BIFURCATION AND CHAOS OF A PARTICLE MOTION SYSTEM WITH HOLONOMIC CONSTRAINT

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Abstract. This paper investigates a holonomic constrained system of a particle moving on a horizontal smooth plane. The equilibrium points, bifurcations and chaotic attractors of the system are analyzed. It shows that the rich dynamic behaviors of the particle motion system, including the degenerate Hopf bifurcations at multiple equilibrium points, the chaotic behaviors of the particle motion. The numerical simulations are carried out to verify theoretical analyses and to exhibit the rich chaotic behaviors.

Keywords: bifurcation; particle motion; chaotic attractor.

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1. INTRODUCTION

Some research shows that the the particle motion becomes complex because of the existence of external force, such as shear force [1] and creep force [2]. Junhong and his cooperators studied a particle motion under external force and investigated the influences of two nonlinear
nonholonomic constraints on the particle motion [3]. In this paper, we will discuss the particle motion system of Ref. [3] with holonomic constraint

\[ x^2 + y^2 = M^2 \quad (M > 0). \]

By applying Lagrange’s method, the equations of holonomic system can be obtained as follows

\[
\begin{align*}
\dot{x} &= ax - bx^2 - cxy + 2\lambda x, \\
\dot{y} &= ay - by^2 - cxy + 2\lambda y,
\end{align*}
\]

Here \( \lambda \) is Lagrange’s multiplier. Differentiate the holonomic constraint equation twice with respect to time and obtain \( \dot{x}^2 + x\ddot{x} + y^2 + y\ddot{y} = 0 \), then \( \lambda = -\frac{x_1^2 + y_1^2 + aM^2 - bx_1^3 - by_1^3 - cxy_1 - cxy_1}{2M^2} \). Thus, the equations of motion of the constrained system become

\[
\begin{align*}
\dot{x} &= x_1, \\
\dot{x}_1 &= -bx_1^2 - cxx_3 + ax_1 - P_1(x, x_1, y, y_1), \\
\dot{y} &= y_1, \\
\dot{y}_1 &= -bx_1^2 - cxx_4 + ax_3 - P_2(x, x_1, y, y_1),
\end{align*}
\]

where

\[
\begin{align*}
P_1(x, x_1, y, y_1) &= \frac{x_1((-bx_1^2 - cxx_3 + ax_1)x_1 + \frac{aM^2}{x_3})}{x_3}, \\
P_2(x, x_1, y, y_1) &= \frac{(\frac{aM^2}{x_3})}{x_3}.
\end{align*}
\]

The stability of equilibrium points, Hopf bifurcation and chaos of the constrained system are investigated as follows.

2. DYNAMIC ANALYSIS

By computations, we can obtain the equilibrium points as follows

\[
\begin{align*}
E_0 &= (0, 0, 0, 0), \quad E_1 = (0, 0, M, 0), \quad E_2 = (0, 0, -M, 0), \quad E_3 = (0, 0, \frac{a}{b}, 0), \\
E_4 &= (M, 0, 0, 0), \quad E_5 = (-M, 0, 0, 0), \quad E_6 = (\frac{a}{b}, 0, 0, 0), \quad E_7 = (\frac{a}{b}, 0, \frac{a}{b}, 0), \\
E_8 &= (-\frac{M}{\sqrt{2}}, 0, -\frac{M}{\sqrt{2}}, 0), \quad E_9 = (\frac{M}{\sqrt{2}}, 0, \frac{M}{\sqrt{2}}, 0).
\end{align*}
\]

The characteristic equations at equilibrium points are obtained as follows

\[
\begin{align*}
f_{E_0}(\lambda) &= \lambda^4 - a\lambda^2 - \lambda^3 + a\lambda, \\
f_{E_1}(\lambda) &= \lambda^4 + (aM - 1)\lambda^3 + (-bM - cM)\lambda^2 + bM\lambda, \\
f_{E_2}(\lambda) &= \lambda^4 + (-cM - 1)\lambda^3 + (bM + cM)\lambda^2 - bM\lambda,
\end{align*}
\]
Then, the system (2) becomes

\[
\begin{align*}
  f_{E_1}(\lambda) &= \lambda^4 + \frac{(ca-b)\lambda^3}{b} - \frac{a(b+c)\lambda^2}{b} + a\lambda, \\
  f_{E_4}(\lambda) &= \lambda^4 + (cM - 1)\lambda^3 + (-2bM - cM + 2a)\lambda^2 + (-2M^2bc + 2Mac + 2bM - 2a)\lambda \\
  &\quad + 2cM(bM - a), \\
  f_{E_5}(\lambda) &= \lambda^4 + (-cM - 1)\lambda^3 + (2bM + cM + 2a)\lambda^2 + (-2M^2bc - 2Mac - 2bM - 2a)\lambda \\
  &\quad + 2cM(bM + a), \\
  f_{E_6}(\lambda) &= \lambda^4 + \frac{(ca-b)\lambda^3}{b} + \frac{a(bM^2 - M^2bc - a^2)\lambda^2}{b^2M^2} + \frac{a(b^2M^2 - a^2)(ca-b)\lambda}{b^3M^2} - \frac{ca^2(b^2M^2 - a^2)}{b^3M^4}, \\
  f_{E_7}(\lambda) &= \lambda^4 - \frac{(ca+b)\lambda^3}{b} - \frac{a(3b^2M^2 - M^2bc - 7a^2)\lambda^2}{b^3M^2} + \frac{a(3b^2M^2 - 7a^2)(ca+b)\lambda}{b^3M^2} - \frac{ca^2(3b^2M^2 - 7a^2)}{b^3M^4}, \\
  f_{E_8}(\lambda) &= \lambda^4 + (cM\sqrt{2}/2 - 1)\lambda^3 + (-cM\sqrt{2}/2 - bM\sqrt{2}/4 + a)\lambda^2 + (-M^2bc/2 \\
  &\quad + M\sqrt{2}ac/2 + bM\sqrt{2}/4 - a)\lambda + M^2bc/2 - 1/2M\sqrt{2}ac, \\
  f_{E_9}(\lambda) &= \lambda^4 + (-cM\sqrt{2}/2 - 1)\lambda^3 + (cM\sqrt{2}/2 + bM\sqrt{2}/4 + a)\lambda^2 + (-M^2bc/2 \\
  &\quad - M\sqrt{2}ac/2 - bM\sqrt{2}/4 - a)\lambda + M^2bc/2 + M\sqrt{2}ac/2.
\end{align*}
\]

From the expression of \( f_{E_6}(\lambda) \), we can obtain the eigenvalues are \( \pm \frac{\sqrt{ab^2M^2-a^3}}{bM}i, 1, \frac{ac}{b} \), where \( \frac{a}{M^2} < 1 \). In this case, the system occurs Hopf bifurcation at the equilibrium \( E_6 \). For \( E_6 \), let

\[
\begin{bmatrix}
  x - \frac{a}{b} \\
  x_1 \\
  y \\
  y_1
\end{bmatrix} =
\begin{bmatrix}
  0 & \frac{bM}{\sqrt{ab^2M^2-a^3}} & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & \frac{ca+b}{ba} \\
  0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4
\end{bmatrix},
\]

Then, the system (2) becomes

\[
\begin{align*}
  \dot{u}_1 &= -\frac{\sqrt{ab^2M^2-a^3}}{bM}u_2 + G_1(u_1, u_2, u_3, u_4), \\
  \dot{u}_2 &= \frac{\sqrt{ab^2M^2-a^3}}{bM}u_1 + G_2(u_1, u_2, u_3, u_4), \\
  \dot{u}_3 &= -cMu_3 + G_3(u_1, u_2, u_3, u_4), \\
  \dot{u}_4 &= u_4 + U_4(u_1, u_2, u_3, u_4),
\end{align*}
\]

where

\[
G_1(u_1, u_2, u_3, u_4) = \frac{b^3M^2u_2^2}{M^2ab^2-a^3} - \frac{cu_1(ac+b)u_3}{ba} + \frac{abMu_2}{\sqrt{M^2ab^2-a^3}} - \frac{bu_2}{M\sqrt{M^2ab^2-a^3}} + \frac{bMu_2}{\sqrt{M^2ab^2-a^3}} - \frac{bu_3}{\sqrt{M^2ab^2-a^3}}.
\]

\[
G_2(u_1, u_2, u_3, u_4) = \frac{b^3M^2u_2^2}{M^2ab^2-a^3} - \frac{cu_1(ac+b)u_3}{ba} + \frac{abMu_2}{\sqrt{M^2ab^2-a^3}} - \frac{bu_2}{M\sqrt{M^2ab^2-a^3}} + \frac{bMu_2}{\sqrt{M^2ab^2-a^3}} - \frac{bu_3}{\sqrt{M^2ab^2-a^3}}.
\]
G_2(u_1, u_2, u_3, u_4) = 0,

G_3(u_1, u_2, u_3, u_4) = cM^3 - \frac{(ac + b)^2 u_4^2}{ba^2} - \frac{cbMu_2(u_3 + u_4)}{\sqrt{M^2 ab^2 - a^2}} + \frac{(ac + b)u_4}{b} - \frac{(ac + b)u_4}{abM^2} \sqrt{M^2 ab^2 - a^2} \\
\times (- \frac{b^3 M^2 u_2^2}{M^2 ab^2 - a^2} - \frac{cu_1 (ac + b)u_4}{ba} + \frac{abMu_2}{\sqrt{M^2 ab^2 - a^2}}) + \frac{(ac + b)u_4}{ba} (\frac{(ac + b)^2 u_2^2}{ba^2} - \frac{cbMu_2(u_3 + u_4)}{\sqrt{M^2 ab^2 - a^2}})

G_4(u_1, u_2, u_3, u_4) = 0.

Furthermore,

\begin{align*}
g_{11} & = \frac{1}{4} \left(- \frac{2b^3 M^2}{M^2 ab^2 - a^2}\right), \\
g_{02} & = \frac{1}{4} \left(\frac{2b^3 M^2}{M^2 ab^2 - a^2}\right), \\
g_{20} & = \frac{1}{4} \left(\frac{2b^3 M^2}{M^2 ab^2 - a^2}\right), \\
g_{21} & = \frac{1}{8} \left(\frac{6b^3 Ma}{(M^2 ab^2 - a^2)^{\frac{3}{2}}} + \frac{2b}{M^2 ab^2 - a^2}\right),
\end{align*}

thus

\begin{align*}
\text{Re}_1(0) = & \text{Re}(\frac{Mbi}{\sqrt{M^2 ab^2 - a^2}} (g_{20} g_{11} - 2|g_{11}|^2 - \frac{1}{3} |g_{02}|^2 + \frac{1}{2} g_{21})) = \frac{1}{8} \frac{b^2}{M^2 ab^2 - a^2} > 0.
\end{align*}

Based on the above analysis and the theorem in [4], we can obtain the conclusion as follows.

**Theorem 1.** The system (2) undergoes degenerate Hopf bifurcations at \(E_6\) and the bifurcating periodic solution is unstable.

According to Routh-Hurwitz criteria, \(E_0, E_1, E_5, E_7\) and \(E_9\) are unstable points. For the other equilibria, we can obtain the same conclusions using the method in [4] when the Hopf bifurcations occur.

### 3. Chaos and Simulations

The above results show that the particle motion system has complex dynamic behaviors. In this section, we give some simulations to study the particle motion and select the parameters \(a = 5, c = 0.001, b = 1\), and initial values \((x_0, x_{10}, y_0, y_{10}) = (0.1, 0.001, 0.1, 0.1)\). In this case, the system (2) has ten equilibrium points. Table 1 indicates the eigenvalues of corresponding Jacobian matrix and the equilibria type and shows the unstable manifold and stable manifold at the equilibrium points of the particle motion system when \(M = 0.2\).

In chaos theory, the equilibrium points of the system are of great importance to understand its nonlinear dynamics [5]. It has been long supposed that the existence of chaotic behaviour in the microscopic motions is responsible for their equilibrium and nonequilibrium properties
and the interconversion of the stable manifolds and the unstable manifolds which can cause complicated dynamics in the system (2).[7-8] The Lyapunov exponents are 0, -0.2 -0.3 and 0.6 using the method in [7], thus the system (2) is chaotic. Fig 1. shows the particle motion trajectory and the chaotic attractor of the system. The poincare maps and chaotic attractors in $x - y$ plane and $x_1 - y_1$ plane are given in Fig. 1-6. If the constrained parameter $M = 1$, the particle motion system also occur chaotic phenomena. Fig. 7-8 show the chaotic attractor in $x - y$ plane and $x - x_1$ plane.

**TABLE 1.** The eigenvalues of corresponding Jacobian matrix and the equilibria type.

<table>
<thead>
<tr>
<th>equilibrium points</th>
<th>eigenvalues of Jacobian matrix</th>
<th>equilibria type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0,0,0)</td>
<td>0, 1,2.2361, -2.2361</td>
<td>unstable equilibrium point</td>
</tr>
<tr>
<td>(0,0,0.2,0)</td>
<td>0, 1, 0.44731, -0.44731</td>
<td>unstable equilibrium point</td>
</tr>
<tr>
<td>(0,0,-0.2,0)</td>
<td>0, 1, ±0.44721i</td>
<td>Hopf bifurcation</td>
</tr>
<tr>
<td>(0,0,5,0)</td>
<td>0, 1, -2.23857, 2.23357</td>
<td>unstable equilibrium point</td>
</tr>
<tr>
<td>(0.2,0,0,0)</td>
<td>1, 0.0024, 0.0612±3.9763i</td>
<td>unstable equilibrium point</td>
</tr>
<tr>
<td>(-0.2,0,0,0)</td>
<td>1, 0.0026, -0.0637±3.22442i</td>
<td>unstable equilibrium point</td>
</tr>
<tr>
<td>(5,0,0,0)</td>
<td>1, -0.005, 55.8569, -55.8569</td>
<td>unstable equilibrium point</td>
</tr>
<tr>
<td>(5,0.5,0)</td>
<td>1, 59.069, -0.01, -52.819</td>
<td>unstable equilibrium point</td>
</tr>
<tr>
<td>($(-\frac{0.2}{\sqrt{2}},0,-\frac{0.2}{\sqrt{2}},0)$</td>
<td>1, 0.0001,±2.25182i</td>
<td>Hopf bifurcation</td>
</tr>
<tr>
<td>($\frac{0.2}{\sqrt{2}},0,\frac{0.2}{\sqrt{2}},0$)</td>
<td>1, -0.000139, 0.00000001±2.2201i</td>
<td>unstable equilibrium point</td>
</tr>
</tbody>
</table>
4. CONCLUSION
The results show that the rich dynamic behaviors of the particle motion system, including Hopf bifurcations, interconversion of the stable manifolds and the unstable manifolds at multiple equilibrium points and chaotic attractors. Thus, the particle motion trajectory has complex dynamic behaviors under the holonomic constraint.

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Conflict of Interests
The authors declare that there is no conflict of interests.

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