WEAKLY C-CONTRACTIVE MAPPINGS IN CONE METRIC SPACES

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Abstract. In this article, we introduce the class of weakly c-contractive mappings in cone metric spaces. A fixed point theorem is established in the framework of cone metric spaces.

Keywords: cone metric space; C-contractive mappings; fixed point.

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1. Introduction

Recently, Huang and Zhang introduced the concept of cone metric spaces by replacing the set of real numbers with an ordered Banach space, for more details; see [4] and the references therein. Subsequently, many fixed point results concerning self mappings in such spaces have been investigated; see [2, 3, 5, 6, 7, 9, 10] and the references therein. In this article, we extend some results in [1] to the framework of cone metric spaces. In this paper, the cones are strongly minihedral and normal to endow the cone metric spaces with an appropriate topology; see [11].

The aim of this paper is to investigate fixed point problems of C-contractive mappings. A fixed point theorem is established in the framework of cone metric spaces.

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The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a fixed point theorem is established in the framework of cone metric spaces. The result presented in this paper mainly generalizes the result of Binayak [1].

2. Preliminaries

We first recall some known definitions, notations and results concerning cones in Banach spaces.

**Definition 2.1.** Let $E$ be a real Banach space with norm $\|\cdot\|$ and let $P$ be a subset of $E$. Then $P$ is called a cone if and only if

1. $P$ is closed, nonempty and $P \neq \{\theta\}$, where $\theta$ is the zero vector in $E$;
2. for any $a, b \geq 0$ (nonnegative real numbers), and $x, y \in P$, we have $ax + by \in P$;
3. for $x \in P$, if $-x \in P$, then $x = \theta$.

Given a cone $P$ in a Banach space $E$, we define on $E$ a partial order $\preceq$ with respect to $P$ by

$$x \preceq y \iff y - x \in P.$$ 

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int}(P)$, where $\text{Int}(P)$ stand for the interior of $P$.

The cone $P$ is called normal if there is a real number $K > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \implies \|x\| \leq K\|y\|.$$ 

The least positive number satisfying this inequality is called the normal constant of $P$. Therefore, we shall say that $P$ is a $K$-normal cone to indicate the fact that the normal constant is $K$.

The cone is said to be regular if every increasing sequence which is bounded from above is convergent. That is, if $(x_n)$ is a sequence such that $x_n \preceq x_2 \preceq \cdots \preceq y$ for some $y \in E$, then there exists $x^* \in E$ such that $\lim_{n \to \infty} \|x_n - x^*\| = 0$. 
Lemma 2.1. [13, 15] Every regular cone is normal. The cone $P$ is regular if every decreasing sequence which is bounded from below is convergent.

Definition 2.2. Let $X$ be a non empty set. A function $d : X \times X \to E$ is called a cone metric on $X$ if:

1. $\theta \preceq d(x, y)$ $\forall x \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ $\forall x, y \in X$;
3. $d(x, z) \preceq d(x, y) + d(y, z)$ $\forall x, y, z \in X$.

The pair $(X, d)$ is called a cone metric space.

From the definition of the order given by a cone $P$, it is obvious that $x \in P \iff \theta \preceq x$. Hence, we can define a concept of positivity on a Banach space as follow.

Definition 2.3. Let $E$ be a real Banach space. Let $P$ be a cone on $E$ and $\preceq$ the partial order with respect to $P$. An element $x \in E$ is said to be a nonnegative vector if $\theta \preceq x$ and positive vector if $\theta \prec x$. Hence $P$ is the set of all nonnegative elements. We shall use the following notations:

- $[\theta, \rightarrow] := \{x \in E : \theta \preceq x\}$
- $]\theta, \rightarrow[ := \{x \in E : \theta \prec x\}$

Definition 2.4. A subset $A$ of $E$ is said to be bounded from above with respect to $P$ (or upper bounded) if there exists $x_0 \in E$ such that $a \preceq x_0$ for all $a \in A$. A subset $A$ of $E$ is said to be bounded from below with respect to $P$ (or lower bounded) if there exists $x_0 \in E$ such that $x_0 \preceq a$ for all $a \in A$.

Definition 2.5. A cone $P$ is said to be minihedral if $x \vee y := \sup\{x, y\}$ exists for all $x, y \in E$ and strongly minihedral if every subset of $E$ which is bounded from above has a supremum.

We also recall the following lemma, which we take from [11] and give the proof as it is there.

Lemma 2.6. Let $(X, d)$ be a quasi-cone metric space. Then for each $c \in E$, $c \gg \theta$, there exists $\sigma > 0$ such that $x \ll c$ whenever $\|x\| < \sigma$, $x \in E$.

Proof. Since $c \gg \theta$, we have $c \in \text{Int}(P)$. Hence, we find $\sigma > 0$ such that $\{x \in E : \|x - c\| < \sigma\} \subset \text{Int}(P)$. If $\|x\| < \sigma$, then $\|(c - x) - c\| = \|-x\| = \|x\| < \sigma$ and hence $(c - x) \in \text{Int}(P)$.

Lemma 2.7. Let $(X, d)$ be a cone metric space over a cone $K$-normal cone $P$. Then one has
a) $\text{Int}(P) + \text{Int}(P) \subset \text{Int}(P)$ and $\lambda \text{Int}(P) \subset \text{Int}(P)$ for any positive real number $\lambda$.

b) For any given $c \gg \theta$ and $c_0 \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $\frac{c_0}{n_0} \ll c$.

c) If $(a_n)$ and $(b_n)$ are sequences in $E$ such that $a_n \to a$, $b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.

**Proposition 2.8.** Let $(X, q)$ be a cone metric space over a cone $P$. If $a \preceq \lambda a$, where $0 \leq \lambda < 1$, then $a = \theta$.

**Definition 2.9.** Let $(x_n)$ be a sequence in a cone metric space $(X, d)$.

(a) $(x_n)$ is convergent to $x \in X$ and we denote $\lim_{n \to \infty} x_n = x$, if for every $c \in E$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \quad d(x_n, x) \ll c;$$

(b) $(x_n)$ is called Cauchy if for every $c \in E$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \quad d(x_n, x_m) \ll c.$$

**Definition 2.10.** A cone metric space $(X, d)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

**Lemma 2.11.** [4] Let $(X, d)$ be a cone metric space over a cone $K$-normal cone $P$. The sequence $(x_n)$ converges to $x \in X$ if and only if $\lim_{n \to \infty} d(x_n, x) = \theta$. The sequence $(x_n)$ is Cauchy if and only if $\lim_{n, m \to \infty} d(x_n, x_m) = \theta$.

Throughout this paper, we shall assume that the cones are strongly minihedral and $K$-normal, hence regular. Except otherwise stated, the notation $\preceq$ designates the partial order with respect to $P$. Furthermore, we shall assume that $\text{Int}(P) \neq \emptyset$.

We conclude this section by the following proposition.

**Proposition 2.12.** [11] Every cone metric space $(X, d)$ is a topological space.

3. Main results

In [1], Binayak proved the following result.
Theorem B. Let $T : X \to X$, where $(X,d)$ is a complete metric space, be a weak $C$-contraction. Then $T$ has a fixed point.

We generalize this result in the setting of cone metric spaces in this section.

Definition 3.1. A mapping $T : X \to X$, where $(X,d)$ is a complete cone metric space, is said to be a weakly $C$-contractive or a weak $C$-contraction if for all $x, y \in X$,

\[(0.1) \quad d(Tx, Ty) \leq \frac{1}{2} [d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)),\]

where $\psi : P \times P \to P$ is a continuous mapping such that $\psi(x, y) = \theta$ if and only if $x = y = \theta$.

Lemma 3.1. Let $(X,d)$ be a cone metric space over a $K$-normal cone $P$. Then for any $c \in P$ and any $a \in E$, $a - c \leq a$.

Proof. Indeed, we have

$$\theta \leq c \iff c \in P \iff a - (a - c) \in P \iff a - c \leq a.$$ 

Theorem 3.3. Let $T : X \to X$, where $(X,d)$ is a complete cone metric space, be a weak $C$-contraction. Then $T$ has a fixed point.

Proof. Let $(x_n)$ be a sequence generated in the iteration $x_{n+1} = Tx_n$. If $x_n = x_{n+1} = Tx_n$, then $x_n$ is a fixed point of $T$. Next, we assume $x_n \neq x_{n+1}$. Using (0.1), we have

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\
\leq \frac{1}{2} [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] - \psi(d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})) \\
= \frac{1}{2} d(x_{n-1}, x_{n+1}) - \psi(d(x_{n-1}, x_{n+1}), \theta) \\
\leq \frac{1}{2} (d(x_{n-1}, x_n) + d(x_n, x_{n+1})) - \psi(d(x_{n-1}, x_{n+1}), \theta).
\]

(0.2)

Using (0.2), we find from Lemma 3.1 that

\[(0.3) \quad d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).\]

Thus $(d(x_n, x_{n+1}))$ is a monotone decreasing sequence in $E$. Moreover, this sequence is bounded below by $\theta$ and since $P$ is regular, the sequence $(d(x_n, x_{n+1}))$ is convergent. Let $d(x_n, x_{n+1}) \to r$
as \( n \to \infty \). Next we prove that \( r = \theta \).

\[
\begin{align*}
    d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
    &\leq \frac{1}{2} (d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})) - \psi(d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\
    &\leq \frac{1}{2} d(x_{n-1}, x_{n+1}) \\
    &\leq \frac{1}{2} (d(x_{n-1}, x_n) + d(x_n, x_{n+1})).
\end{align*}
\]

Letting \( n \to \infty \), we see that \( r \leq \lim_{n \to \infty} \frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2} r + \frac{1}{2} r, \)

or

\[
(0.4) \quad \lim_{n \to \infty} d(x_{n-1}, x_{n+1}) = 2r.
\]

Letting \( n \to \infty \) in (0.2) and using (0.4) and the continuity of \( \psi \), we have

\[
r \leq r - \psi(2r, \theta)
\]

or

\[
-\psi(2r, \theta) \in P,
\]

which is a contradiction unless \( r = \theta \). Thus we have established that

\[
(0.5) \quad d(x_n, x_{n+1}) \to \theta \text{ as } n \to \infty.
\]

Next we show that \((x_n)\) is a Cauchy sequence. If otherwise, then there exists \( \varepsilon \gg \theta \) and increasing sequences of integers \((m(k))\) and \((n(k))\) such that for all integers \(k\), \(n(k) > m(k)\),

\[
d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \text{ and } d(x_{m(k)}, x_{n(k)-1}) \ll \varepsilon.
\]

Then,

\[
\begin{align*}
    \varepsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
    &= d(Tx_{m(k)-1}, Tx_{n(k)-1}) \\
    &\leq \frac{1}{2} (d(x_{m(k)-1}, Tx_{n(k)-1}) + d(x_{n(k)-1}, Tx_{m(k)-1})) \\
    &\quad - \psi(d(x_{m(k)-1}, Tx_{n(k)-1}), d(x_{n(k)-1}, Tx_{m(k)-1})) \text{ by (0.1)} \\
    &= \frac{1}{2} (d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)})) - \psi(d(x_{m(k)-1}, x_{n(k)}), d(x_{n(k)-1}, x_{m(k)})).
\end{align*}
\]

\[
(0.6)
\]
Again, we have

\[ \varepsilon \preceq d(x_{m(k)}, x_{n(k)}) \]

\[ \preceq d(x_{m(k)}, x_{n(k)} - 1) + d(x_{n(k)} - 1, x_{n(k)}) \]

\[ \preceq \varepsilon + d(x_{n(k)} - 1, x_{n(k)}). \]

Letting \( k \to \infty \) in the above inequality and using (0.5), we obtain

(0.7) \[ \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon \]

and

(0.8) \[ \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)} - 1) = \varepsilon. \]

Indeed, we also have

\[ d(x_{m(k)}, x_{n(k)} - 1) \preceq d(x_{m(k)} - 1, x_{m(k)}) + d(x_{m(k)} - 1, x_{n(k)} - 1). \]

Note that

\[ d(x_{m(k)} - 1, x_{n(k)}) \preceq d(x_{m(k)} - 1, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}). \]

Letting \( k \to \infty \) in the above two inequalities and using (0.5), (0.7) and (0.8) we get

(0.9) \[ \lim_{k \to \infty} d(x_{m(k)} - 1, x_{n(k)}) = \varepsilon. \]

Next, letting \( k \to \infty \) in (0.6) and using (0.5), (0.8) and (0.9) we obtain

\[ \varepsilon \preceq \frac{1}{2} (\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon). \]

Or \( \psi(\varepsilon, \varepsilon) \preceq \theta \), which is a contradiction since \( \varepsilon \gg \theta \). Hence \( (x_n) \) is a Cauchy sequence and therefore is convergent in the complete cone metric space \((X, d)\). Let \( x_n \to z \) as \( n \to \infty \). We
prove that \( z \) is a fixed point for \( T \). Indeed, we have

\[
d(z, Tz) \leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \\
\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\
\leq d(z, x_{n+1}) + \frac{1}{2}(d(z, Tx_n)) - \psi(d(z, x_n), d(x_n, Tz)) \\
= d(z, x_{n+1}) + \frac{1}{2}(d(z, x_{n+1}) + d(x_n, Tz)) - \psi(d(z, x_{n+1}), d(x_n, Tz)).
\]

Letting \( n \to \infty \), using the continuity of \( \psi \), we obtain

\[
d(z, Tz) \leq \frac{1}{2}d(z, Tz) - \psi(\theta, d(z, Tz)) \leq \frac{1}{2}d(z, Tz),
\]

which is a contradiction unless \( d(z, Tz) = \theta \). Hence \( z = Tz \).

Next we establish that the fixed point \( z \) is unique. If \( z_1 \) and \( z_2 \) are two fixed points of \( T \), then

\[
d(z_1, z_2) = d(Tz_1, Tz_2) \leq \frac{1}{2}(d(z_1, Tz_2) + d(z_2, Tz_1)) - \psi(d(z_1, Tz_2), d(z_2, Tz_1)).
\]

That is,

\[
d(z_1, z_2) \leq d(z_1, z_2) - \psi(d(z_1, z_2), d(z_1, z_2)) < d(z_1, z_2),
\]

which by property of \( \psi \) is a contradiction unless \( d(z_1, z_2) = \theta \), that is, \( z_1 = z_2 \). This completes the proof.

**Conflict of Interests**

The author declares that there is no conflict of interests.

**REFERENCES**


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