ENEDGE BLOCK DOMINATION IN GRAPHS

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Abstract. For any graph G (V, E), a block graph \( B(G) \) is a graph whose vertices are corresponding to the blocks of G and two vertices in \( B(G) \) are adjacent whenever the corresponding blocks contain a common cutvertex in G. An edge dominating set \( s_e \) of a block graph \( B(G) \) is an endedge block dominating set if \( s_e \) contains all endedges of \( B(G) \). The endedge block domination number \( \gamma_{eb}'(G) \) is the minimum cardinality of an endedge block dominating set. In this paper some bounds for \( \gamma_{eb}'(G) \) are obtained in terms of elements of G. Further exact values of \( \gamma_{eb}'(G) \) for some standard graphs and relationships with other dominating parameters were obtained.

Keywords: block graph; domination number; endedge domination number; endedge block domination number.

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Introduction

In this paper We follow the notations and terminology of Harary [1]. We consider connected, undirected, finite graphs without loops. Let \( G = (V, E) \) be a graph with \( |V| = p \) and \( |E| = q \). \( n \) denotes number of blocks of \( G \). \( N(v) \) and \( N[v] \) denote the open and closed neighborhoods of a vertex \( v \) respectively in \( G \). The degree of an edge \( e = u \ v \) of \( G \) is defined by \( \text{deg } e = \text{deg } u + \text{deg } v - 2 \). The maximum degree of a vertex in \( G \) is denoted by \( \Delta(G) \) and the minimum degree of a vertex in \( G \) is denoted by \( \delta(G) \).

A vertex \( v \) of \( V \) is called a cutvertex if its removal from \( G \) increase the number of components of \( G \). A nontrivial connected graph with no cutvertex is called a block. A block incident with exactly one cutvertex is called an endblock. A block incident with more than one cutvertex is called a nonendblock.

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A set $D$ of a graph $G = (V,E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a minimal dominating set. A set $F$ of edges in a graph $G = (V,E)$ is called an edge dominating set of $G$ if every edge in $E - F$ is adjacent to at least one edge in $F$. The edge domination number $\gamma'(G)$ is the minimum cardinality of an edge dominating set of $G$. Edge domination was introduced by S. Mitchell and S. T. Hedetniemi [2] and is now well studied in graph theory. The edge dominating set is called an endedge dominating set if all endedges belong to edge dominating set of $G$.

The endedge domination number $\gamma_e'(G)$ is the minimum cardinality of endedge dominating set of $G$. Endedge domination is introduced by M.H.Muddebihal and A.R.Sedamkar [3]. A block graph $B(G)$ is a graph whose vertices are corresponding to the blocks of $G$ and two vertices in $B(G)$ are adjacent whenever the corresponding blocks contain a common cutvertex in $G$. A set $D_b$ of a block graph $B(G) = (H,X)$ is a dominating set if every vertex in $H - D_b$ is adjacent to some vertex in $D_b$. The domination number $\gamma(B(G))$ is the minimum cardinality of a minimal block dominating set. Block domination is introduced by M. H. Muddebihal, T. Srinivas and Abdul Majeed [4]. We are introducing endedge block domination in this paper and we obtain certain bounds on $\gamma_e'(G)$ in terms of vertices, blocks and other parameters of $G$.

**Results**

Initially we begin with endedge block domination number of a graph of some standard graphs, which are straight forward in the following theorem.

**Theorem 1**: (i) For any star $K_{1,p}$ with $p \geq 2$, $\gamma_e'(k_{1,p}) = \frac{p-1}{2}$, if $p$ is odd.

$$= \frac{p}{2}, \text{ if } p \text{ is even}.$$  

(ii) For any path $P_p$ with $p \geq 3$, $\gamma_e'(P_p) = \frac{p}{3}$, if $p \equiv 0 \pmod{3}$.

$$= \left\lceil \frac{p}{3} \right\rceil, \text{ otherwise}.$$  

In the following theorem we obtain upper bound for $\gamma_e'(G)$ in terms of number of blocks of $G$.

**Theorem 2**: For any connected graph $G$ with $n \geq 2$ blocks, $\gamma_e'(G) \leq n - 1$. Equality holds for $B(G) \cong K_{1,n-1}$ where $n$ is number of blocks of $G$. 
\textbf{Proof}: Let \( G \) be any nontrivial connected graph with \( A = \{ B_i \}, 2 \leq i \leq n \) blocks. Let \( S_e = \{ q_1, q_2, \ldots, q_l \}, l < n \) be \( y_{eb}^1 \)-set in \( B(G) \) such that \( |S_e| = y_{eb}^1(G) \). We prove the result by induction on number of blocks of \( G \).

Let \( G \) be a graph with \( n = 2 \) blocks. Then \( S_e = \{ q_j \}, j = 1 \) and
\[
|S_e| = y_{eb}^1(G) = n - 1 = 2 - 1.
\]

Let \( G \) be a graph with \( n = 3 \) blocks. Then \( S_e = \{ q_1, q_2 \} \) and
\[
|S_e| = y_{eb}^1(G) = n - 1 = 3 - 1.
\]

Assume that the result is true for \( n = t \) blocks. Then \( y_{eb}^1(G) \leq t - 1 \).

Suppose \( G \) has \( n = t + 1 \) blocks. Then the corresponding block vertex of \((t + 1)\)th block of \( G \) is either an endvertex or a nonendvertex incident with either an endblock or a nonend block respectively in \( B(G) \).

Then clearly \( |S_e| = y_{eb}^1(G) \leq t - 1 + 1 \) gives \( y_{eb}^1(G) \leq (t + 1) - 1 \).

Equality for \( B(G) \cong K_{1,n-1} \) is obvious.

Following corollary gives equality for \( y_{eb}^1(G) \).

\textbf{Corollary 1}: For any connected graph \( G \) with \( n \geq 2 \) blocks with exactly one cutvertex,
\[
y_{eb}^1(G) = \frac{n-1}{2}, \text{ if } n \text{ is odd.}
\]
\[
y_{eb}^1(G) = \frac{n}{2}, \text{ if } n \text{ is even.}
\]

\textbf{Proof}: If \( G \) has exactly one cutvertex, then \( B(G) \) is complete graph and number of vertices of \( B(G) \) is \( n \). By Theorem 1 clearly result follows.

In next theorem we obtain upper bound in terms of \( n \) for \( y_{eb}^1(T) \).

\textbf{Theorem 3}: For any tree with \( n \) blocks \( y_{eb}^1(T) \leq \left\lfloor \frac{n}{2} \right\rfloor \) if and only if \( T \not\cong P_4 \).

\textbf{Proof}: For necessary condition,

Suppose \( y_{eb}^1(T) \leq \left\lfloor \frac{n}{2} \right\rfloor \) for any tree \( T \).

If \( T \cong P_4 \), then \( n = 3 \) and \( \left\lfloor \frac{n}{2} \right\rfloor = 1 \). The corresponding \( B(T) \cong P_3 \) and \( y_{eb}^1(T) = 2 \).

Then \( y_{eb}^1(T) > \left\lfloor \frac{n}{2} \right\rfloor \) a contradiction. Hence \( T \not\cong P_4 \).

For sufficient condition consider \( T \not\cong P_4 \).

Suppose \( T \cong P_p \), \( p \neq 4 \). Then maximum number of blocks in \( T \) are \( n = p - 1 \).

From Theorem 1, \( y_{eb}^1(P_p) = \frac{p}{3} \leq \left\lfloor \frac{p-1}{2} \right\rfloor \), if \( p \equiv 0 \text{ (mod 3)} \).
\begin{align*}
&= \left\lceil \frac{p}{3} \right\rceil \leq \left\lfloor \frac{p-1}{2} \right\rfloor, \ p \neq 4 \text{ otherwise gives the result.}
\end{align*}

**Theorem 4:** For any connected graph $G$ with $m$ endblocks, $\gamma_{eb}^1(G) \leq \gamma_e(G) + \left\lceil \frac{m+1}{2} \right\rceil$. The proof of the Theorem 4 requires some lemmas. Before the lemmas we construct some sets in $G$ as well as in $B(G)$ so that to give the proofs of lemmas.

**Sets in $G$ are**

$E_e = \{e_1, e_2, \ldots., e_e\}$ is a set of all endedges.

$E_n = E(G) \setminus E_e$

$E_g = E_n \cup E_e$

$A_1 = \{B_1, B_2, \ldots., B_i\}$ is a set of endblocks such that each block is adjacent to exactly one block.

$A_2 = \{B_1, B_2, \ldots., B_j\}$ is a set of endblocks such that each block is adjacent to more than one block.

$A_3 = \{B_1, B_2, \ldots., B_k\}$ is a set of all nonendblocks.

$A = A_1 \cup A_2 \cup A_3$.

We define a family as $\mathcal{I} = \{E_e \cup (E(G) \setminus N(E_e))\}$ in $G$.

**Sets in $B(G)$ are as follows.**

$H = \{b_1, b_2, \ldots., b_n\}$ is a set of all vertices.

$H_1 = \{b_1, b_2, \ldots., b_i\}$ is a set of all endvertices of degree 1.

$H_2 = \{b_1, b_2, \ldots., b_j\}$ is a set of all nonend noncutvertices.

$H_3 = \{b_1, b_2, \ldots., b_k\}$ is a set of all cutvertices.

$X_1 = \{q_1, q_2, \ldots., q_t\}$ is a set of all endedges.

$X_2 = E(B(G)) \setminus X_1$

We now define a family $\mathcal{R} = \{X_1 \cup (X_2 \setminus N(X_1))\}$ in $B(G)$.

We consider $A_1 = \phi$ in Lemma 1, lemma 2 and 3.

**Lemma 1:** If $A_1 = \phi$ and each block $B_j \in A_2$ has $p \geq 3$ vertices, then $\gamma_e(G) \equiv \gamma^1(G)$ and $\gamma_{eb}(G) \in X_2$.

**Proof:** Let $F_1 \subseteq E_g$. If $A_1 = \phi$ and each block $B_j \in A_2$ has $p \geq 3$ vertices, then for each edge $e \in F_1$, $\exists$ an edge $e_1 \in \{E_g \setminus F_1\}$ such that $N(e_1) \cap F_1 = \{e\}$. Hence $F_1$ forms
minimal edge dominating set in \( G \). Then \( |F_1| = \gamma^1(G) \). Since \( A_I = \phi \) and each block of \( A_2 \) has \( p \geq 3 \) vertices, \( \gamma^1_e(G) = \gamma^1(G) \).

In \( B(G) \), \( H_1 = \phi \). Then \( X_1 = \phi \) and \( \exists X_2^1 \subset X_2 \) such that every edge in \( X_2 \setminus X_2^1 \) is adjacent to at least one edge in \( X_2^1 \). So \( X_2^1 \) forms \( \gamma^1_{eb} \)-set and \( |X_2^1| = \gamma^1_{eb}(G) \).

**Lemma 2:** If \( A_I = \phi \), each block \( B_j \in A_2 \) has exactly two vertices, then \( \gamma^1_e(G) \in \mathcal{S} \) and
\[
\gamma^1_{eb}(G) \in X_2.
\]

**Proof:** Let \( A_I = \phi \) and each block \( B_j \in A_2 \) has exactly two vertices. Then \( A_2 \cong E_e \) and \( E(G) \setminus A_2 = E_n \). Let \( F_2 \subset E_n \) be the minimal edge dominating set of induced subgraph \( \langle E_n \setminus N(A_2) \rangle \). Then \( F_2 \cup A_2 \) forms \( \gamma^1_e \)-set which belongs to \( \mathcal{S} \) and \( |F_2 \cup A_2| = \gamma^1_e(G) \).

In \( B(G) \), endedge set \( X_1 = \phi \) and \( \exists X_2^1 \subset X_2 \) such that each \( q_i \in X_2 \setminus X_2^1 \) is adjacent to at least one edge \( q_j \in X_2^1 \). Then \( X_2^1 \) forms \( \gamma^1_{eb} \)-set and \( |X_2^1| = \gamma^1_{eb}(G) \).

**Lemma 3:** If \( A_I = \phi \) and some blocks \( B_j \in A_2 \) have exactly two vertices, then \( \gamma^1_e(G) \in \mathcal{S} \) and \( \gamma^1_{eb}(G) \in X_2 \).

**Proof:** Let \( A_1 \subset A_2 \) be set of endblocks and each block of \( A_1 \) has exactly two vertices. Then \( A_2 \cong E_e \) and \( E(G) \setminus A_2 = E_n \). \( \exists \) an edge dominating set \( F_2 \subset E_n \) of induced subgraph \( \langle E_n \setminus N(A_2) \rangle \) such that \( F_2 \cup A_2 \) forms \( \gamma^1_e \)-set which belongs to \( \mathcal{S} \) and \( |F_2 \cup A_2| = \gamma^1_e(G) \).

In \( B(G) \), \( X_2^1 \subset X_2 \) forms \( \gamma^1_{eb} \)-set because each edge in \( X_2 \setminus X_2^1 \) is adjacent to at least one edge in \( X_2^1 \), then \( |X_2^1| = \gamma^1_{eb}(G) \).

In further **Lemma 4, Lemma 5 and 6**, we consider \( A_2 = \phi \).

**Lemma 4:** If each block \( B_l \in A_1 \) has \( p \geq 3 \) vertices and \( A_2 = \phi \), then \( \gamma^1_e(G) \cong \gamma^1(G) \) and
\[
\gamma^1_{eb}(G) \in \mathfrak{R}.
\]

**Proof:** Let \( A_2 = \phi \) and each block \( B_l \in A_1 \) has \( p \geq 3 \) vertices. Then \( E_e = \phi \) and \( \exists \) set of edges \( F_1 \subset E_g \) in \( G \) such that each edge in \( E_g \setminus F_1 \) is adjacent to at least one edge of \( F_1 \).

So \( F_1 \) forms minimal edge dominating set. Since \( E_e = \phi \), \( |F_1| = \gamma^1(G) = \gamma^1_e(G) \).

In \( B(G) \), \( |H_1| = |X_1| \). Let \( X_2^1 \subset X_2 \) be the minimal edge dominating set of \( \langle X_2 \setminus (N(X_1) \cup X_1) \rangle \). Then \( X_1 \cup X_2^1 \) forms \( \gamma^1_{eb} \)-set and \( |X_1 \cup X_2^1| = \gamma^1_{eb}(G) \in \mathfrak{R} \).

**Lemma 5:** If each block \( B_l \in A_1 \) has exactly two vertices and \( A_2 = \phi \), then \( \gamma^1_e(G) \in \mathcal{S} \) and \( \gamma^1_{eb}(G) \in \mathfrak{R} \).
Proof: Let $A_2 = \phi$, each block $B_i \in A_1$ has $p = 2$ vertices. Then $A_1 = E_e$ and $\exists$ a minimal edge dominating set $F_3$ of induced subgraph $\langle E_g \setminus (A_1 \cup N(A_1)) \rangle$ such that $A_1 \cup F_3$ forms $\gamma^1_e$-set. $|A_1 \cup F_3| = \gamma^1_e(G)$ which belongs to $\mathfrak{S}$.

In $B \langle G \rangle$, $|X_1| = |A_1|$ and $X_1^2$ is the minimal edge dominating set of $\langle X_2 \setminus (N(X_1) \cup X_1) \rangle$. Then $X_2 \cup X_1^2$ forms $\gamma^1_{eb}$-set and $|X_1 \cup X_1^2| = \gamma^1_{eb}(G)$ which belongs to $\mathfrak{R}$.

Lemma 6: If some blocks $B_i \in A_1$ have exactly two vertices and $A_2 = \phi$, then $\gamma^1_e(G) \in \mathfrak{S}$

and $\gamma^1_{eb}(G) \in \mathfrak{R}$.

Proof: Let some blocks $B_i \in A_1$ have exactly $p = 2$ vertices and $A_2 = \phi$. Then $\exists$ a minimal edge dominating set $F_4$ of $\langle E(G) \setminus (A_1 \cup N(A_1)) \rangle$ where $A_1 \subset A_1$ is set of all endedges. So $A_1^1 \equiv E_e$ and $A_1 \cup F_4$ forms $\gamma^1_e$-set. Then $|A_1 \cup F_4| = \gamma^1_e(G) \in \mathfrak{S}$.

In $B \langle G \rangle$, $|X_1| = |A_1|$ and $\exists$ $X_2^1 \subset X_2$ such that $X_1 \cup X_2^1$ forms $\gamma^1_{eb}$-set where $X_2^1$ is the minimal edge dominating set of $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$.

Then $|X_1 \cup X_2^1| = \gamma^1_{eb}(G)$ which belongs to $\mathfrak{R}$.

In next lemmas 7, lemma 8 and 9 we consider $A_1 \neq \phi$, $A_2 \neq \phi$.

Lemma 7: If $A_1 \neq \phi$, $A_2 \neq \phi$ and each block $B_i \in A_1$ has $p \geq 3$ vertices, each block $B_j \in A_2$ has $p \geq 3$ vertices then $\gamma^1_e(G) \equiv \gamma^1(G)$ and $\gamma^1_{eb}(G) \in \mathfrak{R}$.

Proof: If each block of $A_1$ and $A_2$ has $3$ or more than three vertices, then $\exists$ a set of edges $F_1 \subset E(G)$ such that every edge in $E(G) \setminus F_1$ is adjacent to at least one edge in $F_1$.

Hence $F_1$ forms $\gamma^1_e$-set in $G$ and since $E_e = \phi$, $|F_1| = \gamma^1_e(G) = \gamma^1_e(G)$.

In $B \langle G \rangle$, $|X_1| = |H_1| = |A_1|$ and $\exists$ a minimal edge dominating set $X_1^2 \subset X_2$ of induced subgraph $\langle X_2 \setminus (X_1 \cup N(X_1)) \rangle$ such that $X_1 \cup X_1^2$ forms $\gamma^1_{eb}$-set $\in \mathfrak{R}$.

Then $|X_1 \cup X_1^2| = \gamma^1_{eb}(G)$.

Lemma 8: If $A_1 \neq \phi$ and each block $B_i \in A_1$ has $p \geq 3$ vertices, $A_2 \neq \phi$ and $A_1^2 \subseteq A_2$ where $A_1^2$ is set of endedges each has degree $\geq 2$, then $\gamma^1_e(G) \in \mathfrak{S}$ and $\gamma^1_{eb}(G) \in \mathfrak{R}$.

Proof: Let $A_1^2 \subseteq A_2$ be set of all endedges in $G$. Let $F_2$ be the minimal edge dominating set of induced subgraph $\langle E(G) \setminus (A_1^2 \cup N(A_1^2)) \rangle$. Then $F_2 \cup A_2$ forms $\gamma^1_e$-set in $G$ and $|F_2 \cup A_2| = \gamma^1_e(G)$. Since $A_1^2 \equiv E_e$, $\gamma^1_e(G) \in \mathfrak{S}$. 

In $B(G)$, $X_1 \cup X_2^1$ forms $\gamma_{eb}^1$-set where $X_2^1 \subset X_2$ is the minimal edge dominating set of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$ which $\in \mathcal{R}$ and $|X_1 \cup X_2^1| = \gamma_{eb}^1(G)$.

**Lemma 9**: If $A_1 \neq \phi$, $A_2 \neq \phi$ and each block $B_j \in A_2$ has $p \geq 3$ vertices, $A_1^1 \subseteq A_1$ is set of all endedges with degree 1 then $\gamma_e^1(G) \in \mathcal{G}$ and $\gamma_{eb}^1(G) \subseteq \mathcal{R}$.

**Proof**: Let , $A_1^1 \subseteq A_1$ is set of all endedges with degree 1 in $G$. Then $A_1^1 \equiv E_e$ and $\exists$ a minimal edge dominating set $F_3$ of $\langle (E(G) \setminus (A_1^1 \cup N(A_1^1))) \rangle$ such that $F_3 \cup A_1^1$ forms $\gamma_{eb}^1$-set $\in \mathcal{S}$ . Then $|F_3 \cup A_1^1| = \gamma_e^1(G)$.

In $B(G)$, $|X_1| = |A_1|$ and $\exists$ a minimal edge dominating set $X_2^1 \subset X_2$ of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$ such that $X_1 \cup X_2^1$ forms $\gamma_{eb}^1$-set $\in \mathcal{R}$ . Then $|X_1 \cup X_2^1| = \gamma_{eb}^1(G)$.

**Lemma 10**: If $A_1 \neq \phi$, $A_2 \neq \phi$ and each block of $A_1$ and $A_2$ has $p = 2$ vertices, then $\gamma_e^1(G) \in \mathcal{S}$ and $\gamma_{eb}^1(G) \subseteq \mathcal{R}$.

**Proof**: Since $A_1$ and $A_2$ have all edges, $\{A_1 \cup A_2\} \equiv E_e$ . Then $\exists$ a minimal edge dominating set $F_5$ of $\langle (E(G) \setminus (E_e \cup N(E_e))) \rangle$ such that $E_e \cup F_5$ forms $\gamma_{eb}^1$–set which belongs to $\mathcal{S}$ . Then $|E_e \cup F_5| = \gamma_e^1(G)$.

In $B(G)$, $|X_1| = |A_1|$ and $\exists$ a minimal edge dominating set $X_2^1 \subset X_2$ of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$ such that $X_1 \cup X_2^1$ forms $\gamma_{eb}^1$-set belongs to $\mathcal{R}$ . Then $|X_1 \cup X_2^1| = \gamma_{eb}^1(G)$.

Now we prove **Theorem 4**

**PROOF OF THE THEOREM 4**:

Let $S_e^1$ be $\gamma_e^1$–set and $S_e$ be $\gamma_{eb}^1$-set in $G$ and $B(G)$ respectively. From Lemma 1, lemma 4 and 7 either $A_1 = \phi$ or each block $B_i \in A_1$ has $p \geq 3$ vertices and either $A_2 = \phi$ or each block $B_j \in A_2$ has $p \geq 3$ vertices in $G$. Then $E_e = \phi$ in $G$ and $\gamma_e^1(G) = \gamma^1(G)$.

In $B(G)$, $|X_1 \cup X_2^1| = \gamma_{eb}^1(G)$ where $X_i$ is set of all endedges and $X_2^1 \subset X_2$ where $X_2 = E(B(G)) \setminus X_1$.

Either $X_1 = \phi$, $X_2^1 \subset X_2$ or $X_1 \neq \phi$, $X_2^1 \subset X_2$ in $B(G)$. Since $|A_1| = |H_1| = |X_1|$ and $|A_2 \cup A_3| = |H_2 \cup H_3| \geq |X_2|$ , Clearly $|X_1 \cup X_2^1| \leq \gamma_e^1(G)$ gives $\gamma_{eb}^1(G) \leq \gamma_e^1(G) + \left\lceil \frac{m-1}{2} \right\rceil$.

From lemma 2 and lemma 3, $A_1 = \phi$ and each block $B_j \in A_2^1 \subseteq A_2$ has exactly two vertices.
Then $A_2^1 \cong E_e \neq \emptyset$ in $G$ and $S_e^1 = A_2^1 \cup F_2$ is $\gamma_e^1$ - set where $F_2$ is minimal edge dominating set of $\langle E(G) \setminus (A_2^1 \cup N(A_2^1)) \rangle$.

In $B(G)$, $X_1 = \emptyset$ and $S_e = X_2^1$ where $X_2^1 \subset X_2$ forms $\gamma_{eb}^1$ - set.

Clearly $|X_2^1| \leq |A_2^1 \cup F_2|$ gives $\gamma_{eb}^1(G) \leq \gamma_e^1(G) + \frac{m-1}{2}$.

From lemma 5, lemma 6 and 9 either each block of $A_1^1 \subseteq A_1$ has exactly two vertices and $A_2 = \emptyset$ or each block of $A_1^1 \subseteq A_1$ has exactly two vertices and each block of $A_2$ has $p \geq 3$ vertices. Then $A_1^1 \cong E_e$ and $S_e^1 = E_e \cup Q_e$ where $Q_e$ is minimal edge dominating set of $\langle E(G) \setminus (E_e \cup N(E_e)) \rangle$.

In $B(G)$, $X_1 \neq \emptyset$ and $S_e = X_1 \cup X_2^1$ where $X_2^1$ is the minimal edge dominating set of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$.

Obviously $|S_e| \leq |S_e^1| + [\frac{m-1}{2}]$ where $m$ is number of end blocks of $G$.

From Lemma 8, each block $B_i \in A_1$ has $p \geq 3$ vertices and each $B_i$ of $A_2^1 \subseteq A_2$ has exactly 2 vertices. Then $A_2^1 \cong E_e$ and $S_e^1 = E_e \cup Q_e$ where $Q_e$ is minimal edge dominating set of $\langle E(G) \setminus (E_e \cup N(E_e)) \rangle$ in $G$.

In $B(G)$, $S_e = X_1 \cup X_2^1$ where $X_2^1$ is the minimal edge dominating set of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$. Then $|S_e| \leq |S_e^1| \leq |S_e^1| + [\frac{m-1}{2}]$ gives the result.

From lemma 10, each block of $A_1$ and $A_2$ has exactly two vertices. Then $A_1 \cup A_2 = E_e$ in $G$. Hence $S_e^1 = E_e \cup Q_e$ where $Q_e$ is minimal edge dominating set of $\langle E(G) \setminus (E_e \cup N(E_e)) \rangle$.

In $B(G)$, $S_e = X_1 \cup X_2^1$ where $|A_1| = |X_1|$ and $X_2^1 \subset X_2$. Clearly, $|S_e| \leq |S_e^1|$ gives $\gamma_{eb}^1(G) \leq \gamma_e^1(G) \leq \gamma_e^1(G) + [\frac{m-1}{2}]$.

**Theorem 5**: Every endblock adjacent to exactly one block in $G$ is in every $\gamma_{eb}^1$ set.

**Proof**: Set of endblocks, each one is adjacent to exactly one block in $G$ forms a set $H_e \subseteq H$ in $B(G)$ where $H_e$ is set of all end vertices of degree 1 and $H$ is set of all vertices in $B(G)$. Clearly $|H_e| = |X_1|$ where $X_1$ is set of all endedges belongs to $\gamma_{eb}^1$ set. Hence the result.

Further theorems provide relations between $\gamma(B(G)), \gamma_{eb}^1(G)$ and number of blocks $n$ of $G$.

**Theorem 6**: For any connected graph $G$ with $n \geq 2$ blocks, $\gamma(B(G)) \leq \gamma_{eb}^1(G)$.
Proof: Let $H = H_e \cup H_n$ be set of all vertices in $B(G)$ where $H_e$ is set of all endvertices and $H_n$ is set of all nonendvertices. Let $X_1$ is set of all endedges and $X_2 = E(B(G)) \setminus X_1$ in $B(G)$. We consider the following cases.

Case 1: Suppose $H_e = \emptyset , H_n \neq \emptyset$. Then $X_1 = \emptyset$. Let $D_b = \{ b_i \}$, $i < n$ be the vertex dominating set of $B(G)$. Let $F \subset X_1^1$ where $X_1^1$ is the minimal edge dominating set of $B(G)$ and $X_2^1 = F \cup Q_f$ where $Q_f$ is the minimal edge dominating set of $E\left(B(G) \setminus (F \cup N(F))\right)$ such that $F$ and $N(F)$ are incident with $b_i \in D_b$. Since $X_1 = \emptyset$, each vertex of $D_b$ is associated with at least one edge of $X_1^1$ clearly $|D_b| \leq |X_2^1|$ gives the result.

Case 2: Suppose $H_e \neq \emptyset , H_n \neq \emptyset$. We consider following subcases.

Subcase 2.1: Suppose $B(G)$ has exactly one cutvertex. Then $X_1 \neq \emptyset, X_2 = \emptyset$ or $X_1 \neq \emptyset, X_2 \neq \emptyset$ or $X_2 \neq \emptyset$.

Clearly $|D_b| = 1 \leq |X_1|$ or $|X_1 \cup X_2^1|$ or $|X_2^1|$ gives $\gamma(B(G)) \leq \gamma_{eb}(G)$.

Subcase 2.2: Suppose $B(G)$ has more than one cutvertices. Then $X_1 \neq \emptyset, X_2 \neq \emptyset$.

Let $N(H_e) = H_e^1$. Then $D_b = H_e^1 \cup H_n^1$ where $H_e^1$ is $\gamma$ - set of $\langle H_n \setminus (N(H_e^1) \cup H_e^1)\rangle$ and $X_1 \cup X_2^1$ forms $\gamma_{eb}^1$ - set where $X_2^1$ is $\gamma^1$ - set of $\langle X_2 \setminus (X_1 \cup N(X_1))\rangle$.

Clearly $|H_e^1 \cup H_n^1| \leq |X_1 \cup X_2^1|$ gives the result.

Theorem 7: For any connected graph $G$ with $n \geq 2$ blocks, $(\gamma(B(G)) + \gamma_{eb}^1(G)) \leq n$.

Proof: We consider the following cases.

Case 1: Suppose $B(G)$ has endedges. Let $X_1 = \{ q_1, q_2, \ldots, q_m \}$ be set of all endedges in $B(G)$.

Let $E(B(G)) \setminus X_1 = X_2$ and $X_1^2 \subseteq X_2$ is $\gamma^1$ - set of $\langle X_2 \setminus (X_1 \cup N(X_1))\rangle$. Then $X_1 \cup X_1^2$ forms $\gamma_{eb}^1$ - set in $B(G)$.

Let $X_1^1 = \{ q_1, q_2, \ldots, q_i \}$, $i \leq m$ be the set of edges adjacent to $X_1$. Then $H_1^2 = \{ b_i \}$, $i \leq m$ denote the $\gamma$ - set of the induced subgraph $\langle X_1 \cup X_1^1 \rangle$.

Further let $E(B(G)) \setminus X_1 \cup X_1^2 = X_3$ such that $H_2^1 = \{ b_j \}$, $j < n$ be the set of vertices incident to the edges of $\langle X_3 \rangle$ but not to the edges of $\langle X_1 \cup X_1^1 \rangle$.

Suppose $H_3^1 \subseteq H_3$ denotes minimal vertex dominating set of $\langle X_3 \rangle$. Then $H_3^1 \cup H_3^1$ is minimal vertex dominating set of $B(G)$.

Clearly, $|X_1 \cup X_1^2| + |H_3^1 \cup H_3^2| \leq n$. Hence $\gamma(B(G)) + \gamma_{eb}^1(G) \leq n$.

Case 2: If $B(G)$ has no endedges, let $D_b$ be the minimal vertex dominating set of $B(G)$.

Let $S_e = \{ q_1, q_2, \ldots, q_l \}$, $l < n$ be the $\gamma_{eb}^1$ - set of $B(G)$.
Suppose $D_2 = \{ b_t \}$, $t < n$ be the set of vertices incident to the edges of $S_e$. Assume $\forall b_t \in D_b$ are incident with some $q_i \in S_e$, $D_2 \setminus D = D_2^1$ and $D_2^1 \cong V - D_b$, then $|D_2^1| + |D_b| = n$ otherwise $|D_2^1| + |D_b| < n$.

Hence from all the cases $\gamma(B(G)) + \gamma_{eb}^1(G) \leq n$.

**Theorem 8:** For any connected graph $G$, $\gamma_{eb}^1(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

**Proof:** Let $S_e = \{ q_1, q_2, \ldots, q_l \}$ be the $\gamma_{eb}^1$-set of $B(G)$ and $X_1 = \{ q_1, q_2, \ldots, q_m \}$ be the set of all endedges in $B(G)$. $X_2 = E(B(G)) \setminus X_1$. We consider the following cases.

**Case 1:** If $X_1 = \phi$, then $\exists$ a set $X_2^1 \subseteq X_2$ such that every edge of $X_2 \setminus X_2^1$ is adjacent to at least one edge of $X_2^1$. Hence $X_2^1$ forms $\gamma_{eb}^1$-set of $B(G)$ and since each block of $G$ contains at least two vertices, the result is obvious.

**Case 2:** If $X_1 \cong S_e$, then clearly $|X_1| \leq \left\lfloor \frac{p}{2} \right\rfloor$ because each $q_i \in X_1$ contains at least two blocks of $G$ and each block of $G$ has at least two vertices.

**Case 3:** If $X_1 \subset S_e$, then $X_1 \cup X_2^1$ forms $\gamma_{eb}^1$-set where $X_2^1 \subset X_2$ is the dominating set of $\langle X_2 \setminus (X_1 \cup N(X_1)) \rangle$. Clearly $|X_1 \cup X_2^1| = |S_e| \leq \left\lfloor \frac{p}{2} \right\rfloor$.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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