{6}\sqrt{-8x^6 + 36x^4 + 27}\right)^{2/3} - 3\times 6^{2/3} + 2\times 6^{2/3}x^2}{3\sqrt[3]{18x + \sqrt{6}\sqrt{-8x^6 + 36x^4 + 27}}} \right|^2 + \left| x^2 + \frac{1}{2} - \frac{\left(\sqrt[3]{6}\left(18x + \sqrt{6}\sqrt{-8x^6 + 36x^4 + 27}\right)^{2/3} - 3\times 6^{2/3} + 2\times 6^{2/3}x^2\right)^2}{36\left(18x + \sqrt{6}\sqrt{-8x^6 + 36x^4 + 27}\right)^{2/3}} \right|^2}\,dx$$

≈ 0.696833

Thus we can take $D^*(C_2, C_1)^- = 0.696$ and $D^*(C_2, C_1)^+ = 0.697$ and finally we have $D(C_1, C_2)^-$ $=0.634$ and $D(C_1, C_2)^+ = 0.635$ so we have secured an ε-proximity between the two curves for $\varepsilon=0.001$.

Example II.6

Let C_1 and C_2, respectively, be the graphs of $y = \frac{1}{x^2} + \frac{1}{2} - \frac{x^2}{2}$ and $y=x^2$, over [1, 1.27].

Here, at the final stage, the calculations lead to cumbersome expressions that require rounding offs. and it will be feasible to produce only upper and a lower bounds $D(C_1, C_2)\pm$.

(i)Under the same symbolism as before and in a similar way, we find that the minimum of $d(P,Q(t))$ is obtained at $t=\frac{1}{x}$, the only acceptable root of the cubic $2t^3 +(1-2y)t - x = (t-\frac{1}{x})(2t^2 + \frac{2}{x}t + x^2) = 0$ in our interval (actually even for $1 \le x \le 2$).Then $d(P, C_2) = \frac{1}{2}\sqrt{-7 + \frac{4}{x^2} + 2x^2 + x^4}$

We thus have to calculate $D(C_1, C_2) = \frac{1}{2}\int_1^2 \frac{x^2+1}{x}\sqrt{x^2+4}\,dx$ .Since $\int x\sqrt{x^2+1}\,dx = (\sqrt{x^2+4})^3$ and $\int \frac{1}{x}\sqrt{x^2+4}\,dx = \sqrt{x^2+4}\ 2\operatorname{arcsinh}\frac{2}{x}$ (modulo the usual constants) we obtain as the exact value of

$$D^*(C_1, C_2) = \frac{\int_1^{1.27} \sqrt{x^4+2x^2+\frac{4}{x^2}-7}\,dx}{2(1.27-1)} \approx 0.809$$

Thus we can take $D^*(C_1, C_2)^- = 0.644$ and $D^*(C_1, C_2)^+ = 0.974$

(ii)It is rather natural to expect that, once again, things will not be as simple as in (i) in order to minimize $d(Q,P)$ for a fixed point $Q=Q(x)$ in C_2 as $P=P(t)$ traces C_1.The critical points $t=\rho(x)$-if any)- require solving in our open interval the octic equation $t^8 - t^6 + (2-x^2)t^4 - 2x\,t^3 - 2t^2 + 4(x^2-1) = 0$.

This requires a numerical approach by use of an advanced scientific computer program. First we make use of Mathematica and check that in the interval (1, 1.27) this octic has always exactly one (real of course) root $\rho = \rho(x)$; consequently we make use of Mathematica again, for a relatively large sample of values of x in (1, 1.27). comparing, each time, for each fixed x, the values of d(Q,P(ρ)), d(Q,P(1)) and d(Q,P(1.27)). Naturally the larger the sample the better the approximation Then we unavoidably resort to the discrete average of all these approximate $d(Q,C_1)$ that we have found, in order to estimate bounds $D^*(C_2,C_1)^{\pm}$ and thus, when combined with part (i) ,we produce bounds $D\,(C_2,C_1)^{\pm}$. More specifically, we will use $x_\kappa=1+0.002\kappa$, $0\leq\kappa\leq 135$ in order to produce lower and upper bounds of $\frac{1}{136}\sum_{\kappa=0}^{135} d\left(Q,P(\rho_\kappa)\right)$ to infer that $D^*(C_2,C_1)\approx 0.695$. Thus we can take $D^*(C_2,C_1)^- = 0.695$ and $D^*(C_2,C_1)^+ = 0.732$. Finally, using (i) we conclude that $0.669< D(C_1,C_2) <0.853$ and so we have secured an ε-proximity between the two curves for ε< 0.2.□

Example II.7

Here we present our only 3-D example:

Let C_1 be the line segment of the points P= (x, 0,1) and C_2 the spiral arc of the points Q=(cost, sint, t) defined over I=[0,1].

(i)Following the same routine we conclude that for any fixed P in C_1, as Q traces C_2, the only critical value of t that minimizes PQ satisfies cost=x and thus $d(P,C_2)=\sqrt{1-x^2+(1-\arccos x)^2}$.

Via Mathematica, we have that $D^*(C_1,C_2)=\int_0^1\sqrt{1-x^2+(1-\arccos x)^2}\,dx\approx 0.886882$. Thus we can take

$D^*(C_1,C_2)^-=0.886$ and $D^*(C_1,C_2)^+=0.887$.

(ii)In a similar way, for any fixed Q=Q(x) in C_2, as P=P(t) traces C_1, the critical t mimizing QP satisfies t=cosx, thus $d(Q,C_1)=\sqrt{(1-x)^2+(\sin x)^2}$ and again via Mathematica $D^*(C_1,C_2)=\int_0^1\sqrt{(1-x)^2+(\sin x)^2}\,dx\approx 0.774255$ Thus, if we take $D^*(C_2,C_1)^-=0.774$ and $D^*(C_2,C_1)^+=0.775$, we have that $0.830 < D(C_1,C_2)< 0.831$, and so we have secured an ε-proximity between the two curves for ε=0.001.

PART III. The topological nature of the metric space V

(A).We start by reminding the concept of a Cauchy sequence $\{s_\kappa, \kappa=1,2,.\}$ within a general metric space (S, *d*).Usually, when this sequence consists of numbers the metric *d* used is simple , i.e the distance between real or complex numbers; but when the sequence consists of functions *d* could be either simple or quite cumbersome (see e.g. [2]).

Generally speaking $\{s_\kappa\}$ will be called a Cauchy sequence if $d(s_m, s_n)\to 0$, whenever m and n $\to\infty$ (independently to each other!).

Though it is a well known result in classical analysis, it is worth reminding also that if $\{ s_\kappa \}$ converges in (S, *d*) then it is a Cauchy sequence but the converse is not true. In case every Cauchy sequence in a metric space converges then we call this space a complete metric space.

We will give below a heuristic argument which indicates that our metric space V is complete, but first let us give a rather simple example of a Cauchy sequence, $\{C_\kappa\}$, in our metric space:

Example III.1

Let C_κ be the line segments $y=x/\kappa$, $\kappa=1,2,\ldots$, over I= [0,1].Then it is immediate that

$D^*(C_m, C_n) = \frac{1}{2}\left|\frac{1}{m}-\frac{1}{n}\right| \frac{1}{m\sqrt{n^2+1}} \to 0$, and $D^*(C_n, C_m) = \frac{1}{2}\left|\frac{1}{m}-\frac{1}{n}\right| \frac{1}{n\sqrt{m^2+1}} \to 0$, and thus $D(C_m, C_n)\to 0$,

as m, n $\to\infty$.□

Remark 7

We should point out that we knew that we have a Cauchy sequence, anyway, since

$d(P, C_n) = \frac{1}{\sqrt{n^2+1}} x_P$, for each point P in I and thus $D^*(I, C_n) = \frac{1}{2\sqrt{n^2+1}} \to 0$, as n $\to\infty$ and since $d(Q,I) = \frac{1}{n} x_Q$ for each point Q in C_n ,$D^*(C_n, I) = \frac{1}{2n} \to 0$, as n $\to\infty$ i.e $\{C_n\}\to C_0$ in our **V** (where C_0 in this case is I itself)).□

A conjecture heuristically motivated

Let us now consider a seemingly rather interesting question whether, for a given nontrivial I, V is a complete metric space or not:

It looks like that the answer should be affirmative if we can materialize the following process for producing the limit curve C in our space (a process which is not genuinely constructive but merely intuitive):

We start with a pair C_1 and C_2 and by choosing middle (or suitable intermediate) points of each minimum distance PQ, for each fixed P in C_1 as Q traces C_2. If necessary we "glue" via line segments the "first" and/or "last" (in the sense of the smaller and/or larger x in I for these intermediate points) in order the new "middle curve" (or intermediate curve) to be defined over the whole interval instead of a subinterval of I .We continue this process for all the consecutive pairs via a "continuous gluing" without worrying about the end of this process since we have started with rectifiable curves. It is plausible that we will be able to produce a continuous and seemingly rectifiable C over I such that, for each n, $D(C_n, C) \leq D(C_n, C_{n+1})$.Since $D(C_n, C_{n+1}) \to 0$, as $n \to \infty$, we conclude $D(C_n, C) \to 0$.

(B)Final Remarks and Open Problems

1. Under the assumption that V is a complete metric space, when examined exclusively topologically (see Remark 4 of part (I)), we then have also a Baire space. This amounts to the property that any countable union of closed sets with empty interior has also an empty interior (see e.g. [3] or [4]). Such a property seems plausible for our metric space: the absence, in each component, of any curves within ε-proximity from I and despite the fact that we are granted a high degree of flexibility since we allow not rectifiable curves, suggests that the union lacks of any acceptable curve (in the sense of Jordan and continuous over the same I).This can be considered as an indication to expect an affirmative answer to the completeness question.

2,An equally challenging question , due to the "exotic" structure of V could have been the following: this metric space seems to be not bounded and thus not totally bounded ; then in order to be (sequentially) compact it is sufficient and necessary to be complete and totally bounded. Thus, even if we have a complete space it would not be a compact space. But, at least is, is it locally compact?

3. Naturally one could had posed also additional questions for our metric *D* and/or our metric space, like whether *D* is an ultrametric, or whether V separable or connected e.t.c.
4. Finally, in case our metric space eventually turned out not to be complete, then naturally rises the question about its completion, in the following sense: what are the additional features that we could had imposed on the curves of $\mathcal{C}_l$ in order our space V to be a dense subspace of the new complete space, with respect to our metric?

Conflict of Interests

The authors declare that there is no conflict of interests.

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