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Engineering Mathematics Letters, 2 (2013), No. 2, 115-123

ISSN 2049-9337

HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS WITH RESPECT TO OTHER POINTS

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Abstract: This paper is concerned with the estimate of second Hankel determinant for the classes of analytic-univalent functions with respect to conjugate points and with respect to symmetric conjugate points in the unit disc $E=\{z:|z|<1\}$.

Keywords: Analytic functions; starlike functions; convex functions; conjugate points; Hankel determinant; coefficient bounds.

Mathematics Subject classification: 30C45, 30C50

1. Introduction

Let A be the class of analytic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \qquad (1.1)$$

in the unit disc $E=\{z:|z|<1\}$.

Let S be the class of functions $f(z)\in A$ and univalent in E.

Received May 9, 2013

In 1959, Sakaguchi [16] introduced the class S_s^* consisting of functions of the form (1.1) and satisfying the condition

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\}>0, z\in E.$$

The functions of the class S_s^* are called starlike functions with respect to symmetric points.

In 1977, Das and Singh [2] defined the class K_s consisting of functions of the form (1.1) and satisfying the condition

$$\operatorname{Re}\left\{\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right\}>0, z\in E.$$

The functions of the class K_s are known as convex functions with respect to symmetric points.

Motivated from the work of Sakaguchi and Das and Singh, El-Ashwah and Thomas [4] defined the following classes:

$$S_c^*=\left\{f(z)\in A:\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right\}>0, z\in E\right\}. \tag{1.2}$$

$$S_{sc}^*=\left\{f(z)\in A:\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}\right\}>0, z\in E\right\}. \tag{1.3}$$

Functions in the classes S_c^* are called starlike functions with respect to conjugate points and that in the class S_{sc}^* are known as starlike functions with respect to symmetric conjugate points.

Again Janteng et al. [7] introduced the following classes:

$$K_c = \left\{ f(z) \in A : \text{Re}\left\{ \frac{2(zf'(z))'}{\left(f(z) + \overline{f(\bar{z})}\right)'} \right\} > 0, z \in E \right\}. \tag{1.4}$$

$$K_{sc} = \left\{ f(z) \in A : \text{Re}\left\{ \frac{2(zf'(z))'}{\left(f(z) - \overline{f(-\bar{z})}\right)'} \right\} > 0, z \in E \right\}. \tag{1.5}$$

Functions of the class K_c are convex functions with respect to conjugate points and that in the class K_{sc} are called convex functions with respect to symmetric conjugate points.

Obviously the functions in these classes are univalent. Various subclasses of analytic functions with respect to conjugate points and with respect to symmetric conjugate points were widely investigated by various authors including Dahar and Janteng [1], Selvaraj and Vasanthi [17], Ravichandran [15] and Tang and Deng [19].

In 1976, Noonan and Thomas [12] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor [13] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by Eq. (1.1) with bounded boundary. Ehrenborg [3] studied the Hankel determinant of exponential polynomials. Also Hankel determinant was studied by various authors including Hayman [6] and Pommerenke[14]. Janteng et al. [8,9] studied the Hankel determinant for the classes of starlike functions, convex functions, starlike functions with respect to symmetric points, convex functions with respect to symmetric points. Recently Singh [18] established the hankel determinant for various subclasses of analytic functions with respect to symmetric points.

Easily, one can observe that the Fekete and Szegö functional is $H_2(1)$. Fekete and Szegö [5] then further generalised the estimate $\left|a_3 - \mu a_2^2\right|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

In this paper, we established the sharp upper bound of the functional $\left|a_2 a_4 - a_3^2\right|$ for functions belonging classes S_c^*, S_{sc}^*, K_c and K_{sc}.

2. Preliminary Results

Let P be the family of all functions p analytic in E for which $\operatorname{Re}(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{2.1}$$

for $z \in E$.

Lemma 2.1.[14] If $p \in P$, then $|p_k| \leq 2 (k = 1,2,3,\ldots)$.

Lemma 2.2.[10,11] If $p \in P$, then

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z,$$

for some x and z satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0,2]$.

3. Main Result

Theorem 3.1 If $f \in S_c^*$, then

$$\left|a_2 a_4 - a_3^2\right| \le 1 \tag{3.1}$$

Proof. As $f \in S_c^*$, so from (1.2)

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}} = p(z). \tag{3.2}$$

On equating coefficients of z, z^2 and z^3 in the expansion of (3.2), we obtain

$$\left.\begin{aligned} a_2 &= p_1, \\ a_3 &= \frac{p_2}{2} + \frac{p_1^2}{2}, \\ a_4 &= \frac{p_3}{3} + \frac{p_1 p_2}{2} + \frac{p_1^3}{6} \end{aligned}\right\}. \tag{3.3}$$

(3.3) yields,

$$a_2 a_4 - a_3^2 = \frac{1}{12}\left\{4p_1 p_3 - p_1^4 - 3p_2^2\right\}. \tag{3.4}$$

Using Lemma 2.1 and Lemma 2.2 in (3.4) , we obtain

$$\left|a_2 a_4 - a_3^2\right| = \frac{1}{48}\left|-3p_1^4 + 2p_1^2\left(4-p_1^2\right)x - \left[12+p_1^2\right]\left(4-p_1^2\right)x^2 + 8p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\right|.$$

Assume that $p_1 = p$ and $p \in [0,2]$, using triangular inequality and $|z| \le 1$, we have

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{48}\left\{3p^4 + 2p^2\left(4-p^2\right)|x| + \left[p^2+12\right]\left(4-p^2\right)|x|^2 + 8p\left(4-p^2\right)\left(1-|x|^2\right)\right\}$$

or

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{48}\left\{3p^4 + 8p\left(4-p^2\right) + 2p^2\left(4-p^2\right)\delta + \left[p^2 - 8p + 12\right]\left(4-p^2\right)\delta^2\right\}$$

$$= \frac{1}{48} F(\delta), \text{ where } \quad \delta = |x| \leq 1 \text{ and}$$

$F(\delta) = 3p^4 + 8p(4 - p^2) + 2p^2(4 - p^2)\delta + [p^2 - 8p + 12](4 - p^2)\delta^2$ is an increasing function.

Therefore $\quad Max.F(\delta) = F(1) = 48.$

Hence
$$|a_2 a_4 - a_3^2| \leq 1.$$

The result is sharp for $\dfrac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} = \dfrac{1+z}{1-z}$ and $\dfrac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} = \dfrac{1+z^2}{1-z^2}$.

On the same lines, we can easily prove the following theorem:

Theorem 3.2 If $f \in K_c$, then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8}.$$

The result is sharp for $p_1 = 1$, $p_2 = -1$ and $p_3 = -2$.

Theorem 3.3 If $f \in S_{sc}^*$, then

$$|a_2 a_4 - a_3^2| \leq 1 \tag{3.5}$$

Proof. As $f \in S_{sc}^*$, so from (1.3)

$$\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} = p(z). \tag{3.6}$$

On equating coefficients of z, z^2 and z^3 in the expansion of (3.6), we obtain

$$\left.\begin{aligned} a_2 &= \frac{p_1}{2}, \\ a_3 &= \frac{p_2}{2}, \\ a_4 &= \frac{p_3}{4} + \frac{p_1 p_2}{8} \end{aligned}\right\}. \tag{3.7}$$

(3.7) yields,

$$a_2 a_4 - a_3^2 = \frac{1}{16}\left\{2p_1 p_3 + p_1^2 p_2 - 4p_2^2\right\}. \tag{3.8}$$

Using Lemma 2.1 and Lemma 2.2 in (3.8) , we obtain

$$\left|a_2 a_4 - a_3^2\right| = \frac{1}{32}\left|-p_1^2\left(4-p_1^2\right)x - \left[8-p_1^2\right]\left(4-p_1^2\right)x^2 + 2p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\right|.$$

Assume that $p_1 = p$ and $p \in [0,2]$, using triangular inequality and $|z| \le 1$, we have

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{32}\left\{2p\left(4-p^2\right) + p^2\left(4-p^2\right)\delta + \left[-p^2 - 2p + 8\right]\left(4-p^2\right)\delta^2\right\}$$

$$= \frac{1}{32}F(\delta), \text{ where } \quad \delta = |x| \le 1 \text{ and}$$

$F(\delta) = 2p\left(4-p^2\right) + p^2\left(4-p^2\right)\delta + \left[-p^2 - 2p + 8\right]\left(4-p^2\right)\delta^2$ is an increasing function.

Therefore $\quad Max.F(\delta) = F(1).$

Consequently
$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{32}G(p), \tag{3.9}$$

where

$$G(p) = F(1).$$

So $\quad G(p) = -8p^2 + 32 \quad$ and $\quad Max.G(p) = G(0).$

Hence from (3.9), we obtain (3.5).

The result is sharp for $p_1 = 0$, $p_2 = 2$ and $p_3 = 0.$

Using the above technique, the proof of the following theorem is obvious.

Theorem 3.4 If $f \in K_{sc}$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{9}.$$

The result is sharp for $p_1 = 0$, $p_2 = 2$ and $p_3 = 0.$

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