Available online at http://scik.org
Eng. Math. Lett. 2017, 2017:6
ISSN: 2049-9337

STABILITY ANALYSIS OF A SYSTEM OF DC SERVO MOTOR WITH LOAD

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Abstract. The problem of stability of electrical circuit systems is very important. One of the main factor that can cause instability in a servo motor is insufficient resolution on the motor's feedback device. This behavior could seriously lead to erratic movement at low speed and the inability of a feedback to detect these small changes causes instability in a servo motor system. In this paper, problems of such system stability provision is considered using Lyapunov's stability theory. By constructing appropriate Lyapunov functional, sufficient conditions which ensure the stability of the state variables chosen $x_i(t)$, $(i = 1,2,3)$ describing the system of the servo motor are obtained.

Keywords: electric circuits; state variables; stability of system circuit; Lyapunov function.

2010 AMS Subject Classification: 34D20, 34K20.

1. Introduction

A servo motor is a useful component in many real control systems. The function of the servo is to receive a control signal that represents a desired output position of the servo shaft and apply power to its DC motor until its shaft turns that position. It uses the position sensing device to determine the rotational position of the shaft. Technically, one can reduce this inability of the

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Received April 19, 2017

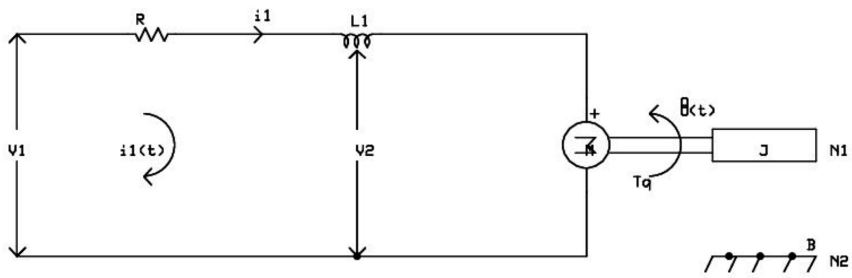

Fig. 1

feedback to detect small changes by using feedback devices like resolver or pulse encoder [1]. This seems to work but is expensive, moreover it does not allow to discover the true cause of instability. Be as it may, if these small changes are detected and eliminated experimentally, it can appear at any moment. This undesirable problem can be detected theoretically and eliminated at development stage. Hence, it is necessary to discover fundamentally using Lyapunov's stability theory. In this paper, we examine the stability of the electrical and control variables behind the operation of the DC servo motor problem. The energy storage elements of a system are what make the system dynamic. The flow of energy into and out of a storage elements occurs at a finite rate is described by a differential equation relating the derivative of the energy storage variable (a state variable) to other power variable of the element. There are three independent energy storages in servo armature circuit, the current source which stores energy in electric voltage, the inductor which stores energy in a magnetic field and the motor angular position which stores mechanical energy. The state variables are the energy storage variables of these elements, V_1, V_2 and $\theta(t)$. According to [5] state variables are the minimum set of variables that fully describe the system circuit and its response to any given set of inputs. By this, it shows that the qualitative property of state-determined servo system represented in Fig. 1 is completely characterized by the stability of the set of $x_i(t), (i = 1, 2, 3)$ variables.

2. Preliminaries

We consider a servo system represented in Fig.1. The input signal to the motor is the armature voltage $V_1(t)$ and the output signal is the angular position $\theta(t)$. The terms R and L_1 are the resistance and inductance of the armature winding in the motor respectively. The voltage V_2 is the back emf generated internally in the motor by the angular motion and k_2 is the motor back emf constant. J is the inertia of the motor and load respectively. Here, the motor and the load assumed not lumped together and B is the damping in the motor and load relative to the fixed chassis. We supposed also that circuit elements (resistance and inductance) are always positive. The first time derivative of the armature current $\frac{di}{dt}$ and the angular position $\frac{d\theta}{dt}$ do exist. The feed current i_1 is finite but undefined. The applied torque Tq is given by

$$Tq : \frac{Jd^2\theta(t)}{dt^2} + \frac{Bd\theta(t)}{dt} - k_m i_1 \frac{N_1}{N_2} = 0, \tag{1}$$

where k_m is the torque constant that relates the torque to the armature current. N_1 and N_2 are the number of turns of the motor and load respectively.

The mathematical equation of servo circuit system can be different depending on the choice of state variables that are defined. The state variables chosen for this analysis are all real variables representing energy stored in the system which are defined as the outputs of the integrators by

$$\begin{aligned} x_1 &= i_1, \\ x_2 &= \theta(t), \\ x_3 &= \frac{d\theta}{dt}. \end{aligned} \tag{2}$$

Applying Kirchhoff's law to the armature circuit in Fig. 1. and by KCL and KVL [4], we obtain the following differential equation

$$\frac{di_1}{dt} = -\frac{k_2 N_2}{L_1 N_1}\frac{d\theta}{dt} - \frac{R}{L_1} i_1 + \frac{1}{L_1} V_1.$$

The case considered here is that of a natural process where dissipation is absent due to external conductive connections. Thus, $\frac{1}{L_1}V_1 = 0$.

It follows that

$$\frac{di_1}{dt} = -\frac{k_2 N_2}{L_1 N_1}\frac{d\theta}{dt} - \frac{R}{L_1} i_1. \tag{3}$$

If we now substitute the state variables in (2) into equations (1) and (3), we obtain the following system of differential equations,

$$\begin{aligned} \dot{x_1} &= -\frac{B}{J}x_1 + \frac{k_m}{J}\frac{N_1}{N_2}x_3, \\ \dot{x_2} &= x_1, \\ \dot{x_3} &= -\frac{k_2}{L_1}\frac{N_2}{N_1}x_1 - \frac{R}{L_1}x_3. \end{aligned} \tag{4}$$

equation (4) is a linear heterogeneous system of three differential equations of the first order which can be represented in matrix notation as

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -\frac{B}{J} & 0 & \frac{k_m}{J}\frac{N_1}{N_2} \\ 1 & 0 & 0 \\ -\frac{k_2}{L_1}\frac{N_2}{N_1} & 0 & -\frac{R}{L_1} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5}$$

So (5) is a linear vector differential equation of order one,

$$\dot{X} = AX, \tag{6}$$

where state vector X is a column vector, A is a 3×3 matrix of constant co-efficients. A is not symmetrical since the circuit is connected through connecting elements.

The Lyapunov functional approach has been a powerful tool to ascertain the stability of critical variables of dynamic systems. Today, this method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamic systems, systems with lag, power system analysis and so on [8]. The major difficulty in applying the method of Lyapunov to the analysis of qualitative properties of critical variables characterizing the system of electrical circuits is the lack of straight forward procedure for finding appropriate Lyapunov functionals. Moreover, how do we construct those appropriate Lyapunov functionals? no author has discussed them thus far. It is in general a difficult task. Similar problem is shared with ordinary differential equations of high orders [6]. Thus, our problem in Fig. 1. is best treated since it has been reduced to stability problem according to Lyapunov's equations in (4). This Lyapunov method lies in constructing a scalar function Ψ such that is positive definite and its derivative $\dot{\Psi}$ along the system (4) under consideration is negative definite. When these properties of Ψ and $\dot{\Psi}$ are shown to be satisfied according to Lyapunov's theory [2, 3, 9], then the behavior of the servo motor system circuit is known. The construction of a Lyapunov function Ψ, which is a quadratic form satisfying the requirements for Ψ and $\dot{\Psi}$ for discussing the stability of the $x_i(t), (i = 1,2,3)$ variables of the linear vector differential equation (6) is obtained. It must be noted here that the physical meaning of Lyapunov function is not considered. In known cases Lyapunov functions were obtained as abstract mathematical approach results. see [7]. Also, in this analysis we assume that physical meaning availability is not connected with function efficiency.

3. Assumptions

In addition to the basic assumption imposed on the elements B, J, L_1, k_2, k_m and R in (4), we suppose that the followings hold,

(i): $Bk_2k_m > BJL_1 + Jk_2k_m$,

(ii): $k_mN_1R > 2JL_1N_2 + BL_1N_2 + JN_2R$.

To establish our result, we use the following scalar function $\Psi = \Psi(x_1, x_2, x_3)$ defined by

$$\begin{aligned} 2\Psi(x_1, \dot{x}_2, x_3) &= \frac{N_2}{k_mL_1N_1}\Big(JL_1 + BL_1 + JR + k_2k_m\Big)x_1^2 \\ &+ \frac{N_2}{N_1}\left(\frac{BR + k_2k_m}{k_mL_1}\right)x_2^2 + \frac{k_mN_1}{JN_2}x_3^2 \\ &+ \frac{2JRN_2}{k_mL_1N_1}x_1x_2 + 2x_2x_3 + 2x_1x_3. \end{aligned} \tag{7}$$

From (7), it is immediate that $\Psi(0,0,0) = 0$. Next, we re-arrange (7) as follows

$$\begin{aligned} 2\Psi(x_1, x_2, x_3) &= \frac{JN_2}{k_mN_1}\left(1 + \frac{k_2k_m}{JL_1}\right)x_1^2 + \frac{BRN_2}{k_2L_1N_1}x_2^2 \\ &+ \left(\frac{k_mN_1}{JN_2} - \frac{L_1N_1}{k_2N_2} - \frac{k_mN_1}{BN_2}\right)x_3^2 + \frac{JRN_2}{k_mL_1N_1}(x_1 + x_2)^2 \\ &+ \frac{L_1N_1}{k_2N_2}\left(\frac{k_2N_2}{L_1N_1}x_2 + x_3\right)^2 + \frac{BN_2}{k_mN_1}\left(x_1 + \frac{k_mN_1}{BN_2}x_3\right)^2. \end{aligned}$$

That is,

$$
\begin{aligned}
2\Psi(x_1,x_2,x_3) &\geq \frac{JN_2}{k_mN_1}\left(1+\frac{k_2k_m}{JL_1}\right)x_1^2+\frac{BRN_2}{k_2L_1N_1}x_2^2 \\
&+ \left(\frac{k_mN_1}{JN_2}-\frac{L_1N_1}{k_2N_2}-\frac{k_mN_1}{BN_2}\right)x_3^2. \qquad (8)
\end{aligned}
$$

It is obvious from (8) that the function Ψ defined in (7) is a positive definite function provided that the Assumption (i) hold. Hence, there is a positive constant $\xi_1 > 0$ sufficiently small such that

$$\Psi(x_1,x_2,x_3) \geq \xi_1(x_1^2+x_2^2+x_3^2).$$

Now, let $\frac{d}{dt}\Psi(x_1(t),x_2(t),x_3(t)) = \dot{\Psi}$ denote the derivative of $\Psi = \Psi(x_1(t),x_2(t),x_3(t))$ along the system of (4). Then, by straight forward calculation from (7) and (4), we obtain after simplification,

$$
\begin{aligned}
\dot{\Psi}(x_1,x_2,x_3) = &- \frac{N_2}{k_mL_1N_1}\left\{\frac{Bk_2k_m+BL_1(2J+1)+BR(J+1)+J^2L_1(2+R)}{JL_1}\right\}x_1^2 \\
&- \left\{\frac{k_mN_1R}{JL_1N_2}-\frac{2JL_1+BL_1+JR}{JL_1}\right\}x_3^2 \\
&- \left\{\frac{2JL_1+BL_1+JR}{2JL_1}\right\}(x_1-x_3)^2. \qquad (9)
\end{aligned}
$$

It follows that,

$$\dot{\Psi}(x_1,x_2,x_3) \leq 0,$$

provided that Assumption (ii) hold and (9) is equal to zero if and only if $x_1 = x_3 = 0$. Since Assumption (ii) hold and the last addend of (9) is certainly negative, then

$$\dot{\Psi}(x_1,x_2,x_3) < -\xi_2(x_1^2+x_3^2),$$

for some $\xi_2 > 0$.

Thus, the analysed servo motor system circuit shown in Fig. 1 is asymptotically stable according to Lyapunov's theory if the inequalities in Assumption (i) and (ii) hold from time t_o to $t = (t-t_o) \rightarrow \infty$.

Conclusion

The application of Lyapunov's theory to analysis of electrical circuit systems is rarely scarce. The Lyapunov's direct or second method allow us to predict and characterize the behaviour of system of servo motor and its response as well eliminate the undesirable cases of small changes due to the motor feedback by relatively simple mathematical approach.

Conflict of Interests

The author declare that there is no conflict of interests.

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