Available online at http://scik.org
Engineering Mathematics Letters, 2 (2013), No. 2, 90-114
ISSN 2049-9337

SOME COUPLED COINCIDENCE POINT RESULTS UNDER C-DISTANCE IN CONE METRIC SPACES

RAKESH BATRA[1,*], SACHIN VASHISTHA[2]

[1]Department of Mathematics, Hans Raj College, University of Delhi, Delhi-110007, India

[2]Department of Mathematics, Hindu College, University of Delhi, Delhi-110007, India

Abstract. A new concept of c-distance in cone metric space has been introduced recently in 2011. Many results in the area of fixed point theory have been proved by different authors using c-distance. In this paper we extend and generalize some coupled coincidence point theorems using functions of two variables taking values in $[0, 1)$ as coefficients in various contractive conditions.

Keywords: coincidence point; cone metric space; c-distance; fixed point.

2010 AMS Subject Classification: 47H10; 54H25; 55M20

1. Introduction

Fixed point theory has variety of interesting applications in disciplines such as chemistry, economics, physics, biology and engineering. In dynamical systems it is used to prove several existence and stability results for the strict fixed points of a set-valued dynamic system F, as well as some conditions that guarantee each dynamic process converges and its limit is a strict fixed point of F. In theoretical economics, such as general equilibrium theory, there comes at point where one needs to know whether the solution to a system of equations necessarily exists; or, more specifically, under which conditions will a solution necessarily exist. The mathematical analysis of this question usually relies

*Corresponding author

Received January 27, 2013

on fixed point theorems. In engineering, fixed point technique has been used in areas like image retrieval and signal processing. In game theory it is used to estblish the existence of Nash equilibrium.

The Banach Contraction Principle is the basic tool in this direction. Due to simplicity and usefulness of this principle, it has become a very important tool in solving the existence problems in many branches of non-linear analysis. Ran and Reurings [26] extended the Banach contraction principle to metric spaces endowed with a partial ordering, and they gave application of their results to matrix equations. In [23] Nieto and L*ó*pez extended the result of Ran and Reurings [26] for non-decreasing mappings and applied their results to get a unique solution for a first order differential equation.

In 2007, Huang and Zhang [17] first introduced the concept of cone metric space. Cone metric space is a generalization of metric space where each pair of points is assigned to a member of a real Banach space having a cone. They also established the existence of fixed point theorems to cone metric spaces. For more study on fixed point theorems in cone metric spaces see [5, 18, 33, 34, 16, 4, 3, 2, 1, 19, 25, 30, 10, 24].

Bhaskar and Lakshmikantham [8] introduced the notion of a coupled fixed point of a mapping $F : X \times X \to X$. They established some coupled fixed point results and applied their results to the study of existence and uniqueness of solution for a periodic boundary value problem. For more results on coupled fixed point theorems see [15, 22, 27, 28, 31, 9, 11].

Lakshmikantham and Ćirić [22] introduced the concept of coupled coincidence points and proved coupled coincidence and coupled common fixed point results for mappings $F : X \times X \to X$ and $g : X \to X$ satisfying nonlinear contraction in ordered metric space.

The studies of asymmetric structures and their application in mathematics are important. Recently Cho et. al. [10](also see [35]) introduced a new concept of c-distance in cone metric spaces which is a cone version of w-distance of Kada et. al. In [29], Shatanawi et. al. proved some coincidence point theorems on cone metric spaces using c-distance for weak contraction mappings satisfying mixed g-monotonicity. In this paper we establish the existence of coupled coincidence point for mappings satisfying contractive conditions having functions taking values in $[0, 1)$ as coefficients and extend the work of Shatanawi et. al. [29] who used scalar coefficients. For more study of related work see [6, 7, 12, 13, 14].

2. Preliminaries

Throughout this paper, $(X, \sqsubseteq)$ denotes a partially ordered set with partial order $\sqsubseteq$.

Definition 2.1. ([8]) A mapping $F : X \times X \to X$ is said to have mixed monotone property if for any $x,\ y \in X$

$$\begin{aligned} &x_1,\ x_2 \in X,\ x_1 \sqsubseteq x_2 \implies F(x_1, y) \sqsubseteq F(x_2, y), \\ &y_1,\ y_2 \in X,\ y_1 \sqsubseteq y_2 \implies F(x, y_1) \sqsupseteq F(x, y_2). \end{aligned} \tag{1}$$

Definition 2.2. ([22]) A mapping $F : X \times X \to X$ is said to have mixed g-monotone property if for any $x,\ y \in X$

$$\begin{aligned} &x_1,\ x_2 \in X,\ gx_1 \sqsubseteq gx_2 \implies F(x_1, y) \sqsubseteq F(x_2, y), \\ &y_1,\ y_2 \in X,\ gy_1 \sqsubseteq gy_2 \implies F(x, y_1) \sqsupseteq F(x, y_2). \end{aligned} \tag{2}$$

Definition 2.3. ([8]) An element $(x, y) \in X \times X$ is called a coupled fixed point of the mappings $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.4. ([22]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.5. ([22]) Let $F : X \times X \to X$ and $g : X \to X$. The mappings F and g are said to commute if $gF(x, y) = F(gx, gy)$ for all $x,\ y \in X$.

In [17], cone metric space was introduced in the following manner: Let $(E, \|.\|)$ be a real Banach space and θ denote the zero element in E. Assume that P is a subset of E. Then P is called a cone if and only if

(i) P is non empty, closed and $P \neq \{\theta\}$,
(ii) If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$.
(iii) $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subseteq E$ and $x, y \in E$, the partial ordering $\preceq$ on E with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in intP$. It can be easily shown that $\lambda.intP \subseteq intP$ for all $\lambda > 0$ and $intP + intP \subseteq intP$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number K satisfying above is called the normal constant of P. In the following we always suppose E is a real Banach space, P is a cone in E with $intP \neq \phi$ and $\preceq$ is partial ordering with respect to P.

Definition 2.6. ([17]) Let X be a non empty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following condition:

(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta \Leftrightarrow x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.7. ([17]) Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$ then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$ then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.8. ([17]) Let (X, d) be a cone metric space, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then

(i) the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ (or equivalently $\|d(x_n, x)\| \to 0$),

(ii) the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ (or equivalently $\|d(x_n, x_m)\| \to 0$).

(iii) the sequence $\{x_n\}$ (respectively, $\{y_n\}$) converges to x (respectively, y) then $d(x_n, y_n) \to d(x, y)$.

Lemma 2.9. ([33]) Every cone metric space (X, d) is a topological space. For $c \gg 0,\ c \in E,\ x \in X$, let $B(x, c) = \{y \in X : d(y, x) \ll c\}$ and $\beta = \{B(x, c) : x \in X,\ c \gg 0\}$. Then $\tau_c = \{U \subseteq X :$ for all $x \in U,$ there exists $B_x \in \beta,$ with $x \in B_x \subseteq U\}$ is a topology on X.

Definition 2.10. ([33]) Let (X, d) be a cone metric space. A map $T : (X, d) \to (X, d)$ is called sequentially continuous if $x_n \in X,\ x_n \to x$ implies $Tx_n \to Tx$.

Lemma 2.11. ([33]) Let (X, d) be a cone metric space, and $T : (X, d) \to (X, d)$ be any map. Then, T is continuous if and only if T is sequentially continuous.

Let (X, d) be a cone metric space and $X^2 = X \times X$. Define a function $\rho : X^2 \times X^2 \to E$ by $\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$ for all (x_1, y_1) and $(x_2, y_2) \in X^2$.Then (X^2, ρ) is a cone metric space [21].

Lemma 2.12. ([21]) Let $z_n = (x_n, y_n) \in X^2$ be a sequence and $z = (x, y) \in X^2$. Then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.

Next we give the notation of c-distance on a cone metric space which is generalization of w-distance of Kada et. al. [20] with some properties.

Definition 2.13. ([10]) Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,
(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,
(q3) for each $x \in X$ and $n \in \mathbb{N}$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,
(q4) For all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.14. The c-distance q is a w-distance on X if we let (X, d) be a metric space, $E = \mathbb{R}$, $P = [0, \infty)$ and (q3) is replaced by the following condition: for any $x \in X$, $q(x, .) : X \to \mathbb{R}$ is lower semicontinuous. Moreover, (q3) holds whenever $q(x, .)$ is lower semi-continuous. Thus, if (X, d) is a metric space, $E = \mathbb{R}$, and $P = [0, \infty)$, then every w-distance is a c-distance. But the converse is not true in the general case. Therefore, the c-distance is a generalization of the w-distance.

Example 2.15. ([32]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = \|x - y\|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

Example 2.16. ([32]) Let (X, d) be a cone metric space and P a normal cone. Define a mapping $q : X \times X \to P$ by $q(x, y) = d(x, y)$ for all $x, y \in X$. Then, q is c-distance.

Example 2.17. ([32]) Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\|_1 = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0,\ t \in [0, 1]\}$. Let $X = [0, +\infty)$(with usual order), and $d(x, y)(t) = \|x - y\|\varphi(t)$ where $\varphi : [0, 1] \to \mathbb{R}$ is given by $\varphi(t) = e^t$ for all $t \in [0, 1]$. Then (X, d) is an ordered cone metric space(see [10] Example 2.9). This cone is not normal. Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\varphi$ for all $x, y \in X$. Then q is a c-distance.

Example 2.18.([32]) Let (X, d) be a cone metric space and P a normal cone. Define a mapping $q : X \times X \to P$ by $q(x, y) = d(u, y)$ for all $x, y \in X$, where u is a fixed point in X. Then q is a c-distance.

Lemma 2.19. [10] Let (X, d) be a cone metric space and q be a c-distance on X. Let$\{x_n\}$ and $\{y_n\}$ be sequences in X and $y, z \in X$. Suppose that u_n is a sequence in P converging to θ. Then the following hold:

(1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.

(2) If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then y_n converges to z.

(3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.20.[29] Let (X, d) be a cone metric space, and let q be a c-distance on X. Let $\{x_n\}$ be a sequence in X. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in P converging to θ. If $q(x_n, y) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$, then $y = z$.

Remark 2.21.([10])

(i) $q(x, y) = q(y, x)$ may not be true for all $x, y \in X$.

(ii) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

3. Main results

Theorem 3.1. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and $k : X \times X \to [0, 1)$ be any given function such that

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ for all $x, y \in X$ and

(ii) $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \preceq k(gx, gy)(q(gx, gu) + q(gy, gv))$ for all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Choose $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$. Since $F(X \times X) \subseteq g(X)$, one can find $x_1, y_1 \in X$ in a way that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Repeating the same argument one can find $x_2, y_2 \in X$ in a way that $gx_2 = F(x_1, y_1)$ and $F(y_1, x_1) = gy_2$. Continuing this process one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X that satisfy $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n \geq 0$. It is asserted that $\{gx_n\}$ is a nondecreasing and $\{gy_n\}$ is a nonincreasing sequence. That is

$$gx_n \sqsubseteq gx_{n+1} \quad and \quad gy_n \sqsupseteq gy_{n+1} \quad for \quad all \quad n \geq 0. \tag{3}$$

For $n = 0$, (3) follows by the choice of x_0 and y_0. Let us assume that (3) holds good for $n = k$, $k \geq 0$. So we have $gx_k \sqsubseteq gx_{k+1}$ and $gy_k \sqsupseteq gy_{k+1}$. Mixed g-monotonicity of F now implies that

$$gx_{k+1} = F(x_k, y_k) \sqsubseteq F(x_{k+1}, y_k) \sqsubseteq F(x_{k+1}, y_{k+1}) = gx_{k+2}.$$

Similarly we have $gy_{k+1} \sqsupseteq gy_{k+2}$. Thus (3) follows for $k+1$. Hence, by induction, our assertion follows. Now for all $n \in \mathbb{N}$

$$\begin{aligned}
q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
&\preceq k(gx_{n-1}, gy_{n-1})(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)) \\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)) \\
&\preceq k(gx_{n-2}, gy_{n-2})(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)) \\
&\vdots \\
&\preceq k(gx_0, gy_0)(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n))
\end{aligned}$$

Put $q_n = q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1})$ and $k = k(gx_0, gy_0)$. Then, we have

$$\begin{aligned}
q_n &= q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \\
&\preceq kq_{n-1} \\
&\vdots \\
&\preceq k^n q_0
\end{aligned}$$

Let $m > n \geq 1$. It follows that

$$q(gx_n, gx_m) \preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m) \quad \textit{and}$$

$$q(gy_n, gy_m) \preceq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \ldots + q(gy_{m-1}, gy_m).$$

Then we have

$$\begin{aligned}
q(gx_n, gx_m) + q(gy_n, gy_m) &\preceq q_n + q_{n+1} + \ldots + q_{m-1} \\
&\preceq k^n q_0 + k^{n+1} q_0 + \ldots + k^{m-1} q_0 \\
&\preceq \frac{k^n}{1-k} q_0 \qquad (4)
\end{aligned}$$

From (4) we have

$$q(gx_n, gx_m) \preceq \frac{k^n}{1-k} q_0 \tag{5}$$

and also

$$q(gy_n, gy_m) \preceq \frac{k^n}{1-k} q_0 \tag{6}$$

Thus, Lemma 2.19.(3) shows that gx_n and gy_n are Cauchy sequences in X. Since X is complete, there exists there exists $x^*, y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of g we get

$$\lim_{n\to\infty} ggx_n = gx^*$$

and

$$\lim_{n\to\infty} ggy_n = gy^*.$$

Commutativity of F and g now implies that

$$ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1}) \quad for \quad all \quad n \in \mathbb{N}$$

$$ggy_n = gF(y_{n-1}, x_{n-1}) = F(gy_{n-1}, gx_{n-1}) \quad for \quad all \quad n \in \mathbb{N}.$$

Since F is continuous, therefore,

$$gx^* = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) = F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) = F(x^*, y^*)$$

$$gy^* = \lim_{n\to\infty} ggy_n = \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) = F(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}) = F(y^*, x^*)$$

Thus (x^*, y^*) is a coupled coincidence point of F and g.

Corollary 3.2. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Let $F : X \times X \to X$ be a continuous function which satisfies mixed monotone property and $k : X \times X \to [0, 1)$ be any given function such that

(i) $k(F(x, y), F(y, x)) \leq k(x, y)$ for all $x, y \in X$ and

(ii) $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \preceq k(x, y)(q(x, u) + q(y, v))$ for all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$.

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $g = I_X$, the identity function on X in Theorem 3.1.

Theorem 3.3. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is complete subspace of X. Let F satisfy mixed g-monotone property and $k : X \times X \to [0, 1)$ be any given function such that

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ for all $x, y \in X$ and
(ii) $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \preceq k(gx, gy)(q(gx, gu) + q(gy, gv))$ for all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$.

Suppose X has the following property

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.
(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Consider Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ as in the proof of Theorem 3.1. Since $(g(X), d)$ is complete, there exists $x^*, y^* \in X$ such that $gx_n \to gx^*$ and $gy_n \to gy^*$. By (q3), (5) and (6) we have

$$q(gx_n, gx^*) \preceq \frac{k^n}{1-k} q_0 \quad for \quad all \quad n \geq 0 \quad and \tag{7}$$

$$q(gy_n, gy^*) \preceq \frac{k^n}{1-k} q_0 \quad for \quad all \quad n \geq 0 \tag{8}$$

Adding (7) and (8) we get

$$q(gx_n, gx^*) + q(gy_n, gy^*) \preceq \frac{2k^n}{1-k} q_0 \quad for \quad all \quad n \geq 0$$

Sequence $\{gx_n\}$ is nondecreasing and converges to gx^*. By given condition (i) we have, therefore, $gx_n \sqsubseteq gx^*$ for all $n \geq 0$ and similarly $gy_n \sqsupseteq gy^*$ for all $n \geq 0$. Thus for all $n \in \mathbb{N}$

$$\begin{aligned}
&q(gx_n, F(x^*, y^*)) + q(gy_n, F(y^*, x^*)) \\
&= q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) + q(F(y_{n-1}, x_{n-1}), F(y^*, x^*)) \\
&\preceq k(gx_{n-1}, gy_{n-1})[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
&\preceq k(gx_{n-2}, gy_{n-2})[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
&\vdots \\
&\preceq k(gx_0, gy_0)[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
&= k[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
&\preceq k\frac{2k^{n-1}}{1-k}q_0 \\
&= \frac{2k^n}{1-k}q_0
\end{aligned}$$

Then

$$q(gx_n, F(x^*, y^*)) \preceq \frac{2k^n}{1-k}q_0 \tag{9}$$

and

$$q(gy_n, F(y^*, x^*)) \preceq \frac{2k^n}{1-k}q_0 \tag{10}$$

By Lemma 2.20., (7) and (9), we have $F(x^*, y^*) = gx^*$. Similarly, by Lemma 2.20. , (8) and (10) we have $F(y^*, x^*) = gy^*$. Thus (x^*, y^*) is a coupled coincidence point of F and g.

Corollary 3.4. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Let $F : X \times X \to X$ be a function which satisfies mixed monotone property and $k : X \times X \to [0, 1)$ be any given function such that

(i) $k(F(x,y),F(y,x)) \leq k(x,y)$ for all $x,y \in X$ and

(ii) $q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)) \preceq k(x,y)(q(x,u)+q(y,v))$ for all $x,y,u,v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$.

Suppose X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0,y_0)$ and $F(y_0,x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*,y^*) = x^*$ and $F(y^*,x^*) = y^*$, that is, F has a coupled fixed point (x^*,y^*).

Proof. Take $g = I_X$, the identity map on X in Theorem 3.3.

Theorem 3.5. Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and $k,\ l : X \times X \to [0,1)$ be any given functions such that

(i) $k(F(x,y),F(y,x)) \leq k(gx,gy)$ and $l(F(x,y),F(y,x)) \leq l(gx,gy)$ for all $x,y \in X$,

(ii) $k(x,y) = k(y,x)$ and $l(x,y) = l(y,x)$ for all $x,y \in X$,

(iii) $(k+l)(x,y) < 1$ for all $x,y \in X$ and

(iv) $q(F(x,y),F(u,v)) \preceq k(gx,gy)q(gx,gu) + l(gx,gy)q(gy,gv)$ for all $x,\ y,\ u,\ v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0,y_0)$ and $F(y_0,x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*,y^*) = gx^*$ and $F(y^*,x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*,y^*).

Proof. Given $x,y,u,v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. Then we have

$$\begin{aligned} & q(F(x,y),F(u,v)) \preceq k(gx,gy)q(gx,gu) + l(gx,gy)q(gy,gv)) \\ \textit{and} \quad & q(F(y,x),F(v,u)) \preceq k(gy,gx)q(gy,gv) + l(gy,gx)q(gx,gu) \\ & \qquad = k(gx,gy)q(gy,gv) + l(gx,gy)q(gx,gu) \end{aligned}$$

Thus $q(F(x,y),F(u,v))+q(F(y,x),F(v,u)) \preceq (k+l)(gx,gy)(q(gx,gu)+q(gy,gv))$ where $(k+l) : X \times X \to [0,1)$ satisfies

$$\begin{aligned}(k+l)(F(x,y),F(y,x)) &= k(F(x,y),F(y,x)) + l(F(x,y),F(y,x)) \\ &\leq k(gx,gy) + l(gx,gy) = (k+l)(gx,gy) \quad for \quad all \quad x,y \in X.\end{aligned}$$

Result follows by Theorem 3.1.

Corollary 3.6. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ be a given function satisfying mixed monotone property and $k,\ l : X \times X \to [0,1)$ be any given functions such that

(i) $k(F(x,y),F(y,x)) \leq k(x,y)$ and $l(F(x,y),F(y,x)) \leq l(x,y)$ for all $x,y \in X$,
(ii) $k(x,y) = k(y,x)$ and $l(x,y) = l(y,x)$ for all $x,y \in X$,
(iii) $(k+l)(x,y) < 1$ for all $x,y \in X$ and
(iv) $q(F(x,y),F(u,v)) \preceq k(x,y)q(x,u) + l(x,y)q(y,v)$ for all $x,y,u,v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$.

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0,y_0)$ and $F(y_0,x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*,y^*) = x^*$ and $F(y^*,x^*) = y^*$, that is, F has a coupled fixed point (x^*,y^*).

Proof. Take $g = I_X$ the identity function on X in Theorem 3.5.

Corollary 3.7. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and $k : X \times X \to [0, \frac{1}{2})$ be any given function such that

(i) $k(F(x,y),F(y,x)) \leq k(gx,gy)$ for all $x,y \in X$,
(ii) $k(x,y) = k(y,x)$ for all $x,y \in X$ and
(iii) $q(F(x,y),F(u,v)) \preceq k(gx,gy)(q(gx,gu)+q(gy,gv))$ for all $x,y,u,v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Take $k(x, y) = l(x, y)$ in Theorem 3.5.

Corollary 3.8. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ be a given function satisfying mixed monotone property and $k : X \times X \to [0, \frac{1}{2})$ be any given functions such that

(i) $k(F(x, y), F(y, x)) \leq k(x, y)$ for all $x, y \in X$,

(ii) $k(x, y) = k(y, x)$ for all $x, y \in X$ and

(iii) $q(F(x, y), F(u, v)) \preceq k(x, y)(q(x, u) + q(y, v))$ for all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$.

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $k(x, y) = l(x, y)$ and $g = I_X$ in Theorem 3.5.

Theorem 3.9. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is complete subspace of X. Let F satisfy mixed g-monotone property and $k,\ l : X \times X \to [0, 1)$ be any given functions such that

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ and $l(F(x, y), F(y, x)) \leq l(gx, gy)$ for all $x, y \in X$,

(ii) $k(x, y) = k(y, x)$ and $l(x, y) = l(y, x)$ for all $x, y \in X$,

(iii) $(k + l)(x, y) < 1$ for all $x, y \in X$ and

(iv) $q(F(x, y), F(u, v)) \preceq k(gx, gy)q(gx, gu) + l(gx, gy)q(gy, gv)$ for all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$.

Suppose X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. It follows form Theorem 3.3 by the similar argument to those given in the proof of Theorem 3.5.

Corollary 3.10. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ be a function satisfying mixed monotone property and $k,\ l : X \times X \to [0, 1)$ be any given functions such that

(i) $k(F(x, y), F(y, x)) \leq k(x, y)$ and $l(F(x, y), F(y, x)) \leq l(x, y)$ for all $x, y \in X$,
(ii) $k(x, y) = k(y, x)$ and $l(x, y) = l(y, x)$ for all $x, y \in X$,
(iii) $(k + l)(x, y) < 1$ for all $x, y \in X$ and
(iv) $q(F(x, y), F(u, v)) \preceq k(x, y)q(x, u) + l(x, y)q(y, v)$ for all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$.

Suppose X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.
(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $g = I_X$ the identity function on X in Theorem 3.9.

Corollary 3.11. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is complete subspace of X. Let F satisfy mixed g-monotone property and $k : X \times X \to [0, \frac{1}{2})$ be any given functions such that

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ for all $x, y \in X$,

(ii) $k(x,y) = k(y,x)$ for all $x, y \in X$ and

(iii) $q(F(x,y), F(u,v)) \preceq k(gx, gy)(q(gx, gu) + q(gy, gv))$ for all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$.

Suppose X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Take $k(x,y) = l(x,y)$ in Theorem 3.9.

Corollary 3.12. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ be a function satisfying mixed monotone property and $k : X \times X \to [0, \frac{1}{2})$ be any given functions such that

(i) $k(F(x,y), F(y,x)) \leq k(x,y)$ for all $x, y \in X$,

(ii) $k(x,y) = k(y,x)$ for all $x, y \in X$ and

(iii) $q(F(x,y), F(u,v)) \preceq k(x,y)(q(x,u) + q(y,v))$ for all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$.

Suppose X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $k(x,y) = l(x,y)$ and $g = I_X$ in Theorem 3.9.

Theorem 3.13. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and

$g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and $k, l : X \times X \to [0, 1)$ be any given functions such that

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ and $l(F(x, y), F(y, x)) \leq l(gx, gy)$ for all $x, y \in X$,

(ii) $(k + l)(x, y) < 1$ for all $x, y \in X$ and

(iii) $q(F(x, y), F(u, v)) \preceq k(gx, gy)q(gx, F(x, y))+l(gx, gy)q(gu, F(u, v))$ for all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. By the similar argument as in Theorem 3.1. we can find the sequences $\{gx_n\}$ and $\{gy_n\}$ satisfying (3). Now for all $n \in \mathbb{N}$

$$
\begin{aligned}
&q(gx_n, gx_{n+1})\\
&= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n))\\
&\preceq k(gx_{n-1}, gy_{n-1})q(gx_{n-1}, F(x_{n-1}, y_{n-1})) + l(gx_{n-1}, gy_{n-1})q(gx_n, F(x_n, y_n))\\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_{n-1}, gx_n) + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_n, gx_{n+1})\\
&\preceq k(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx_n) + l(gx_{n-2}, gy_{n-2})q(gx_n, gx_{n+1})\\
&\vdots\\
&\preceq k(gx_0, gy_0)q(gx_{n-1}, gx_n) + l(gx_0, gy_0)q(gx_n, gx_{n+1})
\end{aligned}
$$

Put $q_n = q(gx_n, gx_{n+1})$ and $d = \frac{k(gx_0,gy_0)}{1-l(gx_0,gy_0)}$. Then $d \in [0, 1)$ and we have

$$q_n = q(gx_n, gx_{n+1}) \preceq dq_{n-1} \preceq \ldots \preceq d^n q_0$$

$$
\begin{aligned}
\textit{Also}\quad &q(gy_n, gy_{n+1})\\
&= q(F(y_{n-1}, x_{n-1}), F(y_n, x_n))\\
&\preceq k(gy_{n-1}, gx_{n-1})q(gy_{n-1}, F(y_{n-1}, x_{n-1})) + l(gy_{n-1}, gx_{n-1})q(gy_n, F(y_n, x_n))
\end{aligned}
$$

$$= k(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(gy_{n-1}, gy_n) + l(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(gy_n, gy_{n+1})$$

$$\preceq k(gy_{n-2}, gx_{n-2})q(gy_{n-1}, gy_n) + l(gy_{n-2}, gx_{n-2})q(gy_n, gy_{n+1})$$

$$\vdots$$

$$\preceq k(gy_0, gx_0)q(gy_{n-1}, gy_n) + l(gy_0, gx_0)q(gy_n, gy_{n+1})$$

Put $r_n = q(gy_n, gy_{n+1})$ and $h = \frac{k(gy_0,gx_0)}{1-l(gy_0,gx_0)}$. Then $h \in [0, 1)$ and we have

$$r_n = q(gy_n, gy_{n+1}) \preceq hr_{n-1} \preceq \ldots \preceq h^n r_0$$

Let $m > n \geq 1$.It follows that

$$\begin{aligned} q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m) \\ &= q_n + q_{n+1} + \ldots + q_{m-1} \\ &\preceq d^n q_0 + d^{n+1} q_0 + \ldots + d^{m-1} q_0 \\ &\preceq \frac{d^n}{1-d} q_0 \end{aligned}$$

$$\begin{aligned} \textit{Also} \quad q(gy_n, gy_m) &\preceq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \ldots + q(gy_{m-1}, gy_m) \\ &= r_n + r_{n+1} + \ldots + r_{m-1} \\ &\preceq h^n r_0 + h^{n+1} r_0 + \ldots + h^{m-1} r_0 \\ &\preceq \frac{h^n}{1-h} r_0 \end{aligned}$$

Thus, Lemma 2.19.(3) shows that gx_n and gy_n are Cauchy sequences in X. since X is complete, there exists there exists $x^*, y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of g we get

$$\lim_{n\to\infty} ggx_n = gx^* \quad \textit{and} \quad \lim_{n\to\infty} ggy_n = gy^*.$$

Commutativity of F and g now implies that

$$ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1}) \; \textit{for all} \; n \in \mathbb{N}$$

$$and \quad ggy_n = gF(y_{n-1}, x_{n-1}) = F(gy_{n-1}, gx_{n-1}) \; for \; all \; n \in \mathbb{N}.$$

Since F is continuous, therefore,

$$\begin{aligned} gx^* &= \lim_{n\to\infty} ggx_n \\ &= \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) \\ &= F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) \\ &= F(x^*, y^*) \end{aligned}$$

$$\begin{aligned} and \quad gy^* &= \lim_{n\to\infty} ggy_n \\ &= \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) \\ &= F(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}) \\ &= F(y^*, x^*) \end{aligned}$$

Thus (x^*, y^*) is a coupled coincidence point of F and g.

Corollary 3.14. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and $k, l \in [0, 1)$ be any given numbers such that $k + l < 1$ and

$$q(F(x, y), F(u, v)) \preceq kq(gx, F(x, y)) + lq(gu, F(u, v))$$

for all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Take $k(x, y) = k$ and $l(x, y) = l$ in Theorem 3.13.

Corollary 3.15. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ be a given function satisfying mixed monotone property and $k, l : X \times X \to [0, 1)$ be any given functions such that

(i) $k(F(x, y), F(y, x)) \leq k(x, y)$ and $l(F(x, y), F(y, x)) \leq l(x, y)$ for all $x, y, \in X$,

(ii) $(k + l)(x, y) < 1$ for all $x, y \in X$ and

(iii) $q(F(x, y), F(u, v)) \preceq k(x, y)q(x, F(x, y)) + l(x, y)q(u, F(u, v))$ for all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$.

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $g = I_X$ in Theorem 3.13.

Corollary 3.16. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ is a continuous function satisfying mixed monotone property and $k, l \in [0, 1)$ be any given numbers such that $k + l < 1$ and

$$q(F(x, y), F(u, v)) \preceq kq(x, F(x, y)) + lq(u, F(u, v))$$

for all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$. If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$,then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $k(x, y) = k, l(x, y) = l$ and $g = I_X$ in Theorem 3.13.

Theorem 3.17. Under the hypothesis of any one of the theorems from Theorem 3.1., Theorem 3.3., Theorem 3.5., Theorem 3.9. and Theorem 3.13. or any one of the corollaries 3.7., 3.11., and 3.14 we have $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$ where (x^*, y^*) is a coincidence point of F and g.

Proof. We prove this theorem under the hypothesis of Theorem 3.1. Proofs are similar for other theorems or corollaries and can be obtained by a little adjustment. We have

$$q(gx^*, gx^*) + q(gy^*, gy^*) = q(F(x^*, y^*), F(x^*, y^*) + q(F(y^*, x^*), F(y^*, x^*))$$

$$\preceq k(x^*, y^*)(q(gx^*, gx^*) + q(gy^*, gy^*))$$

Since $0 \leq k(x^*, y^*) < 1$, we have $q(gx^*, gx^*) + q(gy^*, gy^*) = \theta$. But $q(gx^*, gx^*) \geq \theta$ and $q(gy^*, gy^*) \geq \theta$, hence $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$.

Corollary 3.18. Under the hypothesis of any one of the corollaries 3.2., 3.4., 3.6., 3.8., 3.10., 3.12., 3.15 and 3.16 we have $q(x^*, x^*) = \theta$ and $q(y^*, y^*) = \theta$ where (x^*, y^*) is a coupled fixed point of F.

Proof. Similar to Theorem 3.17. once we work with $g = I_X$.

Theorem 3.19. In addition to the hypothesis of any one of the theorems from Theorem 3.1., Theorem 3.3., Theorem 3.5., Theorem 3.9. and Theorem 3.13. or any one of the corollaries 3.7., 3.11. and 3.14. suppose that any two elements of $g(X)$ are comparable and g is one-one. Then there exists a coupled coincidence point of F and g which is of the form (x^*, x^*) for some $x^* \in X$.

Proof. Again we prove this theorem under the hypothesis of Theorem 3.1. Proofs are similar for other theorems or corollaries and can be obtained by a little adjustment. Consider coupled coincidence point (x^*, y^*) of F and g. Then we have

$$q(gx^*, gy^*) + q(gy^*, gx^*) = q(F(x^*, y^*), F(y^*, x^*) + q(F(y^*, x^*), F(x^*, y^*))$$
$$\preceq k(x^*, y^*)(q(gx^*, gy^*) + q(gy^*, gx^*))$$

Since $0 \leq k(x^*, y^*) < 1$,we have $q(gx^*, gy^*) + q(gy^*, gx^*) = \theta$.But $q(gx^*, gy^*) \geq \theta$ and $q(gy^*, gx^*) \geq \theta$, hence $q(gx^*, gy^*) = \theta$ and $q(gy^*, gx^*) = \theta$. Let $u_n = \theta, x_n = gx^*$ for all $n \geq 0$, then we have $q(x_n, gx^*) \preceq u_n$ for all $n \geq 0$ and $q(x_n, gy^*) \preceq u_n$ for all $n \geq 0$. By Lemma 2.19.(1) we have $gx^* = gy^*$. Since g is one-one, therefore,$x^* = y^*$. Thus there exists a coupled coincidence point of the form (x^*, x^*) for some $x^* \in X$. This completes the proof.

Corollary 3.20. In addition to hypothesis of any one of the corollaries 3.2., 3.4., 3.6., 3.8., 3.10., 3.12., 3.15 and 3.16, suppose that any two elements of X are comparable. Then there exists a coupled fixed point of F which is of the form (x^*, x^*) for some $x^* \in X$.

Proof. Similar to Theorem 3.19. once we work with $g = I_X$.

Example 3.21. Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,1]$(with usual order) and $d(x,y) = |x-y|$. Then (X,d) is an ordered complete cone metric space. Further, let $q : X \times X \to E$ be defined by $q(x,y) = 2d(x,y)$. It is easy to check that q is a c-distance on X. Consider now the function defined by $F(x,y) = x^2/16$ for all $x, y \in X$, $k(x,y) = \frac{1+x+y}{16}$ for all $x, y \in X$ and $gx = x$ for all $x \in X$. Then $F(X \times X) \subseteq g(X)$ and F satisfies mixed g-monotone property. $k(F(x,y), F(y,x)) = \frac{1+\frac{x^2}{16}+\frac{y^2}{16}}{16} \leq \frac{1+x^2+y^2}{16} \leq \frac{1+x+y}{16} = k(gx, gy)$ for all $x, y \in X$. Further $q(F(x,y), F(u,v)) + q(F(y,x), F(v,u)) = 2|\frac{x^2}{16} - \frac{u^2}{16}| + 2|\frac{y^2}{16} - \frac{v^2}{16}| = \frac{1}{8}(x+u)|x-u| + \frac{1}{8}(y+v)|y-v| \leq \frac{x+1}{16}.2|x-u| + \frac{y+1}{16}.2|y-v| \leq \frac{1+x+y}{16}.2|x-u| + \frac{1+x+y}{16}.2|y-v| = k(gx, gy)(q(x,u) + q(y,v))$ for all $x, y, u, v \in X$ with $(gx \leq gu)$ and $(gy \geq gv)$ or$(gx \geq gu)$ and $(gy \leq gv)$. Further F and g are continuous, commuting, $g(0) \leq F(0,1)$ and $g(1) \geq F(1,0)$. Thus, by Theorem 3.1. , F and g have a coincidence point. Here F and g have a coincidence point at $(0,0)$.

Example 3.22. Let $E = \mathbb{R}$ with usual order and $X = [0,1]$. Let $d(x,y) = |x-y|$ for all $x, y \in X$ and $P = \{x \in E : x \geq 0\}$. Then (X,d) is a complete cone metric space. Define $q : X \times X \to E$ by $q(x,y) = y$ for all $x, y \in X$. Then q is a c-distance on X. Let $F : X \times X \to X$ be given by

$$F(x,y) = \begin{cases} \frac{1}{16}(x-y) & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

Define $k : X \times X \to [0,1)$ by $k(x,y) = Max\{\frac{1+x}{16}, \frac{1+y}{16}\}$ for all $x, y \in X$ and let $g(x) = x$ for all $x \in X$. Then F becomes mixed g-monotone function. Now $k(F(x,y), F(y,x)) = Max\{\frac{1+F(x,y)}{16}, \frac{1+F(y,x)}{16}\} \leq Max\{1+F(x,y), 1+F(y,x)\} \leq 1 + F(x,y) + F(y,x) = 1 + \frac{|x-y|}{16} \leq Max\{\frac{1+x}{16}, \frac{1+y}{16}\} = k(x,y)$ for all $x, y \in X$.Also $q(F(x,y), F(u,v)) + q(F(y,x), F(v,u)) = F(u,v) + F(v,u) = \frac{|u-v|}{16} \leq \frac{1}{16}(u+v) \leq Max\{\frac{1+x}{16}, \frac{1+y}{16}\}(q(x,u) + q(y,v)) = k(x,y)(q(x,u) + q(y,v))$ for all $x, y, u, v \in X$. Further $1 \leq F(1,0)$ and $F(0,1) \leq 0$. So all the conditions of Theorem 3.3. are satisfied. We see that $(0,0)$ is a coupled fixed point of F.

Remark 3.23. Theorem 2.2 of [29] is a particular cases of Theorem 3.1. for $k(x,y) = k$ and Corollary 2.9 of [29] is a particular case of Theorem 3.1. for $k(x,y) = k$ and $g = I_X$.

Remark 3.24. Theorem 2.4 of [29] is a particular cases of Theorem 3.3. for $k(x, y) = k$ and Corollary 2.10 of [29] is a particular case of Theorem 3.3. for $k(x, y) = k$ and $g = I_X$.

Remark 3.25. Corollary 2.5 of [29] is a particular cases of Theorem 3.5. for $k(x, y) = k$ and Corollary 2.6 of [29] is a particular case of Theorem 3.5. for $k(x, y) = k$ and $g = I_X$.

Remark 3.26. Corollary 2.7 of [29] is a particular cases of Theorem 3.9. for $k(x, y) = k$ and Corollary 2.8 of [29] is a particular case of Theorem 3.9. for $k(x, y) = k$ and $g = I_X$.

References

[1] M. Abbas, Y.J. Cho and T. Nazir, Common fixed point theorems for four mappings in tvs-valued cone metric spaces, J. Math. Inequal. 5(2) (2011), 287-299.

[2] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341(1) (2008), 416-420.

[3] M. Abbas and B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 22(4) (2009), 511-515.

[4] I. Altun and V. Rakočević, Ordered cone metric spaces and fixed point results, Comput. Math. Appl. 60(5) (2010), 1145-1151.

[5] A. Amini-Harandi and M. Fakhar, Fixed point theory in cone metric spaces obtained via the scalarization method, Comput. Math. Appl. 59(11) (2010), 3529-3534.

[6] R. Batra and S. Vashistha, Coupled coincidence point theorems for nonlinear contractions under c-distance in cone metric spaces, Ann. Funct. Anal. 4(1) (2013), 138-148.

[7] R. Batra and S. Vashistha, Coupled coincidence point theorems for nonlinear contractions under (F, g)-invariant set in cone metric spaces, J. Nonlinear Sci. Appl. 6(2) (2013), 86-96.

[8] T.G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(7) (2006), 1379-1393.

[9] Y.J. Cho, G. He, and N.J. Huang, The existence results of coupled quasi-solutions for a class of operator equations, Bull. Korean Math. Soc. 47(3) (2010), 455-465.

[10] Y.J. Cho, R. Saadati, and S. Wang, *Common fixed point theorems on generalized distance in ordered cone metric spaces*, Comput. Math. Appl. 61(4) (2011), 1254-1260.

[11] Y.J. Cho, M.H. Shah, and N. Hussain, Coupled fixed points of weekly f-contractive mappings in topological spaces, Appl. Math. Lett. 24(7) (2011), 1185-1190.

[12] Z.M. Fadail and A.G.B. Ahmad, Common coupled fixed point theorems of single valued mapping for c-distance in cone metric spaces, Abstra. Appl. Anal. 2012 (2012), Article ID 901792, 24 pages.

[13] Z.M. Fadail and A.G.B. Ahmad, Coupled fixed point theorems of single valued mapping for c-distance in cone metric spaces, J. Appl. Math. 2012 (2012), Article ID 246516, 20 pages.

[14] Z.M. Fadail and A.G.B. Ahmad, New fixed point results of single valued mapping for c-distance in cone metric spaces, Abstra. Appl. Anal. 2012 (2012), Article ID 639713, 12 pages.

[15] M.E. Gordji, Y.J. Cho, and H. Baghani, Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces, Math. Comput. Modelling 54(9-10) (2011), 1897-1906.

[16] E. Graily, S.M. Vaezpour, R. Saadati, and Y.J. Cho, Generalization of fixed point theorems in ordered metric spaces concerning generalized distance, Fixed Point Theory Appl. 2011:30 (2011), 8 Pages.

[17] L.G. Huang and X. Zhang, Cone meric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332(2) (2007), 1468-1476.

[18] D. Ilić and V. Rakočević, Common fixed points for maps on cone metri space, J. Math. Anal. Appl. 341(2) (2008), 876-882.

[19] S. Janković, Z. Kadelburg, and S. Radenović, On cone metric spaces:a survey, Nonlinear Anal. 74(7) (2011), 2591-2601.

[20] O. Kada, T. Suzuki, and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. 44(2) (1996), 381-391.

[21] E. Karapinar, Coupled fixed point theorems for nonlinear contractions in cone metric spaces, Comput. Math. Appl. 59(12) (2010), 3656-3668.

[22] V. Lakshmikantham and L. Cirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70(12) (2009), 4341-4349.

[23] J.J. Nieto and R. Rodriguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sin. 23(12) (2007), 2205-2212.

[24] M. Pavlović Z. Kadelburg and S. Radenović, Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces, Comput. Math. Appl. 59(9) (2010), 3148-3159.

[25] S. Radenović and B.E. Rhoades, Fixed point theorems for two non-self mappings in cone metric spaces, Comput. Math. Appl. 57(10) (2009), 1701-1707.

[26] A.C.M. Ran and M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132(5) (2004), 1435-1443.

[27] W. Shatanawi, Partially ordered cone metric spaces and coupled fixed point results, Comput. Math. Appl. 60(8) (2010), 2508-2515.

[28] W. Shatanawi, Some common coupled fixed point results in cone metric spaces, Intern. J. Math. Analysis 4(45-48) (2010), 2381-2388.

[29] W. Shatanawi, E. Karapinar and H. Aydi, Coupled coincidence points in partially ordered cone metric spaces with a c-distance, J. Appl. Math. 2012 (2012), Article ID 312078, 15 pages.

[30] W. Sintunavarat, Y.J. Cho, and P. Kumam, Common fixed point theorems for c-distance in ordered cone metric spaces, Comput. Math. Appl. 62(4) (2011), 1969-1978.

[31] W. Sintunavarat, Y.J. Cho, and P. Kumam, Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces, Fixed Point Theory Appl. 2011 (2011), p.81.

[32] W. Sintunavarat, Y.J. Cho, and P. Kumam, Coupled fixed point theorems for weak contraction mappings under F-invariant set , Abstra. Appl. Anal., 2012 (2012) 15 pages, doi:10.1155/2012/324874.

[33] D. Turkoglu and M. Abuloha, Cone metric spaces and fixed point theorems in diametrically contractive mappings, Acta Math. Sin. 26(3) (2010), 489-496.

[34] D. Turkoglu, M. Abuloha, and T. Abdeljawad, KKM mappings in cone metric spaces and some fixed point theorems, Nonlinear Anal. 72(1) (2010), 348-353.

[35] S. Wang and B. Guo, Distance in cone metric spaces and common fixed point theorems, Appl. Math. Lett. 24 (2011), 1735-1739.