Available online at http://scik.org

Engineering Mathematics Letters, 1 (2012), No. 1, 44-57

ISSN 2049-9337

CONVERGENCE OF ITERATIVE ALGORITHMS FOR A GENERALIZED VARIATIONAL INEQUALITY AND A NONEXPANSIVE MAPPING

SONGTAO LV[1], AND CHANGQUN WU[2,*]

[1]Department of Mathematics, Shangqiu Normal University, Shangqiu 476000, China

[2]School of Business and Administration, Henan University, Kaifeng 475000, China

Abstract. In this paper, fixed point problems of nonexpansive mappings and solution problems of generalized variational inequalities are investigated based on a viscosity approximate iterative algorithm. Strong convergence theorems for common elements which lie in the fixed point set and in the solution set are established in the framework of Hilbert spaces.

Keywords: nonexpansive mapping; fixed point; relaxed cocoercive mapping; variational inequality.

2000 AMS Subject Classification: 47H05, 47H09, 47J25

1. Introduction

Variational inequalities introduced in the early seventies have witnessed an explosive growth in theoretical advances, algorithmic development and applications across all the discipline of pure and applied sciences; see [1-24] and the references therein. It combines novel theoretical and algorithmic advances with new domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. Now, we have a variety of techniques to suggest and analyze various numerical methods including projection technique and its variant forms, auxiliary

*Corresponding author

Received May 3, 2012

principle, Wiener-Hopf equations and so on. Recently, some classes of generalized variational inequalities involving two or three three nonlinear operators has been studied by many authors; see [1-6] and the references therein. The generalized variational inequalities are useful and important extension and generalizations of the variational inequalities with a wide range of applications in industry, mathematical finance, economics, decision sciences, ecology, mathematical and engineering sciences. For solution problems of the generalized variational inequalities, projection methods which link solution problem of variational inequalities and fixed point problems of nonlinear operators are efficient and popular. Viscosity approximation method which was first introduced by Moudafi [18] has been studied iterative solutions of variational inequalities and fixed points of nonexpansive mappings. In this paper, fixed point problems of a nonexpansive mapping and solution problems of a generalized variational inequality are investigated based on a composite approximate iterative algorithm. Strong convergence theorems for common elements which lie in the fixed point set and in the solution set are established in the framework of Hilbert spaces.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed and convex subset of H and $A : C \to H$ a nonlinear mapping. Recall the following definitions:

(a) A is said to be *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

(b) A is said to be *ν-strongly monotone* if there exists a positive real number $\nu > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \nu \|x - y\|^2, \quad \forall x, y \in C.$$

(c) A is said to be *relaxed μ-cocoercive* if there exists a positive real number $\mu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(d) A is said to be *relaxed (μ, ν)-cocoercive* if there exist positive real numbers $\mu, \nu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)\|Ax - Ay\|^2 + \nu\|x - y\|^2, \quad \forall x, y \in C.$$

Next, we consider the following generalized variational inequality problem. Give nonlinear mappings $T_1 : C \to H$ and $T_2 : C \to H$, find an $u \in C$ such that

$$\langle u - \lambda_1 T_1 u + \lambda T_2 u, v - u\rangle \geq 0, \quad \forall v \in C, \tag{2.1}$$

where λ_1, and λ_2 are constants. In this paper, we use $VI(C, T_1, T_2)$ to denote the solution set of the variational inequality problem (2.1).

It is easy to see that an element $u \in C$ is a solution to the problem (2.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(\lambda_1 T_1 - \lambda T_2)$, where P_C denotes the metric projection from H onto C.

If $T_1 = I$, the identity mapping, and $\lambda_1 = 1$, then the problem (2.1) is reduced to the following. Find $u \in C$ such that

$$\langle T_2 u, v - u\rangle \geq 0, \quad \forall v \in C. \tag{2.2}$$

The variational inequality (2.2) was introduced by Stampacchia [23] in 1964. The problem (2.2) has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences. In this paper, we use $VI(C, T_2)$ to denote the solution set of the variational inequality problem (2.2).

Let $S : C \to C$ be a mapping. Recall that S is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(S)$ to denote the fixed point set of the mapping S.

Recently, many authors studied the problem of approximating a common element in the solution set of variational inequalities (2.1), (2.2), and in the fixed point set of nonexpansive mappings. Motivated by the research work going on in this direction, fixed point problems of nonexpansive mappings and solution problems of generalized variational inequalities are investigated based on a composite approximate iterative algorithm in this paper. Strong convergence theorems for common elements which lie in the fixed point set and in the solution set are established in the framework of Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 ([25]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $S_1 : C \to C$ and $S_2 : C \to C$ be nonexpansive mappings. Suppose that $F(S_1) \cap F(S_2)$ is nonempty. Define a mapping $S : C \to C$ by*

$$Sx = aS_1x + (1-a)S_2x, \quad \forall x \in C.$$

Then S is nonexpansive with $F(S) = F(S_1) \cap F(S_2)$.

Lemma 2.2 ([26]). *Let C be a nonempty closed and convex subset of a real Hilbert space H and $S : C \to C$ a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 2.3 ([27]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H and let $\{\beta_n\}$ be a sequence in $(0,1)$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.4 ([28]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with the contractive constant κ, and $S : C \to C$ a nonexpansive mapping with fixed points. Let $T_{(m,1)} : C \to H$ be a relaxed $(\mu_{(m,1)}, \nu_{(m,1)})$-cocoercive and $L_{(m,1)}$-Lipschitz continuous mapping and $T_{(m,2)} : C \to H$ be a relaxed $(\mu_{(m,2)}, \nu_{(m,2)})$-cocoercive and $L_{(m,2)}$-Lipschitz continuous mapping for each positive integer $m \in [1, N]$, where $N \ge m$ is some positive integer. Assume that $\mathcal{F} = \cap_{m=1}^{N} GVI(C, T_{(m,1)}, T_{(m,2)}) \bigcap F(S) \ne \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm*

$$\begin{cases} x_1 \in C, \\ y_n = \delta_n S x_n + (1-\delta_n) \sum_{m=1}^{N} \eta_{(m,n)} P_C(\lambda_{(m,1)} T_{(m,1)} x_n - \lambda_{(m,2)} T_{(m,2)} x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \ge 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and $\{\eta_{(m,n)}\}$ are sequences in $(0,1)$ satisfying the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \ge 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(d) $\sum_{m=1}^{N} \eta_{(m,n)} = 1$, *and* $\lim_{n\to\infty} \eta_{(m,n)} = \eta_m \in (0,1)$;
(e) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$,

and $\{\lambda_{(m,1)}\}$ and $\{\lambda_{(m,2)}\}$ are sequences such that

(f) $\sqrt{1 - 2\lambda_{(m,1)}\nu_{(m,1)} + 2\lambda_{(m,1)}\mu_{(m,1)} L_{(m,1)}^2 + \lambda_{(m,1)}^2 L_{(m,1)}^2}$
$+ \sqrt{1 - 2\lambda_{(m,2)}\nu_{(m,2)} + 2\lambda_{(m,2)}\mu_{(m,2)} L_{(m,2)}^2 + \lambda_{(m,2)}^2 L_{(m,2)}^2} \le 1.$

Then the sequence $\{x_n\}$ generated by the algorithm converges strongly to $\bar{x}$, where $\bar{x} \in \mathcal{F}$, and solves the following variational inequality: find some point y such that

$$\langle f(y) - y, y - x \rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

Proof. First, we prove that the mapping $P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})$ is nonexpansive for each $m \in \{1, 2, \ldots, N\}$. For each $x, y \in C$, we have

$$\begin{aligned}
&\|P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})x - P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})y\| \\
&\leq \|(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})x - (\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})y\| \\
&\leq \|(x-y) - \lambda_{(m,1)}(T_{(m,1)}x - T_{(m,1)}y)\| + \|(x-y) - \lambda_{(m,2)}(T_{(m,2)}x - T_{(m,2)}y)\|.
\end{aligned} \tag{3.1}$$

It follows from the assumption that $T_{(m,1)} : C \to H$ is relaxed $(\mu_{(m,1)}, \nu_{(m,1)})$-cocoercive and $L_{(m,1)}$-Lipschitz that

$$\begin{aligned}
&\|(x-y) - \lambda_{(m,1)}(T_{(m,1)}x - T_{(m,1)}y)\|^2 \\
&= \|x-y\|^2 - 2\lambda_{(m,1)}\langle T_{(m,1)}x - T_{(m,1)}y, x-y\rangle + \lambda_{(m,1)}^2\|T_{(m,1)}x - T_{(m,1)}y\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_{(m,1)}(-\mu_{(m,1)}\|T_1x - T_1y\|^2 + \nu_{(m,1)}\|x-y\|^2) + \lambda_{(m,1)}^2 L_{(m,1)}^2\|x-y\|^2 \\
&\leq \theta_{(m,1)}^2\|x-y\|^2,
\end{aligned}$$

where $\theta_{(m,1)} = \sqrt{1 - 2\lambda_{(m,1)}\nu_{(m,1)} + 2\lambda_{(m,1)}\mu_{(m,1)}L_{(m,1)}^2 + \lambda_{(m,1)}^2 L_{(m,1)}^2}$. That is,

$$\|(x-y) - \lambda_{(m,1)}(T_{(m,1)}x - T_{(m,1)}y)\| \leq \theta_{(m,1)}\|x-y\|. \tag{3.2}$$

On the other hand, by the the assumption that $T_{(m,2)} : C \to H$ is relaxed $(\mu_{(m,2)}, \nu_{(m,2)})$-cocoercive and $L_{(m,2)}$-Lipschitz, we arrive at

$$\begin{aligned}
&\|(x-y) - \lambda_{(m,2)}(T_{(m,2)}x - T_{(m,2)}y)\|^2 \\
&= \|x-y\|^2 - 2\lambda_{(m,2)}\langle T_{(m,2)}x - T_{(m,2)}y, x-y\rangle + \lambda_{(m,2)}^2\|T_{(m,2)}x - T_{(m,2)}y\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_{(m,2)}(-\mu_{(m,2)}\|T_2x - T_2y\|^2 + \nu_{(m,2)}\|x-y\|^2) + \lambda_{(m,2)}^2 L_{(m,2)}^2\|x-y\|^2 \\
&\leq \theta_{(m,2)}^2\|x-y\|^2,
\end{aligned}$$

where $\theta_{(m,2)} = \sqrt{1 - 2\lambda_{(m,2)}\nu_{(m,2)} + 2\lambda_{(m,2)}\mu_{(m,2)}L^2_{(m,2)} + \lambda^2_{(m,2)}L^2_{(m,2)}}$. This implies that

$$\|(x-y) - \lambda_{(m,2)}(T_{(m,2)}x - T_{(m,2)}y)\| \le \theta_{(m,2)}\|x-y\|. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we see from the restriction (f) that

$$\|P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})x - P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})\| \le \|x-y\|.$$

This shows that $P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})$ is nonexpansive for each $m \in \{1, 2, \dots, N\}$. Fix $p \in \mathcal{F}$ and Put

$$z_n = \sum_{m=1}^{N} \eta_{(m,n)} P_C(\lambda_{(m,1)}T_{(m,1)}x_n - \lambda_{(m,2)}T_{(m,2)}x_n).$$

It follows that $\|z_n - p\| \le \|x_n - p\|$. This in turn implies that

$$\begin{aligned}\|y_n - p\| &= \|\delta_n Sx_n + (1-\delta_n)z_n - p\| \\ &\le \delta_n\|Sx_n - Sp\| + (1-\delta_n)\|z_n - p\| \\ &\le \|x_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\le \alpha_n\kappa\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \big(1 - \alpha_n(1-\kappa)\big)\|x_n - p\| + \alpha_n\|f(p) - p\|.\end{aligned}$$

By mathematical inductions, we arrive at

$$\|x_n - p\| \le \max\{\frac{\|f(p) - p\|}{1-\kappa}, \|x_1 - p\|\}, \quad \forall n \ge 1.$$

This completes the proof that the sequence $\{x_n\}$ is bounded. Since the mapping $P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})$ is nonexpansive for each $m \in \{1, 2, \dots, N\}$, we see that that

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|\sum_{m=1}^{N} \eta_{(m,n+1)} P_C(\lambda_{(m,1)}T_{(m,1)}x_{n+1} - \lambda_{(m,2)}T_{(m,2)}x_{n+1}) \\ &\quad - \sum_{m=1}^{N} \eta_{(m,n)} P_C(\lambda_{(m,1)}T_{(m,1)}x_n - \lambda_{(m,2)}T_{(m,2)}x_n)\| \qquad (3.4) \\ &\le \|x_{n+1} - x_n\| + M_1 \sum_{m=1}^{N} |\eta_{(m,n+1)} - \eta_{(m,n)}|,\end{aligned}$$

where

$$M_1 = \max\{\sup_{n\geq 1} \|P_C(\lambda_{(m,1)}T_{(m,1)}x_n - \lambda_{(m,2)}T_{(m,2)}x_n)\|, \forall 1 \leq m \leq N\}.$$

This in turn implies that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|\delta_{n+1}Sx_{n+1} + (1-\delta_{n+1})z_{n+1} - \delta_n Sx_n - (1-\delta_n)z_n\| \\ &\leq \delta_{n+1}\|Sx_{n+1} - Sx_n\| + (1-\delta_{n+1})\|z_{n+1} - z_n\| + |\delta_{n+1} - \delta_n|\|Sx_n - z_n\| \\ &\leq \|x_{n+1} - x_n\| + M_1 \sum_{m=1}^{N} |\eta_{(m,n+1)} - \eta_{(m,n)}| + |\delta_{n+1} - \delta_n|\|Sx_n - z_n\|.\end{aligned} \tag{3.5}$$

Put $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$. It follows that

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad \forall n \geq 1. \tag{3.6}$$

Now, we estimate $\|l_{n+1} - l_n\|$. In view of

$$\begin{aligned}&l_{n+1} - l_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) + \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} y_{n+1} - \frac{\alpha_n}{1-\beta_n} f(x_n) - \frac{1-\beta_n-\alpha_n}{1-\beta_n} y_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}} (f(x_{n+1}) - y_{n+1}) + \frac{\alpha_n}{1-\beta_n}(y_n - f(x_n)) + y_{n+1} - y_n,\end{aligned}$$

we obtain that

$$\|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - f(x_n)\| + \|y_{n+1} - y_n\|. \tag{3.7}$$

Substituting (3.5) into (3.7), we obtain that

$$\begin{aligned}\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - f(x_n)\| \\ &+ M_1 \sum_{m=1}^{N} |\eta_{(m,n+1)} - \eta_{(m,n)}| + |\delta_{n+1} - \delta_n|\|Sx_n - z_n\|\end{aligned}$$

It follows from the restrictions (b), (c), (d) and (e) that

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_{n+1}\|) < 0.$$

In view of Lemma 2.3, we see that

$$\lim_{n\to\infty} \|l_n - x_n\| = 0. \tag{3.8}$$

Thanks to (3.6), we obtain that

$$x_{n+1} - x_n = (1-\beta_n)(l_n - x_n).$$

It follows from (3.8) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

On the other hand, we have

$$\|y_n - x_n\| \le \frac{1}{\gamma_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{\gamma_n}\|x_n - f(x_n)\|.$$

From the conditions (b) and (c), we obtain that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.10}$$

Define a mapping $R : C \to C$ by

$$Rx = \delta Sx + (1-\delta)\sum_{m=1}^{N} \eta_{(m,n)} P_C(\lambda_{(m,1)}T_{(m,1)}x - \lambda_{(m,2)}T_{(m,2)}x), \quad \forall x \in C,$$

where $\delta = \lim_{n\to\infty}\delta_n$. From Lemma 2.1, we see that R is nonexpansive with $F(R) = F\big(\cap_{m=1}^N P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)}) \bigcap F(S)\big) = \mathcal{F}$.

Next, we show that $Rx_n - x_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}\|Rx_n - x_n\| &\le \|\delta Sx_n + (1-\delta)\sum_{m=1}^{N} \eta_{(m,n)} P_C(\lambda_{(m,1)}T_{(m,1)}x_n - \lambda_{(m,2)}T_{(m,2)}x_n) - y_n\| + \|y_n - x_n\| \\ &\le |\delta - \delta_n| M_2 + \|y_n - x_n\|,\end{aligned}$$

where

$$M_2 = \max\{\sup_{n\ge1} \|Sx_n - \sum_{m=1}^{N} \eta_{(m,n)} P_C(\lambda_{(m,1)}T_{(m,1)}x_n - \lambda_{(m,2)}T_{(m,2)}x_n)\|, \forall 1 \le m \le N\}.$$

In view of the restriction (e), we see from (3.10) that

$$\lim_{n\to\infty} \|Rx_n - x_n\| = 0. \tag{3.11}$$

Since Pf is a contraction with the coefficient κ, we have that there exists a unique fixed point. We use $\bar{x}$ to denote the unique fixed point of the mapping Pf. That is, $Pf(\bar{x}) = \bar{x}$.

Next, we show that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle \le 0$. To show it, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x})-\bar{x}, x_{x_{n_i}}-\bar{x}\rangle. \tag{3.12}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to b. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup \rho$. From Lemma 2.1 and Lemma 2.2, we see that

$$\rho \in F(R) = F\big(\cap_{m=1}^{N} P_C(\lambda_{(m,1)}T_{(m,1)} - \lambda_{(m,2)}T_{(m,2)})\bigcap F(S)\big) = \mathcal{F}.$$

This completes the proof of (3.12).

Finally, we show that $x_n \to \bar{x}$ as $n\to\infty$. Note that

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2\\
&= \alpha_n\langle f(x_n)-f(\bar{x}), x_{n+1}-\bar{x}\rangle + \alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle + \beta_n\langle x_n-\bar{x}, x_{n+1}-\bar{x}\rangle\\
&\quad + \gamma_n\langle y_n-\bar{x}, x_{n+1}-\bar{x}\rangle\\
&\le \alpha_n\kappa\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| + \alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle + \beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\
&\quad + \gamma_n\|y_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\
&\le \frac{1-\alpha_n(1-\kappa)}{2}(\|x_n-\bar{x}\|^2 + \|x_{n+1}-\bar{x}\|^2) + \alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle.
\end{aligned}$$

This in turn implies that

$$\|x_{n+1}-\bar{x}\|^2 \le \big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle.$$

In view of the restriction (b), and (3.12), we find from Lemma 2.4 that This completes the proof.

If $T_{(m,1)} = I$, where I denotes the identity mapping, and $\lambda_{(m,1)} = 1$, then we find from Theorem 2.1 the following.

Corollary 2.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f: C\to C$ be a contractive mapping with the contractive constant κ, and $S: C\to C$ a nonexpansive mapping with fixed points. Let $T_m: C\to H$ be a relaxed (μ_m,ν_m)-cocoercive and L_m-Lipschitz continuous mapping for each positive integer $m\in[1,N]$, where $N\ge m$*

is some positive integer. Assume that $\mathcal{F} = \cap_{m=1}^{N} VI(C, T_m) \bigcap F(S) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by the following algorithm*

$$\begin{cases} x_1 \in C, \\ y_n = \delta_n S x_n + (1-\delta_n) \sum_{m=1}^{N} \eta_{(m,n)} P_C(x_n - \lambda_m T_m x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, *and* $\{\eta_{(m,n)}\}$ *are sequences in* $(0,1)$ *satisfying the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\sum_{m=1}^{N} \eta_{(m,n)} = 1$, *and* $\lim_{n\to\infty} \eta_{(m,n)} = \eta_m \in (0,1)$;

(e) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$,

and $\{\lambda_m\}$ *is a sequence such that*

(f) $\lambda_m \leq \frac{2\nu_m - 2\mu_m L_m^2}{L_m^2}$.

Then the sequence $\{x_n\}$ *generated by the algorithm converges strongly to* $\bar{x}$, *where* $\bar{x} \in \mathcal{F}$, *and solves the following variational inequality: find some point* y *such that*

$$\langle f(y) - y, y - x \rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

If $N = 1$, and $f(x) = u$, where u is a fixed element in C, for all $x \in C$, then we have the following.

Corollary 2.3. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H*. Let* $S : C \to C$ *be a nonexpansive mapping with fixed points. Let* $T_1 : C \to H$ *be a relaxed* (μ_1, ν_1)*-cocoercive and* L_1*-Lipschitz continuous mapping and* $T_2 : C \to H$ *be a relaxed* (μ_2, ν_2)*-cocoercive and* L_2*-Lipschitz continuous mapping. Assume that* $\mathcal{F} =$

$GVI(C,T_1,T_1)\bigcap F(S)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm

$$\begin{cases} x_1 \in C, \\ y_n = \delta_n S x_n + (1-\delta_n) P_C(\lambda_1 T_1 x_n - \lambda_2 T_2 x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and $\{\eta_{(m,n)}\}$ are sequences in $(0,1)$ satisfying the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1;$

(b) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha = \infty;$

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$

(d) $\lim_{n\to\infty} \delta_n = \delta \in (0,1),$

and λ_1 and λ_2 are two constants such that

(e) $\sqrt{1-2\lambda_1\nu_1+2\lambda_1\mu_1 L_1^2+\lambda_1^2 L_1^2}+\sqrt{1-2\lambda_2\nu_2+2\lambda_2\mu_2 L_2^2+\lambda_2^2 L_2^2}\leq 1.$

Then the sequence $\{x_n\}$ generated by the algorithm converges strongly to $\bar{x}$, where $\bar{x}\in\mathcal{F}$, and solves the following variational inequality: find some point y such that

$$\langle f(y) - y, y - x\rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

References

[1] M.A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004) 199-277.

[2] X. Qin, M.A. Noor, GeneralWiener-Hopf equation technique for nonexpansive mappings and general variational inequalities in Hilbert spaces, Appl. Math. Comput. 201 (2008) 716-722.

[3] M.A. Noor, On general quasi-variational inequalities, J. King Saud University-Science, 24 (2012) 81-88.

[4] M.A. Noor, Diffierentiable non-convex functions and general variational inequalities, Appl. Math. Comput. 199 (2008) 623-630.

[5] M.A. Noor, An extragradient algorithm for solving general nonconvex variational inequalities, Appl. Math. Lett. 23 (2010) 917-921.

[6] Y.J. Cho, X. Qin, Systems of generalized nonlinear variational inequalities and its projection methods, Nonlinear Anal. 69 (2008) 4443-4451.

[7] S.Y. Cho, S.M. Kang, Approximation of fixed points of pseudocontraction semigroups based on a viscosity iterative process. Appl. Math. Lett. 24 (2011) 224-228.

[8] D.R. Sahu, J.C. Yao, The prox-Tikhonov regularization method for the proximal point algorithm in Banach spaces. J. Optim. Theory Appl. 151 (2011) 641-655.

[9] O. Güller, On the convergence of the proximal point algorithmfor convex minimization. SIAM J. Control Optim. 29 (1991) 403-419.

[10] L.J. Lin, Y.J. Huang, Generalized vector quasi-equilibrium problems with applications to common fixed point theorems and optimization problems. Nonlinear Anal 66 (2007) 1275-1289.

[11] J. Eckstein, Approximate iterations in Bregman-function-based proximal algorithms. Math. Program. 83 (1998) 113-123.

[12] S.S. Chang, H.W.J. Lee, C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal. 70 (2009) 3307-3319.

[13] S. Husain, S. Gupta, A resolvent operator technique for solving generalized system of nonlinear relaxed cocoercive mixed variational inequalities. Adv. Fixed Point Theory 2 (2012) 18-28.

[14] S.M. Kang, S.Y. Cho, Z. Liu, Convergence of iterative sequences for generalized equilibrium problems involving inverse-strongly monotone mappings. J. Inequal. Appl. 2010 (2010) 827082.

[15] J.K. Kim, S.Y. Cho, X. Qin, Hybrid projection algorithms for generalized equilibrium problems and strictly pseudocontractive mappings. J. Inequal. Appl. 2010 (2010) Article ID 312602.

[16] J.K. Kim, Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings, Fixed Point Theory Appl. 2011 2011:10.

[17] J. Ye, J. Huang, Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces, Journal of Mathematical and Computational Science, 1 (2011) 1-18.

[18] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46-55.

[19] N.K. Mathato, C. Nahak, Equilibrium problem under various types of convexities in Banach space, J. Math. Comput. Sci. 1 (2011) 77-88.

[20] C. Wu, S.Y. Cho, M. Shang, Moudafi's viscosity approximations with demi-continuous and strong pseudo-contractions for non-expansive semigroups, J. Inequal. Appl. 2010 (2010), Article ID 645498.

[21] L. Yu, M. Liang, Convergence theorems of solutions of a generalized variational inequality, Fixed Point Theory Appl. 2011 2011:19.

[22] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (2009) 20-30.

[23] Stampacchia G: Formes bilineaires coercitives sur les ensembles convexes. CR Acad Sci Paris 1964, 258: 4413-4416.

[24] R.U. Verma, Rockafellar's celebrated theorem based on A-maximal monotonicity design, Appl. Math. Lett. 21 (2008) 355-360.

[25] R.E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Tras. Amer. Math. Soc. 179 (1973) 251-262.

[26] F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proc. Symp. Pure. Math. 18 (1976) 78-81.

[27] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter non-expansive semigroups without Bochne integrals, J. Math. Anal. Appl. 305 (2005) 227-239.

[28] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002) 240-256.