Abstract. In this paper, we introduce ∗-g-frames and study the operators associated with a give ∗-g-frames. We also show many useful properties with corresponding notions, g-frames and ∗-frames in Hilbert C*-modules.

Keywords: g-frame; ∗-frame operator; g-frame operator; C*-module; C*-algebra.

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1. Introduction and preliminaries

Frames for Hilbert spaces were introduced in 1952 by Duffin and Schaefer [8]. They abstracted the fundamental notion of Gabor [13] to study signal processing. Many generalizations of frames were introduced, e.g., frames of subspaces [2], Pseudo-frames [18], oblique frames [6], G-frames [15], ∗ frames [1] in Hilbert spaces.

The $\ast$-frames theories in Hilbert $C^*$-modules. In this note, we introduce the $\ast-g$-frames which are generalizations of $g$-frames in Hilbert $C^*$-modules.

The content of the present note is as follow: we continue this introductory section with review of the definitions and basic properties of $C^*$-algebras, Hilbert $C^*$-modules, $\ast$-frame and $g$-frames. In Section 2, we introduce $\ast-g$-frame and present examples of such $\ast-g$-frame. Similar to the ordinary frames, $g$-frames and $\ast$-frames, we introduce the pre-$\ast-g$-frame transform and the $\ast-g$-frames operator. For information about Hilbert $C^*$-module, we refer authors to [10, 14, 17] and about $\ast$-frame, $g$-frame we refer authors to [1, 15]. Our reference for $C^*$-algebras as [7].

Let $A$ be a unitary $C^*$-algebra and $a \in A$. The nonzero element $a$ is called strictly nonzero if zero does not belong $\sigma(a)$, and $a$ is said to be strictly positive if it is strictly nonzero and positive. If $a$ is positive, there is a positive element $b \in A$ such that $b^2 = a$. The relation $\leq'$ given by $a \leq b$ if and only if $b - a$ is positive defines a partial ordering in $A$. Let be $a, b, c \in A$, we have (i) if $a \leq b$, then $ca^* \leq cb^*$. And if $c$ commutes with $a$ and $b$, then $ca \leq cb$ for $0 \leq c$. (ii) $0 \leq a \leq b$ implies $||a|| \leq ||b||, ab \geq 0, a + b \geq 0$, and $a' \leq b'$ for $t \in (0, 1)$ (iii) if $0 \leq a \leq b$ and $a, b$ invertible elements then $0 \leq b^{-1} \leq a^{-1}$.

Now, we are going to introduce some of elementary definitions and the basic properties of Hilbert $C^*$-modules.

**Definition 1.1.** Let $A$ be a $C^*$-algebra, a pre-Hilbert $A$-module is a left $A$-module $X$ equipped with a sesquilinear map $\langle.,.\rangle : X \times X \rightarrow A$ satisfying

1. $\langle x, x \rangle \geq 0; \langle x, x \rangle = 0$ if and only if $x = 0$ for all $x$ in $X$,
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z$ in $X, \alpha, \beta \in \mathbb{C}$,
3. $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y$ in $X$,
4. $\langle ax, y \rangle = a \langle x, y \rangle$ for all $x, y$ in $X, a$ in $A$.

The map $\langle ., . \rangle$ is called an $A$-valued inner product of $X$, and for $x \in X$, we define $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$ is a norm on $X$, where the latter norm denotes that in the $C^*$-algebra $A$. This norm makes $X$ into a left normed module over $A$. A pre-Hilbert module $X$ is called a Hilbert $A$-module if it is complete with respect to its norm. Examples of Hilbert $C^*$-modules are as follows:
(I) Every Hilbert space is a Hilbert $C^*$-module.

(II) Every $C^*$-algebra $A$ is a Hilbert $A$-module via $\langle a, b \rangle = ab^*$ ($a, b \in A$).

(III) Let $\{Y_i, i \in I\}$ be a sequence of $A$-modules and $\oplus Y_i = \{x = (x_i) : i \in Y_i, \sum_{i \in I} \langle x_i, x_i \rangle \text{ is norm convergent in } A\}$. Then $\oplus Y_i$ is a Hilbert $A$-module with $A$-valued inner product $\langle (x_i)_{i \in I}, (y_i)_{i \in I} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$, point wise operations and the norm defined by $||x|| = ||\langle x, x \rangle||^{1/2}$.

Notice that the inner product structure of a $C^*$-algebra is essentially more complicated than complex numbers. One may define an $A$-valued norm $|.|$ by $|x| = \langle x, x \rangle^{1/2}$. Clearly, $||x|| = |||x|||$ for each $x \in X$.

It is known that $|.|$ does not satisfy the triangle inequality in general. Throughout this paper $I$ and $J$ be finite or countable index, sets, $X$ and $Y$ are countably or finitely generated Hilbert $A$-modules and $\{(Y_i) : i \in I\}$ is a sequence of closed sub-modules of $Y$. For each $i \in I, End^*(X, Y_i)$ is the collection of all adjointable $A$-linear maps from $X$ to $Y_i$ and $End^*(X, X)$ is denoted by $End^*(X)$.

**Definition 1.2** [15] A sequence $\{\Lambda_i \in End^*(X, Y_i) : i \in I\}$ is called $g-$frame in $X$ with respect to $\{Y_i : i \in I\}$ if there exist constant reel $C, D > 0$ such that for every $x \in X$,

$$C \langle x, x \rangle \leq \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle \leq D \langle x, x \rangle, \forall x \in X. \quad (1.1)$$

The elements $C$ and $D$ are called the lower and upper $g-$frame bounds respectively of $\{\Lambda_i, i \in I\}$ with respect to $\{Y_i : i \in I\}$.

**Definition 1.3.** [1] A sequence $\{x_i : i \in I\}$ of $X$ is called a $*-\text{frame}$ for $X$ if there exist two strictly nonzero elements $C, D$ in $A$ such that for every $x \in X$,

$$C \langle x, x \rangle C^* \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq D \langle x, x \rangle D^*.$$

The elements $C$ and $D$ are called the lower and upper $*-\text{frame}$ bounds respectively.

Throughout the paper we need the following lemma.

**Lemma 1.4.** [17] Let $X$ and $Y$ two Hilbert $A-$modules and $T \in End^*(X, Y)$. Then

(i) if $T$ is injective and $T$ has closed range, then the adjointable map $T^*T$ is invertible and $||T^*T||^{-1} \leq T^*T \leq ||T||^2$. 


(ii) if $T$ is surjective, then the adjointable map $TT^*$ is invertible and $||(TT^*)^{-1}||^{-1} \leq TT^* \leq ||T||^2$.

2. Main results

Alijani and Dehghan in [1] introduced $*$-frames and A. Khosravi and B. Khosravi in [15] introduced $g$-frames for Hilbert $C^*$-modules. Our next definition and example are generalizations of (3.1) and (3.2) in [15].

**Definition 2.1.** A sequence $\{\Lambda_i \in \text{End}^*(X,Y_i) : i \in I\}$ is called $* - g$-frame with respect to $\{Y_i : i \in I\}$ if there exist $C, D$ strictly nonzero of $A$ such that for every $x \in X$,

$$C \langle x, x \rangle C^* \leq \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle \leq D \langle x, x \rangle D^*.$$  

(2.1)

The elements $C$ and $D$ are called the lower and upper $* - g$- frame bounds respectively in $X$ with respect to $\{Y_i, i \in I\}$. Since $A$ is not a partial ordered set, lower and upper $* - g$- frame bounds may not have order and the optimal bounds may not exist.

If $\lambda = C = D$, then the $* - g$- frame is said to be a $\lambda$-tight $* - g$- frame and if $C = D = 1_A$, it is called a Parseval $* - g$- frame or a normalized $* - g$- frame. The $* - g$- frame is standard if for every $x \in X$, the sum in (2.1) converges in norm.

**Example 2.2.** Let $\{x_i, i \in I\}$ be a $*-$frame of $X$ with lower and upper, $C$ and $D$, respectively. For each $i \in I$, we define $T_{x_i} : X \rightarrow A$, by $T_{x_i}(x) = \langle x, x_i \rangle$ for all $x \in X$. As example in [15], $T_{x_i}$ is adjointable and $T_{x_i}^*(a) = ax_i$ for each $a \in A$. And we have,

$$C \langle x, x \rangle C^* \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq D \langle x, x \rangle D^*, \forall x \in X.$$  

Then

$$C \langle x, x \rangle C^* \leq \sum_{i \in I} \langle T_{x_i}(x), T_{x_i}(x) \rangle \leq D \langle x, x \rangle D^*;$$

for all $x \in X$. So, $\{T_{x_i}, i \in I\}$ is a $* - g$ -frame with bounds $C$ and $D$, respectively, in $X$ with respect to $A$.

Now we studies the corresponding operators of a $* - g$-frame.
Theorem 2.3. Let \( \{ \Lambda_i \in \text{End}^*(X, Y_i) : i \in I \} \) be \(*-g\)-frame with lower and upper, \( C \) and \( D \), respectively. The \(*-g\)-frame transform or pre-\(*-g\)-frame operator \( T : X \rightarrow \oplus Y_i \) defined by \( T(x) = (\Lambda_i(x))_{i \in I} \) is injective, closed range adjointable \( A \)-module map and \( \|T\| \leq ||D|| \). The adjointable \( T^* \) is surjective and it given by: \( T^*(y) = \sum_{i \in I} \Lambda_i^*(y_i) \) where \( y = (y_i)_{i \in I} \in \oplus Y_i \).

Proof. Let \( x \) be a vector of \( X \). We have

\[
\|T(x)\|^2 = \|\langle (\Lambda_i(x))_{i \in I}, (\Lambda_i(x))_{i \in I} \rangle\|, \tag{2.2}
\]

and by definition of the norm in \( \oplus Y_i \), we have

\[
\|\langle (\Lambda_i(x))_{i \in I}, (\Lambda_i(x))_{i \in I} \rangle\| = \|\sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle\| \leq \|D \langle \Lambda_i x, \Lambda_i x \rangle D^*\| = \|D\|^2 \|x\|^2,
\]

for all \( x \in X \). Then

\[
\|T(x)\|^2 \leq ||D||^2 \|x\|^2, \quad \forall x \in X. \tag{2.3}
\]

So \( T \) is well defined and \( \|T\| \leq ||D|| \). Thus \( \Lambda_i \in \text{End}^*(X, Y_i) \), \( T \) is a linear \( A \)-module map.

We now show that \( R_T \) is closed. Let \( \{ Tx_n \} \) be a sequence in the range of \( T \) such that

\[
\lim_{n \to \infty} T{x_n} = y.
\]

The definition of \(*-g\)-frame concludes that

\[
C \langle x_n - x_m, x_n - x_m \rangle C^* \leq \sum_{i \in I} \langle \Lambda_i (x_n - x_m), \Lambda_i (x_n - x_m) \rangle \leq D \langle x_n - x_m, x_n - x_m \rangle D^*, \tag{2.4}
\]

which is equivalent to

\[
C \langle x_n - x_m, x_n - x_m \rangle C^* \leq \langle T(x_n - x_m), T(x_n - x_m) \rangle \leq D \langle x_n - x_m, x_n - x_m \rangle D^*.
\]

Hence, we have

\[
\|C \langle x_n - x_m, x_n - x_m \rangle C^*\| \leq \|T(x_n - x_m)\|^2.
\]

Since

\[
\lim_{n \to \infty} T(x_n - x_m) = 0; \quad \lim_{n \to \infty} C \langle x_n - x_m, x_n - x_m \rangle C^* = 0,
\]

we have
we have
\[
|| \langle x_n - x_m, x_n - x_m \rangle || \leq ||C^{-1}||^2 ||C \langle x_n - x_m, x_n - x_m \rangle C^*||.
\]

Hence, there exists \( x \in X \), such that \( \lim_{n \to \infty} x_n = x \),
\[
||T(x_n - x)||^2 = ||\langle (\Lambda_i(x_n - x))_{i \in I}, (\Lambda_i(x_n - x))_{i \in I} \rangle ||
= ||\sum_{i \in I} \langle \Lambda_i(x_n - x), \Lambda_i(x_n - x) \rangle ||
\leq ||D||^2 ||x_n - x||.
\]
Thus \( \lim_{n \to \infty} T x_n = y \), so range of \( T \) is closed.

We show that \( T \) is injective: Suppose that \( x \in X \) and \( Tx = 0 \). We have
\[
||\langle x, x \rangle || = ||C^{-1}C \langle x, x \rangle C^*(C^*)^{-1}||
\leq ||C^{-1}||^2 ||C \langle x, x \rangle C^*||
\leq ||C^{-1}||^2 ||\sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle ||
= ||C^{-1}||^2 ||\langle (\Lambda_i(x))_{i \in I}, (\Lambda_i(x))_{i \in I} \rangle ||
= ||C^{-1}||^2 ||Tx||^2.
\]
Thus \( x = 0 \), and \( T \) is injective.

We determine \( T^* \): Let be \( x \in X \) and \( (y_i) \in \oplus Y_i \). We have \( \langle Tx, (y_i)_{i \in I} \rangle = \langle (\Lambda_i(x))_{i \in I}, (y_i)_{i \in I} \rangle \).

And by definition of the norm in \( \oplus Y_i \), we have
\[
\langle (\Lambda_i(x))_{i \in I}, (y_i)_{i \in I} \rangle = \sum_{i \in I} \langle \Lambda_i x, y_i \rangle.
\]
Then
\[
\langle Tx, (y_i)_{i \in I} \rangle = \sum_{i \in I} \langle x, \Lambda_i^* y_i \rangle.
\]
So \( T^*((y_i)_{i \in I}) = \sum_{i \in I} \Lambda_i^* y_i \). By injectivity of \( T \), the operator \( T^* \) has closed range and \( X = R_{T^*} \).

This completes the proof.

Now we define \(-g-\) frame operator and studies some of its properties.

Definition 2.4. Let \( \{\Lambda_i \in \text{End}^*(X, V_i) : i \in I\} \) be a \(-g-\) frame with lower and upper \(-g-\) frame, \( C \) and \( D \). Then its \(-g-\) frame operator \( S \) is defined by: \( S(x) = T^*T(x) = \sum_{i \in I} \Lambda_i^* \Lambda_i(x) \), \( (\forall x \in X) \).
**Theorem 2.5.** Let \( \{\Lambda_i \in \text{End}^*(X, Y_i) : i \in I\} \) be a \(-g\)-frame with lower and upper \(-g\)-frame; \( C \) and \( D \), respectively, and with the \(-g\)-frame operator \( S \). Then \( S \), positive, invertible, adjointable and \(|C^{-1}||^2 \leq |S| \leq |D|^2|\).

**Proof.** By Lemma 1.4, and Theorem 2.3, \( S \) is invertible, positive and self-adjointable map. The definition of \(-g\)-frame and of the operator \( S \), concludes that

\[
\langle Sx, x \rangle = \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle \leq D \langle x, x \rangle D^*
\]

and

\[
\langle x, x \rangle \leq C^{-1} \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle (C^*)^{-1}, \forall x \in X.
\]

Then

\[
||C^{-1}||^{-2} ||\langle x, x \rangle|| \leq ||\langle Sx, x \rangle|| \leq ||D||^2 ||\langle x, x \rangle||
\]

for all \( x \in X \). So

\[
||C^{-1}||^{-2} \leq |S| \leq ||D||^2.
\]

This completes the proof.

Now we gave a generalization for Theorem 3.3 in [15].

**Theorem 2.6.** Let for every \( i \in I, \Lambda_i \in \text{End}^*(X, Y_i) \) and \( \{y_{i,j}, j \in I_i\} \) be a \(*\)-frame for \( Y_i \) with frame bounds \( C_i, D_i \), such there exist \( C, D \) strictly nonzero of \( A \) such that

\[
CaC^* \leq C_i a C_i^* \text{ and } D_i a D_i^* \leq Da D^*, \quad (2.5)
\]

for all positive \( a \) of \( A \).

Then the following conditions are equivalent

(i) \( \{\Lambda_i^* (y_{i,j}), j \in I_i\} \) is a \(*\)-frame for \( X \).

(ii) \( \{\Lambda_i, i \in I\} \) is a \(*-g\)-frame for \( X \).

**Proof.** Let \( i \in I \), since \( \{y_{i,j}, j \in I_i\} \) is a \(*\)-frame for \( Y_i \) with bounds \( C_i, D_i \), we have

\[
C_i \langle \Lambda_i x, \Lambda_i x \rangle C_i^* \leq \sum_{j \in I_i} \langle \Lambda_i x, y_{i,j} \rangle \langle y_{i,j}, \Lambda_i x \rangle \leq D_i \langle \Lambda_i x, \Lambda_i x \rangle D_i^*, \quad (2.6)
\]

for all \( x \in X \). So

\[
C_i \langle \Lambda_i x, \Lambda_i x \rangle C_i^* \leq \sum_{j \in I_i} \langle x, \Lambda_i^* y_{i,j} \rangle \langle \Lambda_i^* y_{i,j}, x \rangle \leq D_i \langle \Lambda_i x, \Lambda_i x \rangle D_i^*, \forall x \in X. \quad (2.7)
\]
And by using the conditions (2.5), we deduce

\[ C \langle \Lambda_i x, \Lambda_i x \rangle C^* \leq \sum_{j \in I_i} \langle x, \Lambda_i^* y_{i,j} \rangle \langle \Lambda_i^* y_{i,j}, x \rangle \leq D \langle \Lambda_i x, \Lambda_i x \rangle D^*, \forall x \in X. \]  

(2.8)

Then

\[ C \sum_i \langle \Lambda_i x, \Lambda_i x \rangle C^* \leq \sum_i \sum_{j \in I_i} \langle x, \Lambda_i^* y_{i,j} \rangle \langle \Lambda_i^* y_{i,j}, x \rangle \leq D \sum_i \langle \Lambda_i x, \Lambda_i x \rangle D^*, \forall x \in X. \]  

(2.9)

And, if we suppose the condition \( \{ \Lambda_i^* (y_{i,j}), j \in I_i \} \) is a \(*\)-frame for \( X \) with frame bounds \( C', D' \), we have

\[ C' \langle x, x \rangle C'^* \leq \sum_i \sum_{j \in I_i} \langle x, \Lambda_i^* y_{i,j} \rangle \langle \Lambda_i^* y_{i,j}, x \rangle \leq D' \langle x, x \rangle D'^*, \forall x \in X. \]  

(2.10)

So, by combining (2.9) and (2.10), we get

\[ C \sum_i \langle \Lambda_i x, \Lambda_i x \rangle C^* \leq D' \langle x, x \rangle D'^* \]

and

\[ C' \langle x, x \rangle C'^* \leq D \sum_i \langle \Lambda_i x, \Lambda_i x \rangle D^*, \forall x \in X. \]

Then

\[ D^{-1} C' \langle x, x \rangle C'^* (D^*)^{-1} \leq \sum_i \langle \Lambda_i x, \Lambda_i x \rangle \leq C^{-1} D' \langle x, x \rangle (D'^*)(C^*)^{-1}, \forall x \in X. \]

So (i) \( \Rightarrow \) (ii). Next, we show the Converse. Similarly we have

\[ C \sum_i \langle \Lambda_i x, \Lambda_i x \rangle C^* \leq \sum_i \sum_{j \in I_i} \langle x, \Lambda_i^* y_{i,j} \rangle \langle \Lambda_i^* y_{i,j}, x \rangle \leq D \sum_i \langle \Lambda_i x, \Lambda_i x \rangle D^*, \forall x \in X. \]  

(2.11)

If we suppose \( \{ \Lambda_i, i \in I \} \) is a \(* - g - \) frame, with frame bounds \( C', D' \), so

\[ D \sum_i \langle \Lambda_i x, \Lambda_i x \rangle D^* \leq DD' \langle x, x \rangle D'^* D^* \]

and

\[ C C' \langle x, x \rangle C'^* \leq C \sum_i \langle \Lambda_i x, \Lambda_i x \rangle C^*, \forall x \in X. \]

Then

\[ C C' \langle x, x \rangle C'^* \leq \sum_i \sum_{j \in I_i} \langle x, \Lambda_i^* y_{i,j} \rangle \langle \Lambda_i^* y_{i,j}, x \rangle \leq DD' \langle x, x \rangle D'^* D^*, \forall x \in X. \]

This completes the proof.

The next result is analog to Corollary 3.4 in [15].
**Corollary 2.7.** Let for every $i \in I, \Lambda_i \in \text{End}^\ast(X, Y_i)$ and $\{x_{i,j}, j \in I_i\}$ be a Parseval $\ast$-frames for $Y_i$. Then we have

(i) $\{\Lambda_i, i \in I\}$ is a $\ast - g$--frame (resp. $\ast - g$--Bessel sequence, tight $\ast - g$--frame) for $X$ if only if $\{\Lambda_i^\ast x_{i,j}, i \in I, j \in I_i\}$ is a $\ast$--frames (resp. Bessel sequence, tight frame ) for $X$

(ii) The $\ast - g$--frame operator of $\{\Lambda_i, i \in I\}$ is the $\ast - g$--frame operator of $\{\Lambda_i^\ast x_{i,j}, i \in I, j \in I_i\}$

**Proof.** (i) Follow from the theorem 2.6.

(ii) Letting $x \in X$ and $y \in Y$, we have

$$
\langle \Lambda_i^\ast y, x \rangle = \langle y, \Lambda_i x \rangle = \sum \langle y, x_{i,j} \rangle \langle x_{i,j}, \Lambda_i x \rangle = \sum \langle y, x_{i,j} \rangle \langle \Lambda_i^\ast x_{i,j}, x \rangle = \sum \langle \langle y, x_{i,j} \rangle \Lambda_i^\ast x_{i,j}, x \rangle.
$$

Then $\Lambda_i^\ast y = \sum_j \langle y, x_{i,j} \rangle \Lambda_i^\ast x_{i,j}$. So

$$
\sum_i \Lambda_i^\ast \Lambda_i x = \sum_i \sum_j \langle \Lambda_i x, x_{i,j} \rangle \Lambda_i^\ast x_{i,j} = \sum_i \sum_j \langle x, \Lambda_i^\ast x_{i,j} \rangle \Lambda_i^\ast x_{i,j}.
$$

Since, the $\ast$--frame operator of $\{\Lambda_i^\ast x_{i,j}, i \in I, j \in I_i\}$, is defined by: $S'(x) = \sum_i \sum_j \langle x, \Lambda_i^\ast x_{i,j} \rangle \Lambda_i^\ast x_{i,j}$, see [1]. Then the $\ast - g$--frame operator of $\{\Lambda_i, i \in I\}$ is the $\ast - g$--frame operator of $\{\Lambda_i^\ast x_{i,j}, i \in I, j \in I_i\}$.

This completes the proof.

Now we gave a generalization of Theorem 3.5 in [15].

**Theorem 2.8.** Let $\{\Lambda_i \in \text{End}^\ast(X, Y_i) : i \in I\}$ be a $\ast - g$--frame with lower and upper $\ast - g$--frame; $C$ and $D$, respectively and with The $\ast - g$--frame operator $S$, and $M$ be a Hilbert $A$--module and let $T \in \text{End}^\ast(M, X)$ be invertible. Then $\{\Lambda_i T \in \text{End}^\ast(M, Y_i, i \in I)\}$ is a $\ast - g$--frame with $\ast - g$--frame operator $T^\ast ST$ with bounds $\|T^{-1}\|^{-1}C, \|T\|D$.

**Proof.** We have

$$
C \langle Tx, Tx \rangle C^\ast \leq \sum_{i \in I} \langle \Lambda_i T x, \Lambda_i T x \rangle \leq D \langle Tx, Tx \rangle D^\ast,
$$

(2.12)
for all \(x \in M\). Using Lemma 1.4, we have \(\|TT^*\|^{-1} \|T^{-1}\|^{-1} \langle x,x \rangle \leq \langle Tx,Tx \rangle\). for all \(x \in M\). Or \(\|T^{-1}\|^{-2} \leq \|TT^*\|^{-1}\). This implies

\[
C \|T^{-1}\|^{-2} \langle x,x \rangle C^* \leq C \langle Tx,Tx \rangle C^*.
\]

(2.13)

for all \(x \in M\). And we know that \(\langle Tx,Tx \rangle = \langle T^*Tx,x \rangle \leq \|T^*T\| \langle x,x \rangle\). for all \(x \in M\). This implies that

\[
D \langle Tx,Tx \rangle D^* \leq D \|T\| \|x,x\| (D \|T\|)^*\]

(2.14)

for all \(x \in M\). Using (2.12), (2.13), (2.14) we have

\[
\|T^{-1}\| \|C \langle x,x \rangle (\|T^{-1}\|C)^* \leq \sum_{i} \langle \Lambda_iTx,\Lambda_iTx \rangle \leq \|T\| \|D \langle x,x \rangle (\|T\|D)^*\|.
\]

(2.15)

for all \(x \in M\). So \(\{\Lambda_i \in \text{End}^*(M,Y_i) : i \in I\}\) is a \(*-g\)-frame with bounds \(\|T^{-1}\| \|C,D\|\|T\|\).

Moreover for every \(x \in M\), we have

\[
T^*ST(x) = T^* \sum_{i \in I} \Lambda_i^* \Lambda_jT(x) = \sum_{i \in I} T^* \Lambda_i^* \Lambda_jT(x) = \sum_{i \in I} (\Lambda_iT)^* \Lambda_jT(x).
\]

for all \(x \in M\). This completes the proof.

The next result is a generalization of Corollary 3.6 in [1].

**Corollary 2.9.** Let \(\{\Lambda_i \in \text{End}^*(X,Y_i) : i \in I\}\) be a \(*-g\)-frame with lower and upper \(*-g\)-frame; \(C\) and \(D\), respectively and with The \(*-g\)-frame operator \(S\). Then \(\{\Lambda_iS^{-1} \in \text{End}^*(X,Y_i) : i \in I\}\) is a \(*-g\)-frame with lower and upper \(*-g\)-frame; \(\|D\|^{-2}\) and \(\|C^{-1}\|^2\), respectively.

\(*-g\)-frame operator \(S^{-1}\) and for every \(x \in X\), \(x = \sum \Lambda_iS^{-1} \Lambda_i^* = \sum (\Lambda_iS^{-1})^* \Lambda_i\)

**Proof.** By taking \(M = X\) and \(T = S^{-1}\) in Theorem (2.8), it follow that \(\{\Lambda_iS^{-1} \in \text{End}^*(X,Y_i) : i \in I\}\) is a \(*-g\)-frame where the \(*-g\)-frame operator is \(S^{-1}\) and, we have

\[
\|S\|^{-1} \langle x,x \rangle = \sum \langle \Lambda_iS^{-1}x,\Lambda_iS^{-1}x \rangle \leq \|S^{-1}\| \langle x,x \rangle, \forall x \in X.
\]

Or, we have \(\|S^{-1}\| \leq \|C^{-1}\|^2\) and \(\|S\| \leq \|D\|^2\), so \(\{\Lambda_iS^{-1} \in \text{End}^*(X,Y_i) : i \in I\}\) is a \(*-g\)-frame with lower and upper \(*-g\)-frame; \(\|D\|^{-2}\) and \(\|C^{-1}\|^2\), \(\forall x \in X\). Moreover since for every \(i \in I\), \((\Lambda_iS^{-1})^* = S^{-1} \Lambda_i^*\) and for every \(x \in X, x = S^{-1}Sx = SS^{-1}x\), then \(x = \sum \Lambda_iS^{-1} \Lambda_i^* x = \sum (\Lambda_iS^{-1})^* \Lambda_i x\). This complete the proof.
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