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ON THE CLASS OF TWO DIMENSIONAL KOLMOGOROV SYSTEMS

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Abstract. In this paper we charaterize the integrability and the non-existence of limit cycles of Kolmogorov systems of the form

$$\begin{cases} x' = x\left(P(x,y) + R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right), \\ y' = y\left(Q(x,y) + R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right), \end{cases}$$

where $A(x,y)$, $B(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$ are homogeneous polynomials of degree a, a, n, n, m respectively. Concrete example exhibiting the applicability of our result is introduced.

Keywords: Kolmogorov system; first integral; periodic orbits; limit cycle.

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1. Introduction

Many mathematical models in biology science and population dynamics, frequently involve the systems of ordinary differential equations having the form

(1)
$$\begin{cases} x' = \dfrac{dx}{dt} = xF(x,y), \\ y' = \dfrac{dy}{dt} = yG(x,y), \end{cases}$$

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such systems are called Kolmogorov systems, the derivatives are performed with respect to the time variable, and F , G are two functions in the variables x and y, frequently used to model the iteration of two species occupying the same ecological niche; see $[10,15,18]$. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics; see $[12,19,20]$ chemical reactions, plasma physics; see $[14]$, hydrodynamics; see $[5]$, economics, etc. In the classical Lotka- Volterra-Gause model, F and G are linear and it is well known that there are no limit cycles. There can, of course, only be one critical point in the interior of the realistic quadrant $(x>0, y>0)$ in this case, but this can be a center; however, there are no isolated periodic solutions. We remind that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1). In the qualitative theory of planar dynamical systems; see $[3,4,7,8,9,16,17]$, one of the most important topics is related to the second part of the unsolved Hilbert 16th problem; see $[13]$. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly; see $[1,2,11]$.

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H:\Omega\to\mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt}=\frac{\partial H(x,y)}{\partial x}xF(x,y)+\frac{\partial H(x,y)}{\partial y}yG(x,y)\equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H=h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait; see $[6]$.

A real or complex polynomial $U(x,y)$ is called algebraic solution of the polynomial differential system (1) if

$$\frac{\partial U(x,y)}{\partial x}xF(x,y)+\frac{\partial U(x,y)}{\partial y}yG(x,y)=K(x,y)U(x,y),$$

for some polynomial $K(x,y)$, called the cofactor of $U(x,y)$. The corresponding cofactor of $U(x,y)$ is always polynomial whether $U(x,y)$ is algebraic or non algebraic. If U is real, the curve $U(x,y)=0$ is an invariant under the flow of differential system (1) and the set $\left\{(x,y)\in\mathbb{R}^2,\, U(x,y)=0\right\}$ is formed by orbits of system (1). There are strong relationships between the integrability of system (1) and its number of invariant algebraic solutions.

In this paper we are intersted in studying the integrability and the periodic orbits of the 2-dimensional Kolmogorov systems of the form

$$\left\{\begin{array}{l} x'=x\left(P(x,y)+R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right), \\ y'=y\left(Q(x,y)+R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right), \end{array}\right. \tag{2}$$

where $A(x,y)$, $B(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$ are homogeneous polynomials of degree a, a, n, n, m respectively.

We define the trigonometric functions

$f_1(\theta)=P(\cos\theta,\sin\theta)\left(\cos^2\theta\right)+Q(\cos\theta,\sin\theta)\left(\sin^2\theta\right)$,

$f_2(\theta)=R(\cos\theta,\sin\theta)\ln\left|\frac{A(\cos\theta,\sin\theta)}{B(\cos\theta,\sin\theta)}\right|$,

$f_3(\theta)=(\cos\theta\sin\theta)\left(Q(\cos\theta,\sin\theta)-P(\cos\theta,\sin\theta)\right)$.

2. Main results

Our main result on the integrability and the periodic orbits of the Kolmogorov system (2) is the following.

Theorem 1. *Consider a Komogorov system* (2)*, then the following statements hold.*

(1) *If* $A(x,y)B(x,y)\neq 0$, *then the curve* $U(x,y)=xyQ(x,y)-xyP(x,y)=0$ *is an invariant algebraic curve of the differential system* (2).

(2) *If* $f_3(\theta)A(\cos\theta,\sin\theta)B(\cos\theta,\sin\theta)\neq 0$ *and* $n\neq m$, *then system* (2) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}}\exp\left((m-n)\int^{\arctan\frac{y}{x}}M(\omega)\,d\omega\right)- \\ &\quad (n-m)\int^{\arctan\frac{y}{x}}\exp\left((m-n)\int^{w}M(\omega)\,d\omega\right)N(w)\,dw, \end{aligned}$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, *and the curves which are formed by the trajectories of the differential system* (2)*, in Cartesian coordinates are written as*

$$x^2 + y^2 = \begin{pmatrix} h \exp\left((n-m)\int^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) + \\ (n-m)\exp\left((n-m)\int^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) \\ \int^{\arctan \frac{y}{x}} \exp\left((m-n)\int^{w} M(\omega)\, d\omega\right) N(w)\, dw \end{pmatrix}^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$*. Moreover, the system* (2) *has no limit cycle.*

(3) *If* $f_3(\theta) A(\cos\theta, \sin\theta) B(\cos\theta, \sin\theta) \neq 0$ *and* $n - m \neq 1$*, then system* (2) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}} \exp\left((m-n)\int^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) - \\ &\quad (n-m)\int^{\arctan \frac{y}{x}} \exp\left((m-n)\int^{w} M(\omega)\, d\omega\right) N(w)\, dw, \end{aligned}$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, *and the curves which are formed by the trajectories of the differential system* (2)*, in Cartesian coordinates are written as*

$$x^2 + y^2 = \begin{pmatrix} h \exp\left((n-m)\int^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) + \\ (n-m)\exp\left((n-m)\int^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) \\ \int^{\arctan \frac{y}{x}} \exp\left((m-n)\int^{w} M(\omega)\, d\omega\right) N(w)\, dw \end{pmatrix}^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$*. Moreover, the system* (2) *has no limit cycle.*

(4) *If* $f_3(\theta) A(\cos\theta, \sin\theta) B(\cos\theta, \sin\theta) \neq 0$ *and* $n = m$*, then system* (2) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int^{\arctan \frac{y}{x}} \left(M(\omega) + N(\omega)\right) d\omega\right),$$

and the curves which are formed by the trajectories of the differential system (2)*, in Cartesian coordinates are written as*

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h \exp\left(\int^{\arctan \frac{y}{x}} \left(M(\omega) + N(\omega)\right) d\omega\right) = 0,$$

where $h \in \mathbb{R}$*. Moreover, the system* (2) *has no limit cycle.*

(5) *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$*, then system* (2) *has the first integral* $H = \frac{y}{x}$*, and the curves which are formed by the trajectories of the differential system* (2)*, in Cartesian coordinates are written as* $y - hx = 0$*, where* $h \in \mathbb{R}$*. Moreover, the system* (2) *has no limit cycle.*

Proof. **Proof of statement** (1) **of Theorem 1**

If $A(x,y)B(x,y)\neq 0.$

We prove that $U(x,y)=xyQ(x,y)-xyP(x,y)=0$ is an invariant algebraic curve of the differential system (2).

Indeed, we have

$$\frac{\partial U}{\partial x}x\left(P+R\ln\left|\frac{A}{B}\right|\right)+\frac{\partial U}{\partial y}y\left(Q+R\ln\left|\frac{A}{B}\right|\right)=\frac{\partial U}{\partial x}xR\ln\left|\frac{A}{B}\right|+\frac{\partial U}{\partial y}yR\ln\left|\frac{A}{B}\right|+\frac{\partial U}{\partial x}xP+\frac{\partial U}{\partial y}yQ.$$

Then, taking into account that if $P(x,y)$ and $Q(x,y)$ are homogeneous polynomials of degree n, n respectively, we have

$$x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}=nP \text{ and } x\frac{\partial Q}{\partial x}+y\frac{\partial Q}{\partial y}=nQ.$$

Then, we have

$$\frac{\partial U}{\partial x}xR\ln\left|\frac{A}{B}\right|+\frac{\partial U}{\partial y}yR\ln\left|\frac{A}{B}\right|=x\frac{\partial}{\partial x}(xyQ-xyP)R\ln\left|\frac{A}{B}\right|+y\frac{\partial}{\partial y}(xyQ-xyP)R\ln\left|\frac{A}{B}\right|$$

$$xy(Q-P+xQ_x-xP_x)R\ln\left|\frac{A}{B}\right|+xy(Q-P+yQ_y-yP_y)R\ln\left|\frac{A}{B}\right|$$

$$xy(2Q-2P+(xQ_x+yQ_y)-(xP_x+yP_y))R\ln\left|\frac{A}{B}\right|$$

$$xy(2Q-2P+nQ-nP)R\ln\left|\frac{A}{B}\right|=(n+2)xy(Q-P)R\ln\left|\frac{A}{B}\right|$$

$$=(n+2)U\ln\left|\frac{A}{B}\right|$$

On the other hand, substituting

$$y\frac{\partial P}{\partial y}=nP-x\frac{\partial P}{\partial x}\text{and } y\frac{\partial Q}{\partial y}=nQ-x\frac{\partial Q}{\partial x},$$

in what follows, we get

$$\frac{\partial U}{\partial x}xP+\frac{\partial U}{\partial y}yQ=\frac{\partial}{\partial x}(xyQ-xyP)xP+\frac{\partial}{\partial y}(xyQ-xyP)yQ$$

$$=(yQ-yP+xyQ_x-xyP_x)xP+(xQ-xP+xyQ_y-xyP_y)yQ$$

$$=xy\left(Q^2-P^2+nQ^2-nQP+xPQ_x-xPP_x-xQQ_x+xQP_x\right)$$

$$=((n+1)Q+P-xQ_x+xP_x)U$$

In short, we have

$$\frac{\partial U}{\partial x}x\left(P+R\ln\left|\frac{A}{B}\right|\right)+\frac{\partial U}{\partial y}y\left(Q+R\ln\left|\frac{A}{B}\right|\right)$$

$$=\left(\left((n+1)Q+P-xQ_x+xP_x\right)+(n+2)\ln\left|\frac{A}{B}\right|\right)U.$$

Therefore, $U(x,y)=xyQ(x,y)-xyP(x,y)=0$ is an invariant algebraic curve of the polynomial differential systems (2) with the the cofactor

$$K(x,y)=\left((n+1)Q(x,y)+P(x,y)-x\frac{\partial Q(x,y)}{\partial x}+x\frac{\partial P(x,y)}{\partial x}\right)+(n+2)\ln\left|\frac{A(x,y)}{B(x,y)}\right|.$$

Hence, statement (1) is proved.

In order to prove our results we write the polynomial differential system (2) in Polar coordinates (r,θ), defined by $x=r\cos\theta$ and $y=r\sin\theta$, then system (2) becomes

$$\begin{cases} r'=f_1(\theta)r^{n+1}+f_2(\theta)r^{m+1}, \\ \theta'=f_3(\theta)r^n, \end{cases} \tag{3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction, $r'=\frac{dr}{dt}$ and $\theta'=\frac{d\theta}{dt}$

If $f_3(\theta)A(\cos\theta,\sin\theta)B(\cos\theta,\sin\theta)\neq 0$ and $n\neq m$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta}=M(\theta)r+N(\theta)r^{1+m-n}, \tag{4}$$

where $M(\theta)=\frac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta)=\frac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variables $\rho=r^{n-m}$ we obtain the linear equation

$$\frac{d\rho}{d\theta}=(n-m)\left(M(\theta)\rho+N(\theta)\right). \tag{5}$$

The general solution of linear equation (5) is

$$\begin{aligned}\rho(\theta) &= \exp\left((n-m)\int^{\theta}M(\omega)\,d\omega\right)\\ &\quad\left(\mu+(n-m)\int^{\theta}\exp\left((m-n)\int^{w}M(\omega)\,d\omega\right)N(w)\,dw\right),\end{aligned}$$

where $\mu \in \mathbb{R}$, which has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}} \exp\left((m-n)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + \\ &\quad (m-n)\int^{\arctan\frac{y}{x}} \exp\left((m-n)\int^{w} M(\omega)\,d\omega\right) N(w)\,dw. \end{aligned}$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$x^2+y^2 = \begin{pmatrix} h\exp\left((n-m)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + \\ (n-m)\exp\left((n-m)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int^{\arctan\frac{y}{x}} \exp\left((m-n)\int^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{pmatrix}^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$x^2+y^2 = \begin{pmatrix} h_\Gamma\exp\left((n-m)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + \\ (n-m)\exp\left((n-m)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int^{\arctan\frac{y}{x}} \exp\left((m-n)\int^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{pmatrix}^{\frac{2}{n-m}}.$$

But this curve cannot contain the periodic orbit Γ and consequently no limit cycle contained in the realistic quadrant $(x>0, y>0)$, because this curve in realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in \left]0,+\infty\right[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in \left]0,+\infty\right[$, the abscissa is given by

$$x = \frac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} h_\Gamma\exp\left((n-m)\int^{\arctan\eta} M(\omega)\,d\omega\right) + \\ (n-m)\exp\left((n-m)\int^{\arctan\eta} M(\omega)\,d\omega\right) \\ \int^{\arctan\eta} \exp\left((m-n)\int^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{pmatrix}^{\frac{2}{n-m}},$$

at most a unique value of x on every half straight OX^+, consequently at most a unique point in realistic quadrant $(x>0, y>0)$. So this curve cannot contain the periodic orbit.

Hence statement (2) of Theorem 1 is proved.

Suppose now that $f_3(\theta) A(\cos\theta, \sin\theta) B(\cos\theta, \sin\theta) \neq 0$ and $n - m = 1$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta} = M(\theta) r + N(\theta), \tag{6}$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, which is a linear equation.

The general solution of linear equation (6) is

$$\rho(\theta) = \exp\left(\int^{\theta} M(\omega)\, d\omega\right)\left(\mu + \int^{\theta} \exp\left(-\int^{w} M(\omega)\, d\omega\right) N(w)\, dw\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$H(x,y) = \sqrt{x^2+y^2}\exp\left(-\int^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) - \int^{\arctan\frac{y}{x}} \exp\left(-\int^{w} M(\omega)\, d\omega\right) N(w)\, dw.$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$x^2 + y^2 = \left(\begin{array}{c} h\exp\left(\int^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) + \exp\left(\int^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) \\ \int^{\arctan\frac{y}{x}} \exp\left(-\int^{w} M(\omega)\, d\omega\right) N(w)\, dw \end{array}\right)^2,$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$x^2 + y^2 = \left(\begin{array}{c} h_\Gamma\exp\left(\int^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) + \exp\left(\int^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) \\ \int^{\arctan\frac{y}{x}} \exp\left(-\int^{w} M(\omega)\, d\omega\right) N(w)\, dw \end{array}\right)^2.$$

But this curve cannot contain the periodic orbit Γ and consequently no limit cycle contained in the realistic quadrant $(x > 0, y > 0)$, because this curve in realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in]0, +\infty[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in]0, +\infty[$, the abscissa is given by

$$x = \frac{1}{\sqrt{1+\eta^2}}\left(\begin{array}{c} h_\Gamma\exp\left(\int^{\arctan\eta} M(\omega)\, d\omega\right) + \exp\left(\int^{\arctan\eta} M(\omega)\, d\omega\right) \\ \int^{\arctan\eta} \exp\left(-\int^{w} M(\omega)\, d\omega\right) N(w)\, dw \end{array}\right)^2,$$

at most a unique value of x on every half straight OX^{+}, consequently at most a unique point in realistic quadrant $(x>0, y>0)$. So this curve cannot contain the periodic orbit.

Hence statement (3) of Theorem 1 is proved.

Suppose now that $f_3(\theta)A(\cos\theta,\sin\theta)B(\cos\theta,\sin\theta)\neq 0$ and $n=m$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta}=(M(\theta)+N(\theta))\,r. \tag{7}$$

The general solution of equation (7) is

$$r(\theta)=\mu\exp\left(\int^{\theta}(M(\omega)+N(\omega))\,d\omega\right),$$

where $\mu\in\mathbb{R}$, which has the first integral

$$H(x,y)=\left(x^2+y^2\right)^{\frac{1}{2}}\exp\left(-\int^{\arctan\frac{y}{x}}(M(\omega)+N(\omega))\,d\omega\right).$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the realistic quadrant $(x>0, y>0)$, and let $h_\Gamma=H(\Gamma)$.

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}}-h\exp\left(\int^{\arctan\frac{y}{x}}(M(\omega)+N(\omega))\,d\omega\right)=0,$$

where $h\in\mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$\left(x^2+y^2\right)^{\frac{1}{2}}=h_\Gamma\exp\left(\int^{\arctan\frac{y}{x}}(M(\omega)+N(\omega))\,d\omega\right).$$

But this curve cannot contain the periodic orbit Γ, and consequently no limit cycle contained in the realistic quadrant $(x>0, y>0)$, because this curve in realistic quadrant has at most a unique point on every straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$, the abscissa is given by

$$x=\frac{h_\Gamma}{\sqrt{(1+\eta^2)}}\exp\left(\int^{\arctan\eta}(M(\omega)+N(\omega))\,d\omega\right),$$

at most a unique value of x on every half straight OX^{+}, consequently at most a unique point in realistic quadrant $(x>0,y>0)$. So this curve cannot contain the periodic orbit.

Hence statement (4) of Theorem 1 is proved.

Assume now that $f_3(\theta)=0$ for all $\theta\in\mathbb{R}$, then from system (3) it follows that $\theta'=0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as $y-hx=0$, where $h\in\mathbb{R}$, since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, consequently no limit cycle.

This completes the proof of statement (5) of Theorem 1. $\square$

3. Examples

The following example is given to illustrate our result

Example 1 If we take $A(x,y)=5x^2+4y^2$, $B(x,y)=x^2+y^2$, $P(x,y)=x^4+x^3y+2x^2y^2+xy^3+y^4$, $Q(x,y)=x^4+2x^3y+2x^2y^2+2xy^3+y^4$ and $R(x,y)=3x^2-xy+3y^2$, then system (2) reads

$$(8)\qquad \begin{cases} x'=x\left(\left(x^4+x^3y+2x^2y^2+xy^3+y^4\right)+\left(3x^2-xy+3y^2\right)\ln\left|\frac{5x^2+4y^2}{x^2+y^2}\right|\right), \\ y'=y\left(\left(x^4+2x^3y+2x^2y^2+2xy^3+y^4\right)+\left(3x^2-xy+3y^2\right)\ln\left|\frac{5x^2+4y^2}{x^2+y^2}\right|\right), \end{cases}$$

the Kolmogorov system (8) in Polar coordinates (r,θ) becomes

$$\begin{aligned} r' &= \left(1+\tfrac{3}{4}\sin 2\theta-\tfrac{1}{8}\sin 4\theta\right)r^5+(3-\cos\theta\sin\theta)\ln\left(\tfrac{9}{2}+\tfrac{1}{2}\cos 2\theta\right)r^3, \\ \theta' &= \left(\cos^2\theta\sin^2\theta\right)r^4, \end{aligned}$$

here $f_1(\theta)=1+\frac{3}{4}\sin 2\theta-\frac{1}{8}\sin 4\theta$, $f_2(\theta)=(3-\cos\theta\sin\theta)\ln\left(\frac{9}{2}+\frac{1}{2}\cos 2\theta\right)$ and $f_3(\theta)=\cos^2\theta\sin^2\theta$. In the realistic quadrant $(x>0,y>0)$ it is the case (a) of the Theorem 1, then the Kolmogorov system (8) has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)\exp\left(-2\int^{\arctan\frac{y}{x}}M(\omega)\,d\omega\right)- \\ &\quad 2\int^{\arctan\frac{y}{x}}\exp\left(-2\int^{w}M(\omega)\,d\omega\right)B(w)\,dw, \end{aligned}$$

where $M(\omega)=\dfrac{1+\frac{3}{4}\sin 2\omega-\frac{1}{8}\sin 4\omega}{\cos^2\omega\sin^2\omega}$, $N(w)=\dfrac{(3-\cos w\sin w)\ln\left(\frac{9}{2}+\frac{1}{2}\cos 2w\right)}{\cos^2 w\sin^2 w}$

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (8), in Cartesian coordinates are written as

$$\begin{aligned}x^2+y^2 &= h\exp\left(2\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)+\\ &\quad 2\exp\left(2\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\int^{\arctan\frac{y}{x}}\exp\left(-2\int^{w} N(\omega)\,d\omega\right)N(w)\,dw\end{aligned}$$

where $h\in\mathbb{R}$. The system (8) has no periodic orbits, and consequently no limit cycle.

4. Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar Kolmogorov differential systems of ODEs in order to obtain explicit expression for a first integral and characterizes its trajectories, this is a one of the classical tools in the classification of all trajectories of dynamical systems.

Conflict of Interests

The authors declare that there is no conflict of interests.

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