CR-SUBMANIFOLDS OF A NEARLY HYPERBOLIC KENMOTSU MANIFOLD ADMITTING A QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. We consider a nearly hyperbolic Kenmotsu manifold with a quarter symmetric non-metric connection and study CR- submanifolds of a nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection. We also study parallel distributions on nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection and find the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection.

Keywords: CR-submanifolds, nearly hyperbolic Kenmotsu manifold, quarter symmetric non-metric connection, integrability conditions, parallel distribution.

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1. Introduction

The notion of CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was introduced and studied by A. Bejancu in ([1], [2]). Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J. Hsu in [5] and M. Kobayashi in [18]. CR-submanifolds of Kenmotsu manifold was studied by A. Bejancu and N. Papaghuic in [4]. Later, several geometers (see, [9], [12] [13], [15] [16]) enriched the study of CR-
submanifolds of almost contact manifolds. The almost hyperbolic \((f,g,\eta,\xi)\)-structure was defined and studied by Upadhyay and Dube in [17]. Dube and Bhatt studied CR-submanifolds of trans-hyperbolic Sasakian manifold in [10]. On the other hand, S. Golab introduced the idea of semi-symmetric and quarter symmetric connections in [8]. CR-submanifolds of LP-Sasakian manifold with quarter symmetric non-metric connection were studied by the first author and S.K. Lovejoy Das in [11]. CR-submanifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connection were studied by the first author, M.D. Siddiqi and S. Rizvi in [14]. In this paper, we study some properties of CR-submanifolds of a nearly hyperbolic Kenmotsu manifold with a quarter symmetric non-metric connection.

2. Preliminaries

Let \(\bar{M}\) be an \(n\)-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure-\((\phi,\xi,\eta, g)\), where a tensor \(\phi\) of type \((1,1)\), a vector field \(\xi\), called structure vector field and \(\eta\), the dual 1-form of \(\xi\) satisfying the followings

\[
\begin{align*}
(2.1) \quad & \phi^2 X = X + \eta(X)\xi, \quad g(X, \xi) = \eta(X), \\
(2.2) \quad & \eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta \phi = 0, \\
(2.3) \quad & g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y)
\end{align*}
\]

for any \(X, Y\) tangent to \(M\) [17]. In this case

\[
(2.4) \quad g(\phi X, Y) = -g(Y, \phi X).
\]

An almost hyperbolic contact metric structure-\((\phi,\xi,\eta, g)\) on \(\bar{M}\) is called hyperbolic Kenmotsu manifold [7] if and only if

\[
(2.5) \quad (\nabla_X \phi) Y = g(\phi X, Y) \xi - \eta(Y) \phi X
\]

for all \(X, Y\) tangent to \(\bar{M}\). On a hyperbolic Kenmotsu manifold \(\bar{M}\), we have

\[
(2.6) \quad \nabla_X \xi = X + \eta(X)\xi
\]

for a Riemannian metric \(g\) and Riemannian Connection \(\nabla\).
Further, an almost hyperbolic contact metric manifold $\bar{M}$ on $(\phi, \xi, \eta, g)$ is called nearly-hyperbolic Kenmotsu [7] if

\begin{equation}
(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X.
\end{equation}

Now, Let $M$ be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on $M$ is denoted by the same symbol $g$. Let $TM$ and $T^\perp M$ be the Lie algebra of vector fields tangential to $M$ and normal to $M$ respectively and $\nabla^*$ be the induced Levi-Civita connection on $M$, then the Gauss and Weingarten formulas are given respectively by

\begin{equation}
\nabla_X Y = \nabla_X^* Y + h(X, Y),
\end{equation}

\begin{equation}
\nabla_X N = -A_N X + \nabla_X^* N
\end{equation}

for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^*$ is a connection on the normal bundle $T^\perp M$, $h$ is the second fundamental form and $A_N$ is the Weingarten map associated with $N$ as

\begin{equation}
g(A_N X, Y) = g(h(X, Y), N)
\end{equation}

for any $x \in M$ and $X \in T_x M$. We write

\begin{equation}
X = PX + QX,
\end{equation}

where $PX \in D$ and $QX \in D^\perp$.

Similarly, for $N$ normal to $M$, we have

\begin{equation}
\phi N = BN + CN,
\end{equation}

where $BN$ (resp. $CN$) is the tangential component (resp. normal component) of $\phi N$.

Now, we define a quarter symmetric non-metric connection by

\begin{equation}
(\bar{\nabla}_X Y) = \nabla_X Y + \eta(Y)\phi X
\end{equation}

such that

\begin{equation}
(\bar{\nabla}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y).
\end{equation}

Using (2.13) and (2.7), we have

\begin{equation}
(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi.
\end{equation}
An almost hyperbolic contact manifold $\bar{M}$ satisfying (2.14) is called nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection.

For a nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection, we have

$$(2.15) \bar{\nabla}_x \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi) + \phi X.$$  

Gauss and Weingarten formula for nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection are given respectively by

$$(2.16) \bar{\nabla}_x Y = \nabla_x Y + h(X,Y),$$

$$(2.17) \bar{\nabla}_x N = -A_N X + \nabla^\perp_X N.$$  

**Definition 2.1.** An m-dimensional submanifold $M$ of an n-dimensional nearly hyperbolic Kenmotsu manifold $\bar{M}$ is called a CR-submanifold if there exists a differentiable distribution $D: x \rightarrow D_x$ on $M$ satisfying the following conditions:

i. $D$ is invariant, that is $\phi D_x \subseteq D_x$ for each $x \in M$.

ii. The complementary orthogonal distribution $D^\perp$ of $D$ is anti-invariant, that is $\phi D^\perp_x \subseteq T^\perp_x M$.

If $\dim D^\perp_x = 0$ (resp., $\dim D_x = 0$), then the CR-Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution $D$ (resp., $D^\perp$) is called the horizontal (resp., vertical) distribution. Also, the pair $(D,D^\perp)$ is called $\xi$–horizontal (resp. vertical) if $\xi_x \in D_x$ (resp., $\xi_x \in D^\perp_x$).

3. Some basic lemmas

**Lemma 3.1.** If $M$ be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then

$$(3.1) -\eta(X)\phi PY - \eta(Y)\phi PX - \eta(X)PY - \eta(Y)PX - 2\eta(X)\eta(Y)P\xi + P\phi(\nabla_X Y)$$

$$+P\phi(\nabla_Y X) = P\nabla_X (\phi PY) + P\nabla_Y (\phi PX) - PA_{\phi QY} X - PA_{\phi QX} Y,$$

$$(3.2) -\eta(X)QY - \eta(Y)QX + 2\eta(X)\eta(Y)Q\xi + 2Bh(X,Y) = Q\nabla_X (\phi PY) + Q\nabla_Y (\phi PY)$$
\[-Q\nabla_{\phi}QY \cdot X - Q\nabla_{\phi}QX \cdot Y,\]

\[(3.3) \quad -\eta(X)\phi QY - \eta(Y)\phi QX - \phi Q(\nabla_X Y) + \phi Q(\nabla_Y X) + 2\xi h(X, Y) = h(X, \Phi Y) + h(Y, \Phi X) + \nabla^\perp_X(\phi QY) + \nabla^\perp_Y(\phi QX)\]

for any \(X, Y \in T(M)\).

**Proof.** From (2.11), we have

\[\Phi Y = \Phi PY + \phi QY.\]

Differentiating covariantly and using (2.16) and (2.17), we get

\[(\nabla_X \phi) Y + \phi (\nabla_X Y) + \phi h(X, Y) = \nabla_X (\Phi PY) + h(X, \Phi PY) - A_{\Phi QY}X + \nabla^\perp_X(\phi QY).\]

Interchanging \(X\) and \(Y\), we have

\[(\nabla_Y \phi) X + \phi (\nabla_Y X) + \phi h(Y, X) = \nabla_Y (\Phi PX) + h(Y, \Phi PX) - A_{\Phi QX}Y + \nabla^\perp_Y(\phi QX).\]

Adding above two equations, we obtain

\[
(\nabla_X \phi) Y + (\nabla_Y \phi) X + \phi (\nabla_X Y) + \phi (\nabla_Y X) + 2\phi h(Y, X) = \nabla_X (\Phi PY) + \nabla_Y (\Phi PX) + h(X, \Phi PY) + h(Y, \Phi PX) - A_{\Phi QY}X - A_{\Phi QX}Y + \nabla^\perp_X(\phi QY) + \nabla^\perp_Y(\phi QX).
\]

Using (2.14) in above equation, we get

\[2\phi h(Y, X) = \nabla_X (\Phi PY) + \nabla_Y (\Phi PX) + h(X, \Phi PY) + h(Y, \Phi PX) - A_{\Phi QY}X - A_{\Phi QX}Y + \nabla^\perp_X(\phi QY) + \nabla^\perp_Y(\phi QX).\]

Comparing tangential, vertical and normal components from both sides of (3.4), we get the desired results.

Hence lemma is proved. □

**Lemma 3.2.** If \(M\) be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold \(\bar{M}\) with quarter symmetric non-metric connection. Then

\[(3.5) \quad 2(\nabla_X \phi) Y = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi + \nabla_X \phi Y\]
\[ +h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y], \]

\[(3.6) \quad 2(\nabla_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi - \nabla_X \phi Y - h(X, \phi Y) + \nabla_Y \phi X + h(Y, \phi X) + \phi [X, Y] \]

for any \(X, Y \in D\).

**Proof.** From Gauss formula (2.16), we get

\[(3.7) \quad \nabla_X \phi Y - \nabla_Y \phi X = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X). \]

Also, we have

\[(3.8) \quad \nabla_X \phi Y - \nabla_Y \phi X = (\nabla_X \phi)Y - (\nabla_Y \phi)X + \phi [X, Y]. \]

From (3.7) and (3.8), we have

\[(3.9) \quad (\nabla_X \phi)Y - (\nabla_Y \phi)X = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y]. \]

Adding (3.9) and (2.14), we get

\[ 2(\nabla_X \phi)Y = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi + \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y]. \]

Subtracting (2.14) from (3.9), we get

\[ 2(\nabla_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi - \nabla_X \phi Y - h(X, \phi Y) + \nabla_Y \phi X + h(Y, \phi X) + \phi [X, Y]. \]

Hence lemma is proved. □

**Corollary 3.3.** If \(M\) be a \(\xi\) – vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold \(\hat{M}\) with quarter symmetric non-metric connection. Then

\[ 2(\nabla_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y], \]

and \[ 2(\nabla_Y \phi)X = \nabla_Y \phi X - \nabla_X \phi Y + h(Y, \phi X) - h(X, \phi Y) + \phi [X, Y] \]

for any \(X, Y \in D\).
Lemma 3.4. If $M$ be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\tilde{M}$ with quarter symmetric non-metric connection. Then

\[(3.10)\quad 2(\overline{\nabla}_Y \phi)Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^1_\phi Z - \nabla^1_\phi Y - \phi[Y, Z] - \eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi,
\]

\[(3.11)\quad 2(\overline{\nabla}_Z \phi)Y = A_{\phi Z} Y - A_{\phi Y} Z + \nabla^1_\phi Y - \nabla^1_\phi Z - \phi[Y, Z] - \eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi
\]

for any $Y, Z \in D^\perp$.

Proof. Let $Y, Z \in D^\perp$. From Weingarten formula (2.17), we get

\[(3.12)\quad \overline{\nabla}_Y \phi Z - \overline{\nabla}_Z \phi Y = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^1_\phi Z - \nabla^1_\phi Y.
\]

Also, we have

\[(3.13)\quad \overline{\nabla}_Y \phi Z - \overline{\nabla}_Z \phi Y = (\overline{\nabla}_Y \phi)Z - (\overline{\nabla}_Z \phi)Y + \phi[Y, Z].
\]

From (3.12) and (3.13), we obtain

\[(3.14)\quad (\overline{\nabla}_Y \phi)Z - (\overline{\nabla}_Z \phi)Y = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^1_\phi Z - \nabla^1_\phi Y - \phi[Y, Z].
\]

For nearly hyperbolic Kenmotsu manifold, we have

\[(3.15)\quad (\overline{\nabla}_Y \phi)Z - (\overline{\nabla}_Z \phi)Y = -\eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi.
\]

Adding (3.14) and (3.15), we get

\[
2(\overline{\nabla}_Y \phi)Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^1_\phi Z - \nabla^1_\phi Y - \phi[Y, Z] - \eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi.
\]

Subtracting (3.14) from (3.15), we get

\[
2(\overline{\nabla}_Z \phi)Y = A_{\phi Z} Y - A_{\phi Y} Z + \nabla^1_\phi Y - \nabla^1_\phi Z - \phi[Y, Z] - \eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi
\]

for any $Y, Z \in D^\perp$. 
Hence lemma is proved. □

**Corollary 3.5.** If $M$ be a $\xi$ – horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric non-metric connection. Then

$$2(\overline{\nu}_Y \phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \overline{\nu}_Z^{-1} \phi Z - \overline{\nu}_Z^{-1} \phi Y - \phi[Z, Y],$$

and

$$2(\overline{\nu}_Z \phi)Y = A_{\phi Z}Y - A_{\phi Y}Z + \overline{\nu}_Y^{-1} \phi Y - \overline{\nu}_Z^{-1} \phi Z - \phi[Y, Z]$$

for any $Y, Z \in D^\perp$.

**Lemma 3.6.** If $M$ be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric non-metric connection. Then

$$2(\overline{\nu}_X \phi)Y = -A_{\phi Y}X + \overline{\nu}_X^1 \phi Y - \overline{\nu}_X \phi X - h(Y, \phi X) - \phi[X, Y] - \eta(X) \phi Y - \eta(Y) \phi X - \eta(X)Y - \eta(Y)X - 2 \eta(X) \eta(Y) \xi,$$

for any $X \in D$ and $Y \in D^\perp$.

**Proof.** Let $X \in D, Y \in D^\perp$

From Gauss and Weingarten formulae, we have

$$2(\overline{\nu}_X \phi)Y - \overline{\nu}_Y \phi X = -A_{\phi Y}X + \overline{\nu}_X^1 \phi Y - \overline{\nu}_Y \phi X - h(Y, \phi X).$$

Also, we have

$$2(\overline{\nu}_Y \phi)X - \overline{\nu}_X \phi Y = (\overline{\nu}_X \phi)Y - (\overline{\nu}_Y \phi)X + \phi[X, Y].$$

From (3.18) and (3.19), we get

$$2(\overline{\nu}_X \phi)Y - (\overline{\nu}_Y \phi)X = -A_{\phi Y}Z + \overline{\nu}_X^1 \phi Y - \overline{\nu}_Y \phi X - h(Y, \phi X) - \phi[X, Y].$$

Also, for nearly hyperbolic Kenmotsu manifold we have

$$2(\overline{\nu}_X \phi)Y + (\overline{\nu}_Y \phi)X = -\eta(X) \phi Y - \eta(Y) \phi X - \eta(X)Y - \eta(Y)X - 2 \eta(X) \eta(Y) \xi.$$
Adding (3.20) and (3.21), we obtain

\[ 2(\nabla_X \phi)Y = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] - \eta(X)\phi Y - \eta(Y)\phi X \]
\[ -\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi. \]

Subtracting (3.20) from (3.21), we find

\[ 2(\nabla_Y \phi)X = -A_{\phi Y}X - \nabla_X^\perp \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y] - \eta(X)\phi Y - \eta(Y)\phi X \]
\[ -\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi. \]

Hence lemma is proved. \(\square\)

**Corollary 3.7.** If \(M\) be a \(\xi\) – horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold \(\overline{M}\) with quarter symmetric non-metric connection. Then

\[ 2(\nabla_X \phi)Y = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] - \eta(X)\phi Y - \eta(X)Y, \]
\[ 2(\nabla_Y \phi)X = -A_{\phi Y}X - \nabla_X^\perp \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y] - \eta(X)\phi Y - \eta(X)Y \]

for any \(X \in D\) and \(Y \in D^\perp\).

**Corollary 3.8.** If \(M\) be a \(\xi\) – vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold \(\overline{M}\) with quarter symmetric non-metric connection. Then

\[ 2(\nabla_X \phi)Y = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] - \eta(Y)\phi X - \eta(Y)X, \]

and

\[ 2(\nabla_Y \phi)X = -A_{\phi Y}X - \nabla_X^\perp \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y] - \eta(Y)\phi X - \eta(Y)X \]

for \(X \in D\) and \(Y \in D^\perp\).

4. **Parallel distribution**

**Definition 4.1.** The horizontal (resp., vertical) distribution \(D\) (resp., \(D^\perp\)) is said to be Parallel \cite{3} with respect to the connection on \(M\) if \(\nabla_X Y \in D\) (resp., \(\nabla_Z W \in D^\perp\)) for any vector field \(X, Y \in D\) (resp., \(W, Z \in D^\perp\)).
**Theorem 4.2.** Let $M$ be a $\xi$–vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric non-metric connection. Then

(4.1) \[ h(X, \phi Y) = h(Y, \phi X) \] for any $X, Y \in D$.

**Proof.** Using parallelism of horizontal distribution $D$, we have

(4.2) \[ \nabla_X (\phi Y) \in D \quad \text{and} \quad \nabla_Y \phi X \in D \quad \text{for any} \ X, Y \in D. \]

From (3.2), we have

(4.3) \[ 2Bh(X,Y) = 0, \]

for any $X, Y \in D$.

Also, from (2.12) we have

(4.4) \[ \phi h(X,Y) = Bh(X,Y) + Ch(X,Y). \]

Using (4.3) in (4.4), we get

(4.5) \[ \phi h(X,Y) = Ch(X,Y). \]

Next, from (3.3) we have

\[ h(X, \phi Y) + h(Y, \phi X) = 2Ch(X,Y). \]

Using (4.5) in above equation, we have

(4.6) \[ h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X,Y). \]

Replacing $X$ to $\phi X$, we obtain

(4.7) \[ h(\phi X, \phi Y) + h(Y, X) = 2\phi h(\phi X, Y). \]

Now, replacing $Y$ to $\phi Y$ in (4.6), we get

(4.8) \[ h(X, Y) + h(\phi Y, \phi X) = 2\phi h(X, \phi Y). \]

From (4.7) and (4.8), we have

\[ h(X, \phi Y) = h(Y, \phi X). \]
Hence theorem is proved. □

**Theorem 4.3.** Let \( M \) be a \( \xi \)-vertical \( CR \)-submanifold of a nearly hyperbolic Kenmotsu manifold \( \bar{M} \) with quarter symmetric non-metric connection. If the distribution \( D^\bot \) is parallel with respect to the connection on \( M \). Then

\[
(4.9) \quad A_{\phi\gamma}Z + A_{\phi\delta}Y \in D^\bot
\]

for any \( Y, Z \in D^\bot \).

**Proof.** Let \( Y, Z \in D^\bot \), then using Weingarten formula (2.17), we obtain

\[
(4.10) \quad (\bar{\nabla}_Y\phi)Z + \phi(\bar{\nabla}_YZ) = -A_{\phi\delta}Y + \nabla_{\phi}Y Z.
\]

Using (2.16) in (4.10), we have

\[
(4.11) \quad (\bar{\nabla}_Y\phi)Z = -A_{\phi\delta}Y + \nabla_{\phi}Y Z - \phi(\bar{\nabla}_YZ) - \phi h(Y, Z).
\]

Interchanging \( Y \) and \( Z \), we have

\[
(4.12) \quad (\bar{\nabla}_Z\phi)Y = -A_{\phi\gamma}Z + \nabla_{\phi}Y Z - \phi(\bar{\nabla}_ZY) - \phi h(Z, Y).
\]

Adding (4.11) and (4.12), we get

\[
(4.13) \quad (\bar{\nabla}_Y\phi)Z + (\bar{\nabla}_Z\phi)Y = -A_{\phi\delta}Y - A_{\phi\gamma}Z + \nabla_{\phi}Z + \nabla_{\phi}Y Z - \phi(\bar{\nabla}_YZ) - \phi(\bar{\nabla}_ZY) - 2\phi h(Y, Z).
\]

Using (2.14) in (4.13), we obtain

\[
(4.14) \quad -\eta(Z)\phi Y - \eta(Y)\phi Z - \eta(Y)Z - \eta(Z)Y + 2\eta(Y)\eta(Z)\xi = -A_{\phi\gamma}Z - A_{\phi\delta}Y
\]

\[
+\nabla_{\phi}Z + \nabla_{\phi}Y Z - \phi(\bar{\nabla}_YZ) - \phi(\bar{\nabla}_ZY) - 2\phi h(Y, Z).
\]

Taking inner product with \( X \in D \) in (4.14), we get

\[
g(A_{\phi\gamma}Z + A_{\phi\delta}Y, X) = 0.
\]

This implies that

\[
A_{\phi\gamma}Z + A_{\phi\delta}Y \in D^\bot
\]

for any \( Y, Z \in D^\bot \).
Hence theorem is proved. □

**Definition 4.4.** A CR-submanifold is said to be mixed-totally geodesic if \( h(X, Z) = 0 \) for all \( X \in D \) and \( Z \in D^\perp \).

**Definition 4.5.** A Normal vector field \( N \neq 0 \) is called \( D – parallel \) normal section if \( \nabla_X^\perp N = 0 \) for all \( X \in D \).

**Theorem 4.6.** Let \( M \) be a mixed totally geodesic \( \xi – vertical \) CR-submanifold of a nearly hyperbolic Kenmotsu manifold \( \tilde{M} \) with quarter symmetric non-metric connection. Then the normal section \( N \in \phi D^\perp \) is \( D – parallel \) if and only if \( \nabla_X \phi N \in D \) for all \( X \in D \).

**Proof.** Let \( N \in \phi D^\perp \), then from (3.2) we have

\[
\tag{4.15} 2B h(X, Y) = Q \nabla_Y (\phi X) - QA_{\phi Y} X.
\]

Using definition of mixed geodesic CR-submanifold, we have

\[
\tag{4.16} Q \nabla_Y (\phi X) - QA_{\phi Y} X = 0.
\]

\[
\tag{4.17} Q \nabla_Y \phi X = 0
\]

for \( X \in D \).

In particular, we have

\[
\tag{4.18} Q \nabla_Y X = 0.
\]

Using (4.18) in (3.3), we get

\[
\phi Q \nabla_X Y = \nabla_X^\perp \phi Q Y.
\]

That is, \( \phi Q \nabla_X (\phi N) = \nabla_X^\perp N \). Then by definition of parallelism of \( N \), we have

\[
\phi Q \nabla_X (\phi N) = 0.
\]

Consequently, we get

\[
\nabla_X (\phi N) \in D \text{ for all } X \in D.
\]

Converse part is easy consequence of (4.20). □
REFERENCES