GALOIS CONNECTIONS AND RIGHT CLOSURE OPERATORS

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Abstract. In this paper, we investigate the relations between right (left) closure operators and residuated (Galois) connections on generalized residuated lattices.

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1. Introduction


In this paper, we investigate the relations between right (left) closure operators and residuated (Galois) connections on generalized residuated lattices. We give their examples.

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Definition 1.1. [4,5] A structure \((L, \lor, \land, \oslash, \to, \Rightarrow, \bot, \top)\) is called a \textit{generalized residuated lattice} if it satisfies the following conditions:

(1) \((L, \lor, \land, \top, \bot)\) is a bounded where \(\top\) is the universal upper bound and \(\bot\) denotes the universal lower bound;

(2) \((L, \oslash, \top)\) is a monoid;

(3) it satisfies a residuation, i.e.

\[
a \oslash b \leq c \iff a \leq b \to c \iff b \leq a \Rightarrow c.
\]

We call that a generalized residuated lattice has the law of double negation if \(a = (a^*)^0 = (a^0)^*\) where \(a^0 = a \to \bot\) and \(a^* = a \Rightarrow \bot\).

Remark 1.2. [4-8] (1) A generalized residuated lattice is a residuated lattice \((\to = \Rightarrow)\) iff \(\oslash\) is commutative.

(2) A left-continuous t-norm \(([0, 1], \leq, \oslash)\) defined by \(a \to b = \bigvee\{c \mid a \oslash c \leq b\}\) is a residuated lattice

(3) Let \((L, \leq, \oslash, \bot, \top)\) be a quantale. For each \(x, y \in L\), we define

\[
x \to y = \bigvee\{z \in L \mid z \oslash x \leq y\}, \quad x \Rightarrow y = \bigvee\{z \in L \mid x \oslash z \leq y\}.
\]

Then it satisfies Galois correspondence, that is,

\[
(x \oslash y) \leq z \iff x \leq (y \oslash z) \iff y \leq (x \Rightarrow z).
\]

Hence \((L, \lor, \land, \oslash, \to, \Rightarrow, \bot, \top)\) is a generalized residuated lattice.

(4) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we assume \((L, \land, \lor, \oslash, \to, \Rightarrow, \bot, \top)\) is a generalized residuated lattice with the law of double negation and if the family supremum or infimum exists, we denote \(\bigvee\) and \(\bigwedge\).

Lemma 1.3. [4-8] For each \(x, y, z, x_i, y_i \in L\), we have the following properties.

(1) If \(y \leq z\), \((x \oslash y) \leq (x \oslash z)\), \(x \to y \leq x \to z\) and \(z \to x \leq y \to x\) for \(\to \in \{\to, \Rightarrow\}\).

(2) \(x \oslash y \leq x \land y \leq x \lor y\).

(3) \(x \to (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \to y_i)\) and \((\bigvee_{i \in \Gamma} x_i) \to y = \bigvee_{i \in \Gamma} (x_i \to y)\) for \(\to \in \{\to, \Rightarrow\}\).
(4) \( x \rightarrow (\bigvee_{i \in I} y_i) \geq \bigvee_{i \in I} (x \rightarrow y_i) \), for \( \rightarrow \in \{\Rightarrow, \rightarrow\} \).

(5) \( (\bigwedge_{i \in I} x_i) \rightarrow y \geq \bigwedge_{i \in I} (x_i \rightarrow y) \), for \( \rightarrow \in \{\Rightarrow, \rightarrow\} \).

(6) \( (x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) \) and \( (x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z) \).

(7) \( x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z) \) and \( x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z) \).

(8) \( x \odot (x \Rightarrow y) \leq y \) and \( (x \rightarrow y) \odot x \leq y \).

(9) \( (x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z \) and \( (y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z \).

(10) \( (x \Rightarrow z) \leq (y \odot x) \Rightarrow (y \odot z) \) and \( (x \rightarrow z) \leq (x \odot y) \rightarrow (z \odot y) \).

(11) \( (x \Rightarrow y) \leq (y \Rightarrow x) \Rightarrow (x \rightarrow z) \) and \( (y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z) \).

(12) \( x_i \rightarrow y_i \leq (\bigwedge_{i \in I} x_i) \rightarrow (\bigwedge_{i \in I} y_i) \) for \( \rightarrow \in \{\Rightarrow, \rightarrow\} \).

(13) \( x_i \rightarrow y_i \leq (\bigvee_{i \in I} x_i) \rightarrow (\bigvee_{i \in I} y_i) \) for \( \rightarrow \in \{\Rightarrow, \rightarrow\} \).

(14) \( x \rightarrow y = \top \) iff \( x \leq y \).

(15) \( x \rightarrow y = y^0 \Rightarrow x^0 \) and \( x \Rightarrow y = y^* \rightarrow x^* \).

(16) \( \bigwedge_{i \in I} x_i^* = (\bigvee_{i \in I} x_i)^* \) and \( \bigvee_{i \in I} x_i^* = (\bigwedge_{i \in I} x_i)^* \).

(17) \( \bigwedge_{i \in I} x_i^0 = (\bigvee_{i \in I} x_i)^0 \) and \( \bigvee_{i \in I} x_i^0 = (\bigwedge_{i \in I} x_i)^0 \).

**Definition 1.4.** [7] Let \( X \) be a set. A function \( e_X : X \times X \rightarrow L \) is called a **right preorder** if it satisfies:

- \((E1)\) \( e_X(x, x) = \top \) for all \( x \in X \),

- \((R)\) \( e_X(x, y) \odot e_X(y, z) \leq e_X(x, z) \), for all \( x, y, z \in X \).

A function \( e_X \) is called a **left preorder** if it satisfies \((E1)\) and \( (L) \) \( e_X(y, z) \odot e_X(x, y) \leq e_X(x, z) \), for all \( x, y, z \in X \).

The pair \((X, e_X)\) is a right preorder (resp. left-preorder) set.

**Remark 1.5.** (1) We define two functions \( e^\uparrow_L, e^\downarrow_L : L \times L \rightarrow L \) as \( e^\uparrow_L(x, y) = x \Rightarrow y \) and \( e^\downarrow_L(x, y) = x \rightarrow y \). Then \( e^\uparrow_L \) is a right preorder and \( e^\downarrow_L \) is a left preorder from Lemma 1.3 (9).

(2) We define two functions \( e^\uparrow_{LX}, e^\downarrow_{LX} : L^X \times L^X \rightarrow L \) as

\[
e^\uparrow_{LX}(A, B) = \bigwedge_{x \in X} (A(x) \Rightarrow B(x)), \quad e^\downarrow_{LX}(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)).
\]

Then \( e^\uparrow_{LX} \) is a right preorder and \( e^\downarrow_{LX} \) is a left preorder from Lemma 1.3 (9).
Definition 1.6. [7,10] Let $X$ and $Y$ be two sets. Let $F, H : L^X \to L^Y$ and $G, K : L^Y \to L^X$ be operators.

(1) The pair $(F, G)$ is called a residuated connection between $X$ and $Y$ if for $A \in L^X$ and $B \in L^Y$, $F(A) \leq B$ iff $A \leq G(B)$.

(2) The pair $(H, K)$ is called a Galois connection between $X$ and $Y$ if for $A \in L^X$ and $B \in L^Y$, $B \leq H(A)$ iff $A \leq K(B)$.

Definition 1.7. [7] (1) A map $G : L^X \to L^Y$ is a right isotone map if for all $A, B \in L^X$, $e^\uparrow_{L^X}(A, B) \leq e^\uparrow_{L^Y}(G(A), G(B))$.

(2) A map $G : L^X \to L^Y$ is a left isotone map if for all $A, B \in L^X$, $e^\uparrow_{L^X}(A, B) \leq e^\uparrow_{L^Y}(G(A), G(B))$.

(3) A map $G : L^X \to L^Y$ is a right antitone map if for all $A, B \in L^X$, $e\downarrow_{L^X}(A, B) \leq e\downarrow_{L^Y}(G(B), G(A))$.

(4) A map $G : L^X \to L^Y$ is a left antitone map if for all $A, B \in L^X$, $e\downarrow_{L^X}(A, B) \leq e\downarrow_{L^Y}(G(B), G(A))$.

Definition 1.8. [7] A map $C : L^X \to L^X$ is called a right (resp. left) closure operator if it satisfies the following conditions:

(C1) $A \leq C(A)$, for all $A \in L^X$.

(C2) $C(C(A)) = C(A)$, for all $A \in L^X$.

(C3) $C$ is a right (resp. left) isotone map.

A map $I : L^X \to L^X$ is called a right (resp. left) interior operator if it satisfies the following conditions:

(I1) $I(A) \leq A$, for all $A \in L^X$.

(I2) $I(I(A)) = I(A)$, for all $A \in L^X$.

(I3) $I$ is a right (resp. left) isotone map.

Theorem 1.9. [7] Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be two maps.

(1) A pair $(G, H)$ is a residuated connection with two right isotone maps $G$ and $H$ iff for all $A \in L^X$ and $B \in L^Y$, $e^\uparrow_{L^Y}(G(A), B) = e^\uparrow_{L^X}(A, H(B))$. 
(2) A pair \((G, H)\) is a residuated connection with two left isotone maps \(G\) and \(H\) iff for all \(A \in L^X\) and \(B \in L^Y\), \(e_{LY}^+(G(A), B) = e_{LX}^+(A, H(B))\).

(3) A pair \((G, H)\) is a Galois connection with right antitone map \(G\) and left antitone map \(H\) iff for all \(A \in L^X\) and \(B \in L^Y\), \(e_{LX}^+(A, H(B)) = e_{LY}^+(B, G(A))\).

(4) A pair \((G, H)\) is a Galois connection with left antitone map \(G\) and right antitone map \(H\) iff for all \(A \in L^X\) and \(B \in L^Y\), \(e_{LX}^+(A, H(B)) = e_{LY}^+(B, G(A))\).

**Theorem 1.10.** [7] Let \(G : L^X \to L^Y\) and \(H : L^Y \to L^X\) be right isotone maps with a residuated connection \((G, H)\). Then the following statements hold:

1. \(H \circ G\) is a right closure operator.
2. \(G \circ H\) is a right interior operator.

**Corollary 1.11.** [7] Let \(G : L^X \to L^Y\) and \(H : L^Y \to L^X\) be left isotone maps with a residuated connection \((G, H)\). Then the following statements hold:

1. \(H \circ G\) is a left closure operator.
2. \(G \circ H\) is a left interior operator.

**Theorem 1.12.** [7] Let \(G : L^X \to L^Y\) be a right antitone map and \(H : L^Y \to L^X\) be a left antitone map with a Galois connection \((G, H)\). Then

1. \(H \circ G\) is a left closure operator.
2. \(G \circ H\) is a right closure operator.

**Corollary 1.13.** [7] Let \(G : L^X \to L^Y\) be a left antitone map and \(H : L^Y \to L^X\) be a right antitone map with a Galois connection \((G, H)\). Then

1. \(H \circ G\) is a right closure operator.
2. \(G \circ H\) is a left closure operator.

2. Galois connections and right closure operators

**Theorem 2.1.** Let \((X, e_X)\) be a left preordered set. Let \((e_X)^\#_\wedge, (e_X)^\circ_\vee : L^X \to L^X\) be maps as follows:

\[
(e_X)^\#_\wedge(A)(x) = \bigwedge_{y \in X} (e_X(x, y) \Rightarrow A(y)), \quad (e_X)^\circ_\vee(A)(x) = \bigvee_{y \in X} (e_X(y, x) \odot A(y)).
\]
Then the following statements hold.

1. \((e_X)^\dagger\) is a right interior operator.
2. \((e_X)^\odot\) is a right closure operator.
3. \(e_{LX}^\dagger((e_X)^\odot(A), B) = e_{LX}^\dagger(A, (e_X)^\dagger(B))\).
4. \((e_X)^\dagger \circ (e_X)^\odot = (e_X)^\odot\).
5. \((e_X)^\odot \circ (e_X)^\dagger = (e_X)^\dagger\).

**Proof.**

1. (I1) \((e_X)^\dagger(A)(x) \leq (e_X(x, x) \Rightarrow A(x)) = A(x)\).

1. Since \(e_X\) is a left preorder, \(\bigvee_{y \in X}((e_X(y, z) \odot e_X(x, y))) = e_X(x, z)\). Thus

\[
(e_X)^\dagger((e_X)^\dagger(B))(x) = \bigwedge_{y \in X}(e_X(x, y) \Rightarrow (e_X)^\dagger(B)(y)) = \bigwedge_{y \in X}(e_X(x, y) \Rightarrow e_X(y, z) \Rightarrow B(z)) = \bigwedge_{y \in X}(e_X(x, z) \Rightarrow B(z)) = (e_X)^\dagger(B)(x).
\]

1. Since \((e_X(x, y) \Rightarrow A(y)) \odot (A(y) \Rightarrow B(y)) \leq e_X(x, y) \Rightarrow B(y)\), then

\[
e_{LX}^\dagger(A, B) \leq e_{LX}^\dagger((e_X)^\dagger(A), (e_X)^\dagger(B)).
\]

Thus \((e_X)^\dagger\) is a right interior operator.

2. (C1) \(A \leq (e_X)^\odot(A)\).

2. Since \(e_X\) is a left preorder, \(\bigvee_{y \in X}((e_X(y, z) \odot e_X(x, y))) = e_X(x, z)\).

\[
(e_X)^\odot((e_X)^\odot(A))(y) = \bigvee_{z \in X}(e_X(z, y) \odot (e_X)^\odot(A)(z)) = \bigvee_{z \in X}(e_X(z, y) \odot \bigvee_{x \in X}(e_X(x, z) \odot A(x))) = \bigvee_{z \in X}(e_X(z, y) \odot e_X(x, z) \odot A(x)) = \bigvee_{x \in X}(e_X(x, y) \odot A(x)) = (e_X)^\odot(A)(y).
\]
(C3) Since \(e_X(x, y) \odot A(x) \odot (A(x) \Rightarrow B(x)) \leq e_X(x, y) \odot B(y)\), then

\[
(A(x) \Rightarrow B(x)) \leq (e_X(x, y) \odot A(x)) \Rightarrow (e_X(x, y) \odot B(y)),
\]

\[
e_L^\hat{\cdot} (A, B) \leq e_L^\hat{\cdot} ((e_X)^\odot (A), (e_X)^\odot (B)).
\]

Thus \((e_X)^\odot\) is a right closure operator.

(3)

\[
e_L^\hat{\cdot} ((e_X)^\odot (A), B) = \bigwedge_{y \in X} ((e_X)^\odot (A)(y) \Rightarrow B(y))
\]

\[
= \bigwedge_{y \in X} \left( \bigvee_{x \in X} (e_X(x, y) \odot A(x)) \Rightarrow B(y) \right)
\]

\[
= \bigwedge_{y \in X} \bigwedge_{x \in X} \left( (e_X(x, y) \odot A(x)) \Rightarrow B(y) \right)
\]

\[
= \bigwedge_{x \in X} \left( A(x) \Rightarrow \bigwedge_{y \in X} (e_X(x, y) \Rightarrow B(y)) \right)
\]

\[
= \bigwedge_{x \in X} (A(x) \Rightarrow (e_X)^\hat{\cdot} (B)(x))
\]

\[
= e_L^\hat{\cdot} (A, (e_X)^\hat{\cdot} (B)).
\]

(4) By (1), since \((e_X)^\odot \geq (e_X)^\hat{\cdot} \circ (e_X)^\odot\), we only show \((e_X)^\odot \leq (e_X)^\hat{\cdot} \circ (e_X)^\odot\) from:

\[
\top = e_L^\hat{\cdot} ((e_X)^\odot (A), (e_X)^\odot (A)) = e_L^\hat{\cdot} ((e_X)^\odot ((e_X)^\odot (A)), (e_X)^\odot (A))
\]

\[
= e_L^\hat{\cdot} ((e_X)^\odot (A), (e_X)^\hat{\cdot} ((e_X)^\odot (A))) \quad \text{(by (3))}.
\]

(5) By (2), since \((e_X)^\odot \circ (e_X)^\hat{\cdot} \geq (e_X)^\hat{\cdot}\), we only show \((e_X)^\odot \circ (e_X)^\hat{\cdot} \leq (e_X)^\hat{\cdot}\) from:

\[
\top = e_L^\hat{\cdot} ((e_X)^\hat{\cdot} (B), (e_X)^\hat{\cdot} (B)) = e_L^\hat{\cdot} ((e_X)^\hat{\cdot} (B), (e_X)^\hat{\cdot} ((e_X)^\hat{\cdot} (B)))
\]

\[
= e_L^\hat{\cdot} ((e_X)^\odot ((e_X)^\hat{\cdot} (B)), (e_X)^\hat{\cdot} (B)) \quad \text{(by (3))}.
\]

**Theorem 2.2.** Let \((X, e_X)\) be a right preordered set. Let \((e_X)^\uparrow, \odot (e_X): L^X \to L^X\) be maps as follows:

\[
(e_X)^\uparrow (A)(x) = \bigwedge_{y \in X} (e_X(x, y) \Rightarrow A(y)), \quad \odot (e_X)(A)(x) = \bigvee_{y \in X} (A(y) \odot e_X(y, x)).
\]

Then the following statements hold.

(1) \((e_X)^\uparrow\) is a left interior operator.

(2) \((e_X)^\odot\) is a left closure operator.

(3) \(e_L^\uparrow ((e_X)^\odot (A), B) = e_L^\uparrow (A, (e_X)^\uparrow (B))\).

(4) \((e_X)^\uparrow \circ (e_X) = (e_X)^\odot\).
(5) \( \circ (e_X) \circ (e_X)^\dagger = (e_X)^\dagger \).

**Proof.** (1) (I1) \( (e_X)^\dagger(A) \leq A \). (I2) Since \( e_X \) is a right preorder, \( \bigvee_{y \in X} ((e_X(x, y) \circ e_X(y, z)) = e_X(x, z) \). Thus

\[
(e_X)^\dagger((e_X)^\dagger(B))(x) = \bigwedge_{y \in X} (e_X(x, y) \to (e_X)^\dagger(B)(y))
= \bigwedge_{y \in X} \left( e_X(x, y) \to \bigwedge_{z \in X} (e_X(y, z) \to B(z)) \right)
= \bigwedge_{y \in X} \bigwedge_{z \in X} \left( (e_X(x, y) \circ e_X(y, z)) \to B(z) \right) \text{ (by Lemma 1.3(6))}
= \bigwedge_{z \in X} \left( e_X(x, z) \to B(z) \right)
= (e_X)^\dagger(B)(x).
\]

(I3) Since \( (A(y) \to B(y)) \circ (e_X(x, y) \to A(y)) \leq e_X(x, y) \to B(y) \), then

\[
e^\dagger_{L_X}(A, B) \leq e^\dagger_{L_X}((e_X)^\dagger(A), (e_X)^\dagger(B)).
\]

Thus \( (e_X)^\dagger \) is a left interior operator.

(2) (C1) \( A \leq (e_X)^\circ (A) \).

(C2) Since \( e_X \) is a right preorder, \( \bigvee_{y \in X} \left( (e_X(x, y) \circ e_X(y, z)) = e_X(x, z) \right) \). Thus

\[
\circ (e_X)(\circ (e_X)(A))(y) = \bigvee_{z \in X} \left( (e_X)(A)(z) \circ e_X(z, y) \right)
= \bigvee_{z \in X} \left( \bigvee_{x \in X} \left( A(x) \circ e_X(x, z) \right) \circ (e_X(z, y)) \right)
= \bigvee_{z \in X} \left( A(x) \circ \bigvee_{z \in X} \left( (e_X(x, z) \circ e_X(z, y)) \right) \right)
= \bigvee_{z \in X} \left( A(x) \circ e_X(x, y) \right)
= \circ (e_X)(A)(y).
\]

(C3) Since \( (A(y) \to B(y)) \circ A(y) \circ e_X(x, y) \leq B(y) \circ e_X(x, y) \),

\[
e^\dagger_{L_X}(A, B) \leq e^\dagger_{L_X}(\circ (e_X)(A), \circ (e_X)(B)).
\]

Thus \( \circ (e_X) \) is a left closure operator.
\[ e_{LX}^\dagger(\circ(e_X)(A), B) = \bigwedge_{y \in X} (\circ(e_X)(A)(y) \to B(y)) \]
\[ = \bigwedge_{y \in X} \left( \bigvee_{x \in X} (A(x) \circ e_X(x, y)) \to B(y) \right) \]
\[ = \bigwedge_{y \in X} \bigwedge_{x \in X} (A(x) \circ e_X(x, y)) \to B(y) \]
\[ = \bigwedge_{x \in X} \bigwedge_{y \in X} (A(x) \to (e_X(x, y) \to B(y))) \]
\[ = \bigwedge_{x \in X} (A(x) \to (e_X)^\dagger(B))(x) \]
\[ = e_{LX}^\dagger(A, (e_X)^\dagger(B)). \]

(4) By (1), since \( \circ(e_X) \geq (e_X)^\dagger \circ \circ(e_X) \), we only show \( \circ(e_X) \leq (e_X)^\dagger \circ \circ(e_X) \) from:

\[ \top = e_{LX}^\dagger(\circ(e_X)(A), (\circ(e_X)(A)) = e_{LX}^\dagger(\circ(e_X)(\circ(e_X)(A)), (\circ(e_X)(A)) \]
\[ = e_{LX}^\dagger(\circ(e_X)(A), (e_X)^\dagger(\circ(e_X)(A)). \]

(5) By (2), since \( \circ(e_X) \circ (e_X)^\dagger \geq (e_X)^\dagger \), we only show \( \circ(e_X) \circ (e_X)^\dagger \leq (e_X)^\dagger \) from:

\[ \top = e_{LX}^\dagger((e_X)^\dagger(B), (e_X)^\dagger(B)) = e_{LX}^\dagger((e_X)^\dagger(B), (e_X)^\dagger((e_X)^\dagger(B))) \]
\[ = e_{LX}^\dagger(\circ(e_X)((e_X)^\dagger(B)), (e_X)^\dagger((e_X)^\dagger(B)). \]

**Theorem 2.3.** For each \( A \in L^X \) and \( B \in L^Y \) and \( R \in L^{X \times Y} \), we define: \( R^\rightharpoonup, R^\leadsto : L^X \to L^Y \) is defined as:

\[ R^\rightharpoonup(A)(y) = \bigwedge_{x \in X} (A(x) \to R(x, y)), \quad R^\leadsto(A)(y) = \bigwedge_{x \in X} (A(x) \Rightarrow R(x, y)) \]

and \( R^\leftrightharpoons, R^\rightharpoons : L^Y \to L^X \) is defined as:

\[ R^\leftrightharpoons(B)(x) = \bigwedge_{y \in Y} (B(y) \to R(x, y)), \quad R^\rightharpoons(B)(x) = \bigwedge_{y \in Y} (B(y) \Rightarrow R(x, y)) \]

(1) \( R^\rightharpoonup \) is a left antitone map and \( R^\rightharpoons \) is a right antitone map.

(2) \( R^\rightharpoons \) is a left antitone map and \( R^\rightharpoons \) is a right antitone map.

(3) Let \( R^\rightharpoonup \) be a left antitone map and \( R^\rightharpoons \) a right antitone map with a Galois connection \( (R^\rightharpoonup, R^\rightharpoons) \).

(4) \( R^\rightharpoonup \circ R^\rightharpoons : L^Y \to L^Y \) is a left closure operator and \( R^\rightharpoons \circ R^\rightharpoonup : L^X \to L^X \) is a right closure operator.

(5) \( e_{LY}^\dagger(B, R^\rightharpoonup(A)) = e_{LX}^\dagger(A, R^\rightharpoons(B)). \)
(6) $R^\rightarrow \circ R^\leftarrow : L^Y \to L^Y$ is a right closure operator and $R^\leftarrow \circ R^\rightarrow : L^X \to L^X$ is a left closure operator.

**Proof.** (1) Since $(A(x) \Rightarrow B(x)) \odot (B(x) \Rightarrow R(x, y)) \leq (A(x) \Rightarrow R(x, y))$,

$$e_{L^X}^\Delta(A, B) \leq e_{L^Y}^\uparrow(R^\leftarrow(B), R^\Rightarrow(A)).$$

(2) Since $(B(y) \Rightarrow R(x, y)) \odot (A(y) \Rightarrow B(y)) \leq (A(y) \Rightarrow R(x, y))$,

$$e_{L^Y}^\uparrow(A, B) \leq e_{L^X}^\Delta(R^\leftarrow(B), R^\Rightarrow(A)).$$

(3) From Theorem 1.9(4), we only show that $e_{L^X}^\Delta(A, R^\leftarrow(B)) = e_{L^Y}^\uparrow(B, R^\Rightarrow(A))$ from:

$$e_{L^X}^\Delta(A, R^\leftarrow(B))
= \bigwedge_{x \in X} (A(x) \Rightarrow R^\leftarrow(B)(x))
= \bigwedge_{x \in X} (A(x) \Rightarrow \bigwedge_{y \in X} (B(y) \Rightarrow R(x, y)))
= \bigwedge_{x \in X} \bigwedge_{y \in X} (A(x) \Rightarrow (B(y) \Rightarrow R(x, y)))
= \bigwedge_{y \in X} (B(y) \Rightarrow \bigwedge_{x \in X} (A(x) \Rightarrow R(x, y)))
= \bigwedge_{y \in X} (B(y) \Rightarrow R^\Rightarrow(A)(y)) = e_{L^Y}^\uparrow(B, R^\Rightarrow(A)).$$

(5) From Theorem 1.9(3), $e_{L^Y}^\Delta(B, R^\leftarrow(A)) = e_{L^X}^\Delta(A, R^\leftarrow(B))$ from

$$e_{L^X}^\Delta(A, R^\leftarrow(B)) = \bigwedge_{x \in X} (A(x) \Rightarrow R^\leftarrow(B)(x))
= \bigwedge_{x \in X} (A(x) \Rightarrow \bigwedge_{y \in X} (B(y) \Rightarrow R(x, y)))
= \bigwedge_{x \in X} \bigwedge_{y \in X} (A(x) \Rightarrow (B(y) \Rightarrow R(x, y)))
= \bigwedge_{y \in X} (B(y) \Rightarrow \bigwedge_{x \in X} (A(x) \Rightarrow R(x, y)))
= \bigwedge_{y \in X} (B(y) \Rightarrow R^\Rightarrow(A)(y)) = e_{L^Y}^\Delta(B, R^\leftarrow(B)).$$

(4) and (6) are proved from Corollary 1.12 and Theorem 1.13, respectively.

**Theorem 2.4.** Let $F, G : L^X \to L^X$ be maps such that

$$e_{L^X}^\Delta(F(A), B) = e_{L^X}^\Delta(A, G(B)).$$

Then the following statements are equivalent.

(1) $F$ is a right interior operator.

(2) $G$ is a right closure operator.

(3) $F \circ G = F$. 
(4) $G \circ F = G$.

**Proof.** Since $e_{L^X}^\uparrow (F(A), B) = e_{L^X}^\uparrow (A, G(B))$, by Theorem 1.9(1), $F$ and $G$ are right isotone maps.

(1) $\Rightarrow$ (2). Since $\top = e_{L^X}^\uparrow (F(A), A) = e_{L^X}^\uparrow (A, G(A))$, then $A \leq G(A)$.

$$
e_{L^X}^\uparrow (G(G(A)), G(A)) = e_{L^X}^\uparrow (F(G(G(A))), A) = e_{L^X}^\uparrow (F(F(G(G(A))))), A)$$

$$= e_{L^X}^\uparrow (F(G(G(A))), G(A)) = e_{L^X}^\uparrow (G(G(A)), G(G(A))) = \top.$$

Thus $G$ is a right closure operator.

(2) $\Rightarrow$ (3). Since $F$ is a right isotone map, $\top = e_{L^X}^\uparrow (A, G(A)) \leq e_{L^X}^\uparrow (F(A), F(G(A)))$.

Then $F(A) \leq F(G(A))$. Moreover, $F(A) = F(G(A))$ from:

$$e_{L^X}^\uparrow (F(G(A)), A) = e_{L^X}^\uparrow (G(A), G(F(A))) = e_{L^X}^\uparrow (G(A), G(F(A))))$$

$$\geq e_{L^X}^\uparrow (A, G(F(A))) = \top. \quad (G \text{ is a right isotone map})$$

(3) $\Rightarrow$ (4). Let $F \circ G = F$. Then $G \circ F \circ G = G \circ F$. Since $G \circ F \circ G \geq G$ and $F \circ G(A) \leq A$ implies $G \circ F \circ G(A) \leq G(A)$. So, $G \circ F = G \circ F \circ G = G$.

(4) $\Rightarrow$ (3). It follows from $F \circ G \circ F = F$.

(3) and (4) $\Rightarrow$ (1). $e_{L^X}^\uparrow (F(A), A) \geq e_{L^X}^\uparrow (F(A), F(G(A))) \circ e_{L^X}^\uparrow (F(G(A)), A) = \top \circ \top = \top$. Moreover, $e_{L^X}^\uparrow (F(A), F(F(A))) = e_{L^X}^\uparrow (A, G(F(F(A)))) = e_{L^X}^\uparrow (A, G(F(A))) = \top.$

The following corollary are similarly proved as Theorem 2.4.

**Corollary 2.5.** Let $F, G : L^X \to L^X$ be maps such that

$$e_{L^X}^\uparrow (F(A), B) = e_{L^X}^\uparrow (A, G(B)).$$

Then the following statements are equivalent.

(1) $F$ is a left interior operator.

(2) $G$ is a left closure operator.

(3) $F \circ G = F$.

(4) $G \circ F = G$.

**Theorem 2.6.** Let $F, G : L^X \to L^X$ be maps such that

$$e_{L^X}^\uparrow (F(A), B) = e_{L^X}^\uparrow (A, G(B)).$$
Then the following statements are equivalent.

(1) $F$ is a left closure operator.

(2) $G$ is a left interior operator.

(3) $G \circ F = F$.

(4) $F \circ G = G$.

**Proof.** Since $e_{LX}^\uparrow (F(A), B) = e_{LX}^\uparrow (A, G(B))$, by Theorem 1.9(2), $F$ and $G$ are left isotone maps.

(1) $\Rightarrow$ (3). Since $F(A) = F(G(F(A)))$, we have

$$e_{LX}^\uparrow (G(F(A)), F(A)) = e_{LX}^\uparrow (G(F(A)), F(G(F(A)))) = \top.$$

Then $G(F(A)) \leq F(A)$. Moreover,

$$e_{LX}^\uparrow (F(A), G(F(A))) = e_{LX}^\uparrow (F(F(A)), F(G(F(A)))) = e_{LX}^\uparrow (F(F(A)), F(A)) = \top.$$

Then $G(F(A)) \geq F(A)$.

(3) $\Rightarrow$ (1). Since $F$ is a left isotone map and $A \leq G(F(A))$, we have

$$e_{LX}^\uparrow (A, F(A)) = e_{LX}^\uparrow (A, G(F(A))) = \top.$$

Then $A \leq F(A)$.

$$e_{LX}^\uparrow (F(F(A)), F(A)) = e_{LX}^\uparrow (F(A), G(F(A))) = e_{LX}^\uparrow (F(A), F(A)) = \top.$$

Thus $F(A) = F(F(A))$.

(3) $\Leftrightarrow$ (4). It follows from $G \circ F \circ G = G$ and $F \circ G \circ F = F$.

(2) $\Leftrightarrow$ (4). We prove a similar method as (1) $\Leftrightarrow$ (3).

The following corollary are similarly proved as Theorem 2.5.

**Corollary 2.7.** Let $F, G : L^X \to L^X$ be maps such that

$$e_{LX}^\hat\downarrow (F(A), B) = e_{LX}^\hat\downarrow (A, G(B)).$$

Then the following statements are equivalent.

(1) $F$ is a right closure operator.

(2) $G$ is a right interior operator.
(3) $G \circ F = F$.
(4) $F \circ G = G$.

**Example 2.8.** Let $K = \{(x, y) \in R^2 \mid x > 0\}$ be a set and we define an operation
$\otimes : K \times K \rightarrow K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2, x_1y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1, 0)$, $(x, y)^{-1} = (1 \frac{y}{x^2}, -\frac{y}{x^2})$.

We have a positive cone $P = \{(a, b) \in R^2 \mid a = 1, b \geq 0, \text{ or } a > 1\}$ because $P \cap P^{-1} =\{(1, 0)\}$, $P \circ P \subset P$, $(a, b)^{-1} \circ P \circ (a, b) = P$ and $P \cup P^{-1} = K$. For $(x_1, y_1), (x_2, y_2) \in K$, we define

$$(x_1, y_1) \leq (x_2, y_2) \iff (x_1, y_1)^{-1} \circ (x_2, y_2) \in P, \ (x_2, y_2) \circ (x_1, y_1)^{-1} \in P$$
$$\iff x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2.$$

Then $(K, \leq \otimes)$ is a lattice-group.

The structure $(L, \otimes, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ is a generalized residuated lattice with strong negation where $\bot = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the greatest element from the following statements:

$$(x_1, y_1) \circ (x_2, y_2) = (x_1, y_1) \otimes (x_2, y_2) \lor \left(\frac{1}{2}, 1\right) = (x_1x_2, x_1y_2 + y_1) \lor \left(\frac{1}{2}, 1\right),$$
$$(x_1, y_1) \Rightarrow (x_2, y_2) = ((x_1, y_1)^{-1} \otimes (x_2, y_2)) \land \left(1, 0\right) = \left(\frac{x_2}{x_1}, \frac{y_2 - y_1}{x_1}\right) \land \left(1, 0\right),$$
$$(x_1, y_1) \rightarrow (x_2, y_2) = ((x_2, y_2) \otimes (x_1, y_1)^{-1}) \land \left(1, 0\right) = \left(\frac{x_2}{x_1}, -\frac{y_2y_1}{x_1} + y_2\right) \land \left(1, 0\right).$$

Furthermore, we have $(x, y) = (x, y)^{**} = (x, y)^{**}$ from:

$$(x, y)^* = (x, y) \Rightarrow \left(\frac{1}{2}, 1\right) = \left(\frac{1}{2x}, \frac{1 - y}{x}\right),$$
$$(x, y)^{**} = \left(\frac{1}{2x}, \frac{1 - y}{x}\right) \rightarrow \left(\frac{1}{2}, 1\right) = (x, y).$$

Let $X = \{a, b, c\}$ be a set. Define $(c_X^1(a, b), (c_X^2(a, b)) \in L^{X \times X}$ as

$$c_X^1 = \begin{pmatrix}
(1, 0) & (\frac{5}{8}, \frac{5}{2}) & (\frac{5}{8}, \frac{5}{3}) \\
(\frac{5}{7}, \frac{30}{7}) & (1, 0) & (\frac{5}{8}, -\frac{5}{4}) \\
(1, -2) & (\frac{5}{7}, \frac{10}{3}) & (1, 0)
\end{pmatrix}$$
$$c_X^2 = \begin{pmatrix}
(1, 0) & (\frac{2}{3}, 5) & (\frac{5}{6}, 1) \\
(\frac{5}{7}, 3) & (1, 0) & (\frac{6}{7}, 4) \\
(\frac{5}{6}, -1) & (\frac{3}{4}, 2) & (1, 0)
\end{pmatrix}$$
We easily show that $e_1^X$ is a right partial order and $e_2^X$ is a left partial order. But $e_2^X$ is not a right partial order because

$$e_2^X(b, c) \odot e_2^X(c, a) = (\frac{6}{7}, 4) \odot (\frac{5}{6}, -1) = (\frac{5}{7}, \frac{22}{7}) \not\leq e_2^X(b, a) = (\frac{5}{7}, 3).$$

For $A = ((\frac{2}{3}, 1), (\frac{3}{5}, -1), (1, -1))^t$,

$$\circ(e_2^X)(A) = ((\frac{5}{6}, -2), (\frac{3}{4}, 1), (1, -1))^t$$

$$(e_2^X)^\uparrow(\circ(e_2^X)(A)) = \circ(e_2^X)(A),$$

$$(e_2^X)^\circ(A) = ((\frac{5}{6}, -\frac{11}{6}), (\frac{3}{4}, 1), (1, -1))^t,$$

$$(e_2^X)^\circ((e_2^X)^\circ(A)) = ((\frac{5}{6}, -\frac{11}{6}), (\frac{3}{4}, 1), (1, -\frac{4}{3}))^t \neq (e_2^X)^\circ(A).$$

Since $e_2^X$ is not a right partial order, by Theorem 2.1 (4), $(e_2^X)^\circ((e_2^X)^\circ(A)) \neq (e_2^X)^\circ(A)$.

Let $X = \{a, b, c\}$ and $Y = \{u, v\}$ be sets. Define $R \in L^{X \times Y}$ as

$$R = \begin{pmatrix}
(1, 0) & (\frac{5}{8}, \frac{5}{2}) \\
(\frac{5}{7}, \frac{30}{7}) & (\frac{5}{8}, -\frac{5}{4}) \\
(\frac{1}{2}, 2) & (\frac{5}{6}, \frac{10}{3})
\end{pmatrix}$$

For $A = ((\frac{2}{3}, 1), (\frac{1}{2}, 2), (\frac{2}{3}, -1))^t$,

$$R^\rightarrow(A) = ((\frac{3}{4}, 11\frac{1}{4}), (\frac{15}{16}, -25\frac{1}{16})^t, \quad R^\leftarrow(A) = ((\frac{3}{4}, 9\frac{1}{2}, (\frac{15}{16}, 9\frac{1}{4}))^t$$

$$R^\rightarrow(R^\rightarrow(A)) = ((\frac{2}{3}, 13\frac{2}{3}), (\frac{2}{3}, 1), (\frac{2}{3}, -1))^t, \quad R^\leftarrow(R^\leftarrow(A)) = ((\frac{2}{3}, 85\frac{2}{24}), (\frac{2}{3}, -\frac{5}{24}), (\frac{2}{3}, 1))^t.$$

REFERENCES

