SOME RESULTS FOR STRONG DIFFERENTIAL SUBORDINATION OF
ANALYTIC FUNCTIONS DEFINED BY DIFFERENTIAL OPERATOR

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Abstract: The purpose of this paper is to introduce some results associated with the strong differential subordinations of analytic functions defined in the open unit disk and closed unit disk of the complex plane.

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1. INTRODUCTION

Denote by \( \mathcal{H}(U \times \overline{U}) \) the family of all analytic functions in \( U \times \overline{U} \). Let \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \overline{U} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) indicate the open unit disk and the closed unit disk of the complex plane, respectively. For \( n \in \mathbb{N} = \{1, 2, \ldots \} \) and \( a \in \mathbb{C} \), let \( \mathcal{H}^*[a, n, \zeta] = \{ f \in \mathcal{H}(U \times \overline{U}) : f(z, \zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \ldots, z \in U, \zeta \in \overline{U} \} \), where \( a_k(\zeta) \) are holomorphic functions in \( \overline{U} \) for \( k \geq n \).

Also, let \( \mathcal{A}_{n, \zeta}^* = \{ f \in \mathcal{H}(U \times \overline{U}) : f(z, \zeta) = z + a_{n+1}(\zeta)z^{n+1} + \ldots, z \in U, \zeta \in \overline{U} \} \), where \( a_k(\zeta) \) are holomorphic functions in \( \overline{U} \) for \( k \geq n + 1 \).

A function \( f \in \mathcal{H}^*[a, n, \zeta] \) is said to be starlike in \( U \times \overline{U} \) if
\[ \text{Re} \left\{ \frac{zf_z'(z, \zeta)}{f(z, \zeta)} \right\} > 0, \quad (z \in U, \zeta \in \overline{U}). \]

Denote the class of all starlike functions in \( U \times \overline{U} \) by \( S_\zeta^* \).

Similar, \( f \in \mathcal{H}^*[a, n, \zeta] \) is said to be convex in \( U \times \overline{U} \) if
\[
\text{Re} \left\{ \frac{zf_z''(z, \zeta)}{f_z'(z, \zeta)} + 1 \right\} > 0, \quad (z \in U, \zeta \in \overline{U}).
\]

Denote the class of all convex functions in \( U \times \overline{U} \) by \( K_\zeta^* \).

**Definition 1.1 (Oros and Gh Oros, [9])** Let \( f(z, \zeta), g(z, \zeta) \) analytic in \( U \times \overline{U} \). The function \( f(z, \zeta) \) is said to be strongly subordinate to \( g(z, \zeta) \), written \( f(z, \zeta) \prec \prec F(z, \zeta) \), \( z \in U, \zeta \in \overline{U} \), if there exists an analytic function \( w \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1, z \in U \) such that
\[
f(z, \zeta) = g(w(z), \zeta) \quad \text{for all } \zeta \in \overline{U}.
\]

**Remark 1.1 (Oros and Gh Oros, [9])**

1) Since \( f(z, \zeta) \) is analytic in \( U \times \overline{U} \), for all \( \zeta \in \overline{U} \) and univalent in \( U \), for all \( \zeta \in \overline{U} \), Definition 1.1 is equivalent to \( f(0, \zeta) = g(0, \zeta) \) for all \( \zeta \in \overline{U} \) and \( f(U \times \overline{U}) \subset g(U \times \overline{U}) \).

2) If \( f(z, \zeta) = f(z) \) and \( g(z, \zeta) = g(z) \), the strong subordination becomes the usual notion of subordination.

Let \( A_\zeta^* \) denote the subclass of the functions \( f(z, \zeta) \in \mathcal{H}(U \times \overline{U}) \) of the form:
\[
f(z, \zeta) = z + \sum_{k=2}^{\infty} a_k(\zeta)z^k, \quad z \in U, \zeta \in \overline{U}
\]
which are analytic and univalent in \( U \times \overline{U} \).

For \( \eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \alpha, \tau \geq 0, \mu, \lambda, \beta > 0 \) and \( \alpha \neq \lambda \), we consider the differential operator
\[
A_{\mu, \lambda, \tau}^\eta(\alpha, \beta) : A^*_\zeta \rightarrow A^*_\zeta, \text{ introduced by Amourah and Darus [2], where}
\]
\[
A_{\mu, \lambda, \tau}^\eta(\alpha, \beta)f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + \frac{(n-1)(\lambda - \alpha)\beta + \tau}{\mu + \lambda} \right] \eta a_k(\zeta)z^k. \quad (1.2)
\]
It is readily verified from (1.2) that
\[
z \left( A_{\mu, \lambda, \tau}^\eta(\alpha, \beta)f(z, \zeta) \right)_z'
\]
Some of the special cases of the operator defined by (1.2) can be found in [1,4,10,11].

In recent years, many authors obtained various interesting results associated with strong differential subordination and superordination for example (see [3,5,6,12,13,14]).

In order to derive our main results, we need the following Lemmas.

**Lemma 1.1 (Miller and Mocanu, [8])** Let \( h(z, \zeta) \) be a convex function with \( h(0, \zeta) = a \), for every \( \zeta \in \overline{U} \) and let \( \gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) with \( Re(\gamma) \geq 0 \). If \( p \in \mathcal{H}^*[a, n, \zeta] \) and

\[
p(z, \zeta) + \frac{1}{\gamma} zp'_z(z, \zeta) \ll h(z, \zeta), \quad (z \in U, \zeta \in \overline{U}),
\]

then

\[
p(z, \zeta) \ll q(z, \zeta) \ll h(z, \zeta), \quad (z \in U, \zeta \in \overline{U}),
\]

where \( q(z, \zeta) = \frac{\gamma}{n\pi} \int_0^\zeta t^{n-1} h(t, \zeta) dt \) is convex and it is the best dominant of (1.4).

**Lemma 1.2 (Miller and Mocanu, [7])** Let \( q(z, \zeta) \) be a convex function in \( U \times \overline{U} \) for all \( \zeta \in \overline{U} \) and let \( h(z, \zeta) = q(z, \zeta) + n\delta zq'_z(z, \zeta) \), \( z \in U, \zeta \in \overline{U} \), where \( \delta > 0 \) and \( n \) is a positive integer. If

\[
p(z, \zeta) = q(0, \zeta) + p_n(\zeta)z^n + p_{n+1}(\zeta)z^{n+1} + \ldots,
\]

is analytic in \( U \times \overline{U} \) and

\[
p(z, \zeta) + \delta zp'_z(z, \zeta) \ll h(z, \zeta), \quad (z \in U, \zeta \in \overline{U}),
\]

then

\[
p(z, \zeta) \ll q(z, \zeta), \quad (z \in U, \zeta \in \overline{U}),
\]

and this result is sharp.

### 2. Main Result

**Theorem 2.1.** Let \( h(z, \zeta) \) be a convex function such that \( h(0, \zeta) = 1 \). If \( f \in \mathcal{A}^*_\zeta \) satisfies the strong differential subordination:

\[
\left(A_{\mu,\lambda,\tau}^{\eta}(\alpha, \beta)f(z, \zeta)\right)_z' \ll h(z, \zeta),
\]

where

\[
A_{\mu,\lambda,\tau}^{\eta}(\alpha, \beta)f(z, \zeta) = \frac{\mu + \lambda}{(\lambda - \alpha)\beta + \pi} A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) - \left(1 - \frac{\mu + \lambda}{(\lambda - \alpha)\beta + \pi}\right) A^{\eta}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta).
\]
then
\[ \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{z} \ll q(z, \zeta) \ll h(z, \zeta), \]
where \( q(z, \zeta) = \frac{1}{z} \int_0^z h(t, \zeta)dt \) is convex and it is the best dominant.

**Proof.** Suppose that
\[ p(z, \zeta) = \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{z}, \quad z \in U, \zeta \in \overline{U}. \]  
(2.2)
Then the function \( p(z, \zeta) \) is analytic in \( U \times \overline{U} \) and \( p(0, \zeta) = 1 \).

Simple computations from (2.2), we get
\[ p(z, \zeta) + zp_z'(z, \zeta) = \left( A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \right)'_z. \]  
(2.3)
Using (2.3), (2.1) becomes
\[ p(z, \zeta) + zp_z'(z, \zeta) \ll h(z, \zeta). \]

An application of Lemma 1.1 with \( n = 1, \gamma = 1 \) yields
\[ \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{z} \ll q(z, \zeta) = \frac{1}{z} \int_0^z h(t, \zeta)dt \ll h(z, \zeta). \]

By taking \( h(z, \zeta) = \frac{\zeta + (2\rho - \zeta)z}{1 + z}, 0 \leq \rho < 1 \) in Theorem 2.1, we obtain the following corollary:

**Corollary 2.1.** If \( f \in \mathcal{A}^{*}_{\zeta} \) satisfies the strong differential subordination:
\[ \left( A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \right)'_z \ll \frac{\zeta + (2\rho - \zeta)z}{1 + z}, \]
then
\[ \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{z} \ll \frac{1}{z} \int_0^z \frac{\zeta + (2\rho - \zeta)t}{1 + t}dt = 2\rho - \zeta + \frac{2(\zeta - \rho)}{z} \ln(1 + z). \]

**Theorem 2.2.** Let \( q(z, \zeta) \) be a convex function such that \( q(0, \zeta) = 1 \) and let \( h \) be the function \( h(z, \zeta) = q(z, \zeta) + zq_z'(z, \zeta) \). If \( f \in \mathcal{A}^{*}_{\zeta} \) satisfies the strong differential subordination:
\[ \left( \frac{z A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)} \right)'_z \ll h(z, \zeta), \]  
(2.4)
then
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\[
\frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)} \ll q(z, \zeta).
\]

**Proof.** Suppose that

\[
p(z, \zeta) = \frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}, \quad z \in U, \zeta \in \bar{U}.
\] (2.5)

Then the function \( p(z, \zeta) \) is analytic in \( U \times \bar{U} \) and \( p(0, \zeta) = 1 \).

Differentiating both sides of (2.5) with respect to \( z \) and using (2.4), we have

\[
p(z, \zeta) + zp'_z(z, \zeta)
= \frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}
+ \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)} \bigg( A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \bigg)'_z - A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \bigg( A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \bigg)'_z
+ \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)} \bigg( z A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \bigg)'_z - z A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \bigg( A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \bigg)'_z
= \left( \frac{z A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)} \right)'_z \ll h(z, \zeta).
\]

An application of Lemma 1.2, we obtain

\[
\frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)} \ll q(z, \zeta).
\]

**Theorem 2.3.** Let \( h(z, \zeta) \) be a convex function such that \( h(0, \zeta) = 1 \). If \( 0 \leq \sigma < p, \theta \in \mathbb{C} \)
and \( f \in \mathcal{A}^\eta_{\zeta} \) satisfies the strong differential subordination:

\[
\frac{1 - \theta}{1 - \sigma} \left( \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{z} - \sigma \right) + \frac{\theta}{1 - \sigma} \left( \left( A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta) \right)'_z - \sigma \right) \ll h(z, \zeta),
\] (2.6)

then

\[
\frac{1}{1 - \sigma} \left( \frac{A^\eta_{\mu,\lambda,\tau}(\alpha, \beta)f(z, \zeta)}{z} - \sigma \right) \ll q(z, \zeta) \ll h(z, \zeta),
\]
where \( q(z, \zeta) = \frac{1}{\theta} z^{-\frac{1}{\theta}} \int_0^z t^{\frac{1}{\theta}-1} h(t, \zeta) dt \) is convex and it is the best dominant.

**Proof.** Suppose that

\[
p(z, \zeta) = \frac{1}{1 - \sigma} \left( \frac{A^\eta_{\mu, \lambda, \tau}(\alpha, \beta) f(z, \zeta)}{z} - \sigma \right), \quad z \in U, \zeta \in \overline{U}.
\]

Then the function \( p(z, \zeta) \) is analytic in \( U \times \overline{U} \) and \( p(0, \zeta) = 1 \).

Differentiating both sides of (2.7) with respect to \( z \), we have

\[
p(z, \zeta) + \theta z p'_z(z, \zeta) = \frac{1 - \theta}{1 - \sigma} \left( \frac{A^\eta_{\mu, \lambda, \tau}(\alpha, \beta) f(z, \zeta)}{z} - \sigma \right) + \frac{\theta}{1 - \sigma} \left( \left( A^\eta_{\mu, \lambda, \tau}(\alpha, \beta) f(z, \zeta) \right)' - \sigma \right).
\]

From (2.6) and (2.8), we get

\[
p(z, \zeta) + \theta z p'_z(z, \zeta) \ll h(z, \zeta).
\]

An application of Lemma 1.1 with \( n = 1, \gamma = \frac{1}{\theta} \) yields

\[
\frac{1}{1 - \sigma} \left( \frac{A^\eta_{\mu, \lambda, \tau}(\alpha, \beta) f(z, \zeta)}{z} - \sigma \right) \ll q(z, \zeta) = \frac{1}{\theta} z^{-\frac{1}{\theta}} \int_0^z t^{\frac{1}{\theta}-1} h(t, \zeta) dt \ll h(z, \zeta).
\]

**Theorem 2.4.** Let \( q(z, \zeta) \) be a convex function such that \( q(0, \zeta) = 1 \) and let \( h \) be the function \( h(z, \zeta) = q(z, \zeta) + \frac{1}{1 + (\lambda - \alpha) \beta + n \tau} z q'_z(z, \zeta) \), where \( \frac{\mu + \lambda}{(\lambda - \alpha) \beta + n \tau} > 0 \). Suppose that

\[
F(z, \zeta) = \frac{1 + \frac{\mu + \lambda}{(\lambda - \alpha) \beta + n \tau}}{z^{\frac{\mu + \lambda}{(\lambda - \alpha) \beta + n \tau}}} \int_0^z t^{\frac{\mu + \lambda}{(\lambda - \alpha) \beta + n \tau}-1} f(t, \zeta) dt, \quad z \in U, \zeta \in \overline{U}.
\]

If \( f \in \mathcal{A}^*_\zeta(p) \) satisfies the strong differential subordination

\[
\left( A^\eta_{\mu, \lambda, \tau}(\alpha, \beta) f(z, \zeta) \right)'_z \ll h(z, \zeta),
\]

then

\[
\left( A^\eta_{\mu, \lambda, \tau}(\alpha, \beta) F(z, \zeta) \right)'_z \ll q(z, \zeta).
\]

**Proof.** Suppose that

\[
p(z, \zeta) = \left( A^\eta_{\mu, \lambda, \tau}(\alpha, \beta) F(z, \zeta) \right)'_z, \quad z \in U, \zeta \in \overline{U}.
\]
Then the function \( p(z, \zeta) \) is analytic in \( U \times \bar{U} \) and \( p(0, \zeta) = 1 \).

From (2.9), we have

\[
z^{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}} F(z, \zeta) = \left(1 + \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau}\right) \int_0^z t^{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}-1} f(t, \zeta) dt .
\] (2.12)

Differentiating both sides of (2.12) with respect to \( z \), we get

\[
\left(1 + \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau}\right) f(z, \zeta) = \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau} F(z, \zeta) + zf'_z(z, \zeta)
\]

and

\[
\left(1 + \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau}\right) A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) f(z, \zeta)
\]

\[
= \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau} A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) F(z, \zeta) + z \left( A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) F(z, \zeta) \right)' .
\]

So

\[
\left( A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) f(z, \zeta) \right)' = \left( A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) F(z, \zeta) \right)' + \frac{z \left( A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) F(z, \zeta) \right)''}{1 + \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau}} .
\] (2.13)

From (2.11) and (2.13), we obtain

\[
p(z, \zeta) + \frac{1}{1 + \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau}} z p'_z(z, \zeta) = \left( A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) f(z, \zeta) \right)' .
\] (2.14)

Using (2.14), (2.10) becomes

\[
p(z, \zeta) + \frac{1}{1 + \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau}} z p'_z(z, \zeta) \ll q(z, \zeta) + \frac{1}{1 + \frac{\mu + \lambda}{(\lambda-\alpha)\beta+n\tau}} z q'_z(z, \zeta).
\]

An application of Lemma 1.2 yields \( p(z, \zeta) \ll q(z, \zeta) \). By using (2.10), we obtain

\[
\left( A_{\mu,\lambda,\tau}^\eta(\alpha, \beta) F(z, \zeta) \right)' \ll q(z, \zeta).
\]

**Conflict of Interests**

The authors declare that there is no conflict of interests.
REFERENCES


