STRONG CONVERGENCE THEOREMS FOR FIXED POINT PROBLEMS AND GENERALIZED EQUILIBRIUM PROBLEMS OF THREE RELATIVELY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to introduce a new hybrid projection algorithm for finding a common element of the set of common fixed points of three relatively quasi-nonexpansive mappings and the set of solutions of a generalized equilibrium problem in Banach space. Our results improve and extend the corresponding results announced by many others.

Keywords: Relatively quasi-nonexpansive mappings; Generalized equilibrium problem; Fixed points.

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1. Introduction

Let $E$ be a real Banach space with the dual space of $E^*$ and let $\langle \cdot, \cdot \rangle$ be the generalized duality pairing between $E$ and $E^*$. Let $C$ be a nonempty closed convex subset of $E$. We denote the sets of nonnegative integers and real numbers by $N$ and $R$ respectively. Let $A : C \to E^*$ be a nonlinear mapping and $f : C \times C \to R$ be a bifunction. The generalized

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equilibrium problem is to find $u \in C$, such that

$$f(u, y) + \langle Au, y - u \rangle \geq 0, \forall y \in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by $GEP$.

Whenever $A \equiv 0$, problem (1.1) is equivalent to finding $u \in C$, such that

$$f(u, y) \geq 0, \forall y \in C. \quad (1.2)$$

The set of its solutions is denoted by $EP$.

Whenever $f \equiv 0$, problem (1.1) is equivalent to finding $u \in C$, such that

$$\langle Au, y - u \rangle \geq 0, \forall y \in C. \quad (1.3)$$

The set of its solutions is denoted by $VI(C, A)$.

We recall some definitions and results which will be needed in this paper. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$. Denote by $F(T)$ the set of fixed points of $T$, that is $F(T) = \{x \in C : Tx = x\}$. A mapping $A : C \to E^*$ is called $\alpha-$ inverse-strongly monotone, if there exits an $\alpha > 0$, such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C.$$ 

It is easy to see that if $A$ is $\alpha-$ inverse-strongly monotone mapping, then it is $\frac{1}{\alpha}$-Lipschitzian, i.e. $\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|, \forall x, y \in C$.

The mapping $J : E \to 2^{E^*}$ defined by $Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \forall x \in E$ is called the normalized duality mapping. It is well-known that if $E^*$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$. We also defined the function $\phi$ as following

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \forall x, y \in E. \quad (1.4)$$

Following Alber[1], the generalized projection $\Pi_C$ from $E$ onto $C$ is defined by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x), \forall x \in E.$$ 

It is clear that in Hilbert space $H$, (1.4) reduces to $\phi(x, y) = \|x - y\|^2$ and $\Pi_C$ is the metric projection of $H$ onto $C$. 

Very recently, Takhashi and Zembayashi[2] proposed the following iteration for a relatively nonexpansive mapping:

\[
\begin{align*}
    x_0 &= x \in C, \quad C_0 = C, \\
    y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n), \\
    u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
    H_n &= \{ z \in C : \phi(z, u_n) \leq \phi(z, x_n) \}, \\
    W_n &= \{ z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0 \}, \\
    x_{n+1} &= \Pi_{H_n \cap W_n}x_0,
\end{align*}
\]

and proved that the sequence \( \{x_n\} \) converges strongly to \( \Pi_{F(S) \cap EP}x_0 \).

In 2008, Qin et al.[3] introduced the following iterative for two closed relatively quasi-nonexpansive mappings in Banach space:

\[
\begin{align*}
    x_0 &\in E, \quad C_1 = C, \\
    y_n &= J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\
    u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
    C_{n+1} &= \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\
    x_{n+1} &= \Pi_{C_{n+1}}x_0,
\end{align*}
\]

and proved that the sequence \( \{x_n\} \) converges strongly to \( \Pi_{F(T) \cap F(S) \cap EP}x_0 \).

In 2009, K.Wattanawitoon and P.Kuman[4] introduced the following iterative for two closed relatively quasi-nonexpansive mappings in Banach space:

\[
\begin{align*}
    x_0 &\in E, \quad C_1 = C, \\
    y_n &= J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\
    z_n &= J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\
    u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jz_n \rangle \geq 0, \forall y \in C, \\
    C_{n+1} &= \{ z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
    x_{n+1} &= \Pi_{C_{n+1}}x_0,
\end{align*}
\]

and proved that the sequence \( \{x_n\} \) converges strongly to \( \Pi_{F(T) \cap F(S) \cap EP}x_0 \).

In 2010, S.S. Chang[5] introduced the following iterative for two relatively nonexpansive
mappings in Banach space:

\[
\begin{cases}
  x_0 \in E, \ C_1 = C, \\
  y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) JSz_n), \\
  z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JTz_n), \\
  u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
  H_n = \{v \in C : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1 - \beta_n) \phi(v, z_n) \leq \phi(v, x_n)\}, \\
  W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
  x_{n+1} = \Pi_{H_n \cap W_n}x_0,
\end{cases}
\] (1.8)

and proved that the sequence \( \{x_n\} \) converges strongly to \( \Pi_{F(T) \cap F(S) \cap \text{GEP}}x_0 \).

In this paper, motivated by K.Wattanawitoon and P.Kuman\[4\], we modified iterations of (1.7) to obtain strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces.

2. Preliminaries

Let \( C \) be a nonempty closed convex subset of \( E \), and let \( T \) be a mapping from \( C \) into itself. A point \( p \) in \( C \) is said to be an asymptotic fixed point of \( T \) if \( C \) contains a sequence \( \{x_n\} \) which converges weakly to \( p \) such that \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \). The set of asymptotic fixed points of \( T \) will be denoted by \( \widehat{F(T)} \). A mapping \( T \) from \( C \) into itself is said to be relatively nonexpansive if \( \widehat{F(T)} = F(T) \) and \( \phi(p, Tx) \leq \phi(p, x) \) for all \( x \in C \) and \( p \in F(T) \). \( T \) is said to be \( \phi \)-nonexpansive, if \( \phi(Tx, Ty) \leq \phi(x, y) \) for \( x, y \in C \). \( T \) is said to be relatively quasi-nonexpansive if \( F(T) \neq \emptyset \) and \( \phi(p, Tx) \leq \phi(p, x) \) for all \( x \in C \) and \( p \in F(T) \).

**Remark 2.1** The class of relatively quasi-nonexpansive is more general than the class of relatively nonexpansive mapping which requires the strong restriction: \( \widehat{F(T)} = F(T) \).

**Lemma 2.2** (Kamimura and Takahashi\[9\]) Let \( E \) be a uniformly convex and smooth Banach space and let \( \{x_n\} \) and \( \{y_n\} \) be two sequences of \( E \). If \( \phi(x_n, y_n) \to 0 \) and either \( \{x_n\} \) or \( \{y_n\} \) is bounded, then \( \|x_n - y_n\| \to 0 \).

**Lemma 2.3** (Alber\[1\]) Let \( C \) be a nonempty closed convex subset of a smooth Banach
space $E$ and $x \in E$. Then $x_0 = \Pi_C x$ if and only if
\[
\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \forall y \in C.
\]

**Lemma 2.4** (Alber[1]) Let $E$ be a reflexive, strictly convex subset of a smooth Banach space and let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then
\[
\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(x, y), \forall y \in C.
\]

**Lemma 2.5** (Qin et al.[3]) Let $E$ be a uniformly convex and smooth Banach space, let $C$ be a closed convex subset of $E$, and let $T$ be a closed and relatively quasi-$\phi$-nonexpansive mapping from $C$ into itself. Then $F(T)$ is a closed convex subset of $C$.

**Lemma 2.6** (Cho et al.[10]) Let $E$ be a uniformly convex Banach space and $B_r(0)$ be a closed ball of $E$. Then there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that
\[
\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|),
\]
for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

**Lemma 2.7** (Kamimura and Takahashi[9]) Let $E$ be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$.

For solving the generalized equilibrium problem, let us assume a bifunction $f$ satisfied the following conditions
\begin{enumerate}
\item[(A1)] $f(x, x) = 0$ for all $x \in C$;
\item[(A2)] $f$ is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
\item[(A3)] for all $x, y, z \in C, \limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;
\item[(A4)] for all $x \in C, f(x, \cdot)$ is convex and lower semicontinuous.
\end{enumerate}

**Lemma 2.8** (Blum and Oetti [11]) Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $f : C \times C \to R$ be a bifunction satisfying (A1)-(A4), and let $r > 0$ and $x \in E$, then there exists $z \in C$ such that
\[
f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C.
\]
Lemma 2.9 (Takahashi and Zembayashi [2]) Let $C$ be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$, and let $f : C \times C \to R$ be a bifunction satisfying (A1)-(A4), for $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows

$$T_r(x) = \{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \}.$$ 

for all $x \in E$. Then

(i) $T_r$ is single-valued;

(ii) $T_r$ is a firmly nonexpansive-type mapping, i.e.,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle, \quad \forall x, y \in E;$$

(iii) $F(T_r) \supseteq \overline{F(T_r)} = EP$;

(iv) $EP$ is a closed convex subset of $C$.

Lemma 2.10 (Takahashi and Zembayashi [2]) Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $f : C \times C \to R$ be a bifunction satisfying (A1)-(A4), and $r > 0$. Then for $x \in E$ and $q \in F(T_r)$,

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.11 (S.S. Chang [5]) Let $E$ be a smooth, strictly convex and reflexive Banach space and $C$ be a nonempty closed convex of $E$. Let $A : C \to E^*$ be an $\alpha$—inverse-strongly monotone mapping, let $f$ be a function from $C \times C \to R$ satisfying (A1)-(A4), and let $r > 0$. Then the following statements hold.

(I) for $x \in E$, there exists $u \in C$ such that

$$f(u, y) + \langle Au, y - u \rangle + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \forall y \in C;$$

(II) if $E$ is additionally uniformly smooth and $K_r : E \to C$ is defined as

$$K_r(x) = \{ u \in C : f(u, y) + \langle Au, y - u \rangle + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \forall y \in C \},$$

then the mapping $K_r$ has the following properties

(i) $K_r$ is single-valued.
(ii) $K_r$ is a firmly nonexpansive-type mapping, i.e.,

$$
\langle K_r x - K_r y, J K_r x - J K_r y \rangle \leq \langle K_r x - K_r y, J x - J y \rangle, \quad \forall x, y \in E,
$$

(iii) $F(K_r) = \overline{F(K_r)} = GEP$,

(iv) $GEP$ is a closed convex subset of $C$,

(v) $\phi(q, K_r x) + \phi(K_r x, x) \leq \phi(q, x), \forall q \in F(K_r)$.

3. Main results

Theorem 3.1 Let $C$ be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $A : C \to E^*$ be an $\alpha-$inverse-strong monotone mapping and let $f$ be a bifunction from $C \times C \to R$ satisfying (A1)-(A4), let $T, S, R : C \to C$ be three closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap F(R) \cap GEP \neq \emptyset$. \{x_n\}, \{y_n\}, \{z_n\} and \{u_n\} are the sequences generated by the following,

$$
\begin{cases}
  x_0 \in E, \quad C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\
  y_n = J^{-1}(\delta_n Jx_n + (1 - \delta_n) JRz_n), \\
  z_n = J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n J Sx_n), \\
  u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \forall y \in C, \\
  C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1 - \delta_n) \phi(z, z_n) \leq \phi(z, x_n)\}, \\
  x_{n+1} = \Pi_{C_{n+1}} x_0.
\end{cases}
$$

(3.1)

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the restrictions,

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n \to \infty} \alpha_n \beta_n > 0, \lim_{n \to \infty} \alpha_n \gamma_n > 0, \lim_{n \to \infty} \delta_n (1 - \delta_n) > 0$;

(c) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F$, where $z = \Pi_F x_0$. 

Proof. We split the proof into six steps.

Step 1. We show that $C_n$ is closed and convex for all $n \geq 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that $C_k$ is closed and convex for some $k \in N$. For $z \in C_k$, one obtains that $$\phi(z, u_k) \leq \delta_k \phi(z, x_k) + (1 - \delta_k) \phi(z, z_k),$$
is equivalent to

$$2 \langle z, (1 - \delta_k)Jz_k + \delta_k Jx_k - Ju_k \rangle \leq (1 - \delta_k) \|z_k\|^2 - \|u_k\|^2 + \delta_k \|x_k\|^2,$$

and

$$\delta_k \phi(z, x_k) + (1 - \delta_k) \phi(z, z_k) \leq \phi(z, x_k),$$
is equivalent to

$$2 \langle z, Jx_k - Jz_k \rangle \leq \|x_k\|^2 - \|z_k\|^2.$$

It implies that $C_{k+1}$ is closed and convex. Then, for all $n \geq 0$, $C_n$ is closed and convex. This show that $\Pi_{C_n+1} x_0$ is well defined. Notice that $u_n = K_{r_n}y_n$ for all $n \geq 1$.

Step 2. Let us show that $F \subset C_n$ for each $n \geq 0$.

$F \subset C_1 = C$ is obvious, suppose $F \subset C_k$ for some $k \in N$, then for any $w \in F \subset C_k$, one has,

$$\phi(w, z_k) = \phi(w, J^{-1}(\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k))$$

$$= \|w\|^2 - 2\alpha_k \langle w, Jx_k \rangle - 2\beta_k \langle w, JTx_k \rangle - 2\gamma_k \langle w, JSx_k \rangle + \|\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k\|^2$$

$$\leq \|w\|^2 - 2\alpha_k \langle w, Jx_k \rangle - 2\beta_k \langle w, JTx_k \rangle - 2\gamma_k \langle w, JSx_k \rangle$$

$$+ \alpha_k \|Jx_k\|^2 + \beta_k \|JTx_k\|^2 + \gamma_k \|JSx_k\|^2$$

$$= \alpha_k \phi(w, x_k) + \beta_k \phi(w, T x_k) + \gamma_k \phi(w, S x_k)$$

$$\leq \phi(w, x_k),$$

(3.2)
and

\[
\phi(w, u_k) = \phi(w, K_{rk} y_k) \\
\leq \phi(w, y_k) \\
= \phi(w, J^{-1}(\delta_k Jx_k + (1 - \delta_k)JRz_k)) \\
\leq \|w\|^2 - 2\delta_k \langle w, Jx_k \rangle - 2(1 - \delta_k) \langle w, JRz_k \rangle + \delta_k \|x_k\|^2 + (1 - \delta_k) \|Rz_k\|^2 \\
= \delta_k \phi(w, x_k) + (1 - \delta_k) \phi(w, Rz_k) \\
\leq \delta_k \phi(w, x_k) + (1 - \delta_k) \phi(w, z_k) \\
\leq \delta_k \phi(w, x_k) + (1 - \delta_k) \phi(w, x_k) \\
= \phi(w, x_k),
\]

(3.3)

that is \( w \in C_{k+1} \). This implies that \( F \subset C_n \) for all \( n \geq 0 \).

Step 3. We claim that \( \{x_n\} \) is bounded, and \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \). Indeed, by the definition of \( x_n = \Pi_{C_n} x_0 \), from Lemma 2.4 it follows that for each \( w \in F \) and each \( n \geq 1 \), we obtain

\[
\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, \Pi_{C_n} x_0) \leq \phi(w, x_0).
\]

This implies that \( \{\phi(x_n, x_0)\} \) is bounded, and so \( \{x_n\}, \{u_n\}, \{z_n\}, \{Tx_n\}, \{Sx_n\}, \{Rz_n\} \) are all bounded. Furthermore, noticing that \( x_n = \Pi_{C_n} x_0 \) and \( x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n \), we get \( \phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \), for all \( n \geq 0 \). Thus, \( \phi(x_n, x_0) \) is nondecreasing, so the limit of \( \phi(x_n, x_0) \) exists, from Lemma 2.4 we have

\[
\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0),
\]

which leads to \( \lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0 \), it follows that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

Step 4. We will prove that \( \{x_n\} \) is a cauchy sequence.

By the construction of \( C_n \), one has that \( C_m \subset C_n \) and \( x_m = \Pi_{C_m} x_0 \in C_n \) for any positive integer \( m \geq n \). It follows that,

\[
\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \phi(x_m, x_0) - \phi(x_n, x_0),
\]

letting \( m, n \to \infty \), one has \( \phi(x_m, x_n) \to 0 \), it follows \( \lim_{n \to \infty} \|x_m - x_n\| = 0 \). Hence \( \{x_n\} \) is a Cauchy sequence. We can assume that \( x_n \to p \in C \), as \( n \to \infty \).
Step 5. We claim that $p \in F$.

In fact, for $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, from the definition of $C_{n+1}$ we conclude that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n),$$

and

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n).$$

It follows that $\phi(x_{n+1}, u_n) \to 0$ and $\phi(x_{n+1}, z_n) \to 0$. One has

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \|x_{n+1} - u_n\| = \lim_{n \to \infty} \|x_{n+1} - z_n\| = 0, \quad (3.4)$$

and so

$$\lim_{n \to \infty} \|x_n - u_n\| = \lim_{n \to \infty} \|x_n - z_n\| = \lim_{n \to \infty} \|u_n - z_n\| = 0 \quad (3.5).$$

Since $E$ is uniformly smooth, and $J$ is uniformly norm-norm continuous on bounded sets, we have

$$\lim_{n \to \infty} \|Jx_n - Ju_n\| = \lim_{n \to \infty} \|Jx_n - Jz_n\| = 0. \quad (3.6)$$

Let $r = \sup_{n \geq 0}\{\|x_n\|, \|Tx_n\|, \|Sx_n\|, \|Rz_n\|\}$. From Lemma 2.9 and Lemma2.6, one has

$$\phi(w, z_n) = \phi(w, J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n))$$

$$= \|w\|^2 - 2\alpha_n \langle w, Jx_n \rangle - 2\beta_n \langle w, JTx_n \rangle - 2\gamma_n \langle w, JSx_n \rangle - 2\alpha_n \|Jx_n + \beta_n JTx_n + \gamma_n JSx_n\|^2$$

$$\leq \|w\|^2 - 2\alpha_n \langle w, Jx_n \rangle - 2\beta_n \langle w, JTx_n \rangle - 2\gamma_n \langle w, JSx_n \rangle + \alpha_n \|Jx_n\|^2 + \beta_n \|JTx_n\|^2$$

$$+ \gamma_n \|JSx_n\|^2 - \alpha_n \beta_n g(\|JTx_n - Jx_n\|)$$

$$= \alpha_n \phi(w, x_n) + \beta_n \phi(w, Tx_n) + \gamma_n \phi(w, Sx_n) - \alpha_n \beta_n g(\|JTx_n - Jx_n\|)$$

$$\leq \phi(w, x_n) - \alpha_n \beta_n g(\|JTx_n - Jx_n\|). \quad (3.7)$$

Then

$$\alpha_n \beta_n g(\|JTx_n - Jx_n\|) \leq \phi(w, x_n) - \phi(w, z_n). \quad (3.8)$$

On the other hand, we have

$$\phi(w, x_n) - \phi(w, z_n) = \|x_n\|^2 - \|z_n\|^2 - 2\langle w, Jx_n - Jz_n \rangle$$

$$= (\|x_n\| - \|z_n\|)(\|x_n\| + \|z_n\|) - 2\langle w, Jx_n - Jz_n \rangle \quad (3.9)$$

$$\leq (\|x_n - z_n\|)(\|x_n\| + \|z_n\|) + 2\|w\|\|Jx_n - Jz_n\|. $$
It follows from (3.5) and (3.6), we have

\[
\lim_{n \to \infty} [\phi(w, x_n) - \phi(w, z_n)] = 0,
\]

hence,

\[
\lim_{n \to \infty} g(\|JT x_n - Jx_n\|) = 0.
\]

From the property of \( g \) that

\[
\lim_{n \to \infty} \|JT x_n - Jx_n\| = 0.
\]

Since \( J^{-1} \) is also uniformly norm-norm continuous on bounded sets, we see that

\[
\lim_{n \to \infty} \|T x_n - x_n\| = 0.
\]

(3.11)

Similarly,

\[
\lim_{n \to \infty} \|S x_n - x_n\| = 0.
\]

(3.12)

From Lemma 2.6, one also has

\[
\phi(w, y_n) = \phi(w, J^{-1}(\delta_n Jx_n + (1 - \delta_n)JRz_n))
\]

\[
= \|w\|^2 - 2\langle w, \delta_n Jx_n + (1 - \delta_n)JRz_n \rangle + \|\delta_n Jx_n + (1 - \delta_n)JRz_n\|^2
\]

\[
\leq \|w\|^2 - 2\delta_n \langle w, Jx_n \rangle - 2(1 - \delta_n)\langle w, JRz_n \rangle + \delta_n \|x_n\|^2 + (1 - \delta_n)\|Rz_n\|^2
\]

\[
- \delta_n (1 - \delta_n)g(\|JRz_n - Jx_n\|)
\]

\[
= \delta_n \phi(w, x_n) + (1 - \delta_n)\phi(w, Rz_n) - \delta_n (1 - \delta_n)g(\|JRz_n - Jx_n\|)
\]

\[
\leq \delta_n \phi(w, x_n) + (1 - \delta_n)\phi(w, z_n) - \delta_n (1 - \delta_n)g(\|JRz_n - Jx_n\|)
\]

\[
\leq \phi(w, x_n) - \delta_n (1 - \delta_n)g(\|JRz_n - Jx_n\|).
\]

(3.13)

Hence

\[
\delta_n (1 - \delta_n)g(\|JRz_n - Jx_n\|) \leq \phi(w, x_n) - \phi(w, y_n).
\]

(3.14)

On the other hand, we get

\[
\phi(w, x_n) - \phi(w, y_n) = \|x_n\|^2 - \|y_n\|^2 - 2\langle w, Jx_n - Jy_n \rangle
\]

\[
= (\|x_n\| - \|y_n\|)(\|x_n\| + \|y_n\|) - 2\langle w, Jx_n - Jy_n \rangle
\]

\[
\leq (\|x_n - y_n\|)(\|x_n\| + \|y_n\|) + 2\|w\|\|Jx_n - Jy_n\|.
\]

(3.15)
From Lemma 2.8, we have
\[
\phi(u_n, y_n) = \phi(K_{r_n}y_n, y_n)
\leq \phi(w, y_n) - \phi(w, K_{r_n}y_n)
\leq \phi(w, x_n) - \phi(w, u_n)
\leq \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n \rangle
\leq (\|x_n - u_n\|)(\|x_n\| + \|u_n\|) + 2\|w\||Jx_n - Ju_n|.
\]
(3.16)

Hence
\[
\lim_{n \to \infty} \phi(u_n, y_n) = 0,
\]
and so,
\[
\lim_{n \to \infty} \|u_n - y_n\| = 0.
\]
(3.17)

Combining with (3.5), we conclude that
\[
\lim_{n \to \infty} \|x_n - y_n\| = 0, \lim_{n \to \infty} \|Jx_n - Ju_n\| = 0.
\]
(3.18)

It follows from (3.14), (3.15), (3.18), we obtain
\[
g(\|JRz_n - Jx_n\|) \to 0, n \to \infty.
\]
and so,
\[
\lim_{n \to \infty} \|Rz_n - x_n\| = 0.
\]
Noticing (3.5), we have
\[
\lim_{n \to \infty} \|Rz_n - z_n\| = 0.
\]
(3.19)

From the closedness of \(S, T\) and \(R\), we get \(p \in F\). Next, we show \(p \in GEP = F(K_r)\).

Since \(J\) is uniformly norm-to-norm continuous on bounded subsets of \(E\), from (3.17) we have \(\lim_{n \to \infty} \|Ju_n - Jy_n\| = 0\). From \(r_n \geq a > 0\), then
\[
\lim_{n \to \infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0.
\]
(3.20)

Let \(F(u, y) = f(u, y) + \langle Au, y - u \rangle\), for \(u_n = K_{r_n}y_n\), we have
\[
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C.
\]
Therefore,
\[ \left\| y - u_n \right\| \left\| Ju_n - Jy_n \right\| r_n \geq \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq -F(u_n, y) \geq F(y, u_n). \]

By taking the limit as \( n \to \infty \) in the above inequality and from (A4) and (3.21), one has
\[ F(y, p) \leq 0, \forall y \in C, \]

For all \( 0 < t < 1 \) and \( y \in C \), define \( y_t = ty + (1 - t)p \). Noticing that \( y, p \in C \), then \( y_t \in C \), which yields that \( F(y_t, p) \leq 0 \). From (A1) and (A4) that
\[ 0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, p) \leq tF(y_t, y). \]

That is
\[ F(y_t, y) \geq 0. \]

Let \( t \downarrow 0 \), we obtain \( F(p, y) \geq 0, \forall y \in C \). This implies that \( p \in GEP \). This shows that \( p \in F \).

Step 6. We prove \( p = \Pi_F x_0 \).

In fact, by Lemma 2.5,
\[ \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \forall z \in C_n. \]

Since \( F \subset C_n \) for all \( n \geq 1 \), we arrive at
\[ \langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \forall w \in F. \]

By taking the limit in the above inequality, one has
\[ \langle p - w, Jx_0 - Jp \rangle \geq 0, \forall w \in F. \]

At this point, in view of Lemma 2.3, we can get \( p = \Pi_F x_0 \). This completes the proof of theorem3.1.

Putting \( A = 0 \) in Theorem3.1, we can get,

**Corollary 3.2** Let \( C \) be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space \( E \). Let \( f \) be a bifunction from \( C \times C \to R \) satisfying (A1)-(A4) and let \( T, S, R : C \to C \) be three closed relatively quasi-nonexpansive mappings
such that $F := F(T) \cap F(S) \cap F(R) \cap EP \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$\begin{cases}
  x_0 \in E, \ C_1 = C, \ x_1 = \Pi_{C_1} x_0, \\
  y_n = J^{-1}(\delta_n Jx_n + (1 - \delta_n)JRz_n), \\
  z_n = J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n JSx_n), \\
  u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
  C_{n+1} = \{ z \in C : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1 - \delta_n) \phi(z, z_n) \leq \phi(z, x_n) \}, \\
  x_{n+1} = \Pi_{C_{n+1}} x_0.
\end{cases} \tag{3.21}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n \to \infty} \alpha_n \beta_n > 0, \lim_{n \to \infty} \alpha_n \gamma_n > 0, \lim_{n \to \infty} \delta_n (1 - \delta_n) > 0$;

(c) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F$, where $z = \Pi_F x_0$.

Putting $f = 0$ in Theorem 3.1, we can obtain,

**Corollary 3.3** Let $C$ be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $A : C \to C^*$ be an $\alpha-$ inverse- strong monotone mapping and let $T, S, R : C \to C$ be three closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap F(R) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$\begin{cases}
  x_0 \in E, \ C_1 = C, \ x_1 = \Pi_{C_1} x_0, \\
  y_n = J^{-1}(\delta_n Jx_n + (1 - \delta_n)JRz_n), \\
  z_n = J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n JSx_n), \\
  u_n \in C \text{ such that } \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
  C_{n+1} = \{ z \in C : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1 - \delta_n) \phi(z, z_n) \leq \phi(z, x_n) \}, \\
  x_{n+1} = \Pi_{C_{n+1}} x_0.
\end{cases} \tag{3.22}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n \to \infty} \alpha_n \beta_n > 0, \lim_{n \to \infty} \alpha_n \gamma_n > 0, \lim_{n \to \infty} \delta_n (1 - \delta_n) > 0$;

(c) $\{r_n\} \subset [a, \infty)$ for some $a > 0$. 
Then \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in F \), where \( z = \Pi_F x_0 \).

**Corollary 3.4** Let \( C \) be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space \( E \). Let \( f \) be a bifunction from \( C \times C \to R \) satisfying (A1)-(A4) and let \( R : C \to C \) be a closed relatively quasi-nonexpansive mapping such that \( F := F(R) \cap EP \neq \emptyset \). Let \( \{x_n\} \) be the sequences generated by the following:

\[
\begin{align*}
  x_0 & \in E, \ C_1 = C, \ x_1 = \Pi_{C_1} x_0, \\
  y_n & = J^{-1}(\delta_n Jx_n + (1 - \delta_n)JRx_n), \\
  u_n & \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \quad (3.23) \\
  C_{n+1} & = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\
  x_{n+1} & = \Pi_{C_{n+1}} x_0.
\end{align*}
\]

Suppose that \( \delta_n \) are sequences in \([0,1]\) satisfying the restrictions:

(a) \( \lim_{n \to \infty} \delta_n (1 - \delta_n) > 0 \);
(b) \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \).

Then \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in F \), where \( z = \Pi_F x_0 \).

**Corollary 3.5** Let \( C \) be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space \( E \). Let \( f \) be a bifunction from \( C \times C \to R \) satisfying (A1)-(A4) and let \( T, S : C \to C \) be two closed relatively quasi-nonexpansive mappings such that \( F := F(T) \cap F(S) \cap EP \neq \emptyset \). Let \( \{x_n\} \) and \( \{u_n\} \) be the sequences generated by the following:

\[
\begin{align*}
  x_0 & \in E, \ C_1 = C, \ x_1 = \Pi_{C_1} x_0, \\
  y_n & = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\
  u_n & \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \quad (3.24) \\
  C_{n+1} & = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\
  x_{n+1} & = \Pi_{C_{n+1}} x_0.
\end{align*}
\]

Suppose that \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \([0,1]\) satisfying the restrictions:

(a) \( \alpha_n + \beta_n + \gamma_n = 1 \);
(b) \( \lim_{n \to \infty} \alpha_n \beta_n > 0, \lim_{n \to \infty} \alpha_n \gamma_n > 0 \),
(c) \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \).

Then \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in F \), where \( z = \Pi_F x_0 \).
Corollary 3.6 Let $C$ be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $f$ be a bifunction from $C \times C \to R$ satisfying (A1)-(A4) and let $T, S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$
\begin{aligned}
x_0 & \in E, \ C_1 = C, \ x_1 = \Pi_{C_1}x_0, \\
y_n & = J^{-1}(\delta_n Jx_n + (1 - \delta_n)Jz_n), \\
z_n & = J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n JS x_n), \\
u_n & \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
C_{n+1} & = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} & = \Pi_{C_{n+1}}x_0.
\end{aligned}
$$

(3.25)

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n \to \infty} \alpha_n \beta_n > 0, \lim_{n \to \infty} \alpha_n \gamma_n > 0, \lim_{n \to \infty} \delta_n (1 - \delta_n) > 0$;

(c) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F$, where $z = \Pi_Fx_0$.

Corollary 3.7 Let $C$ be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $A : C \to E^*$ be an $\alpha-$ inverse-strong monotone mapping and $f$ be a bifunction from $C \times C \to R$ satisfying (A1)-(A4) and let $T, R : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(R) \cap GEP \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$
\begin{aligned}
x_0 & \in E, \ C_1 = C, \ x_1 = \Pi_{C_1}x_0, \\
y_n & = J^{-1}(\delta_n Jx_n + (1 - \delta_n)JRz_n), \\
z_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\
u_n & \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
C_{n+1} & = \{z \in C_n : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1 - \delta_n) \phi(z, z_n) \leq \phi(z, x_n)\}, \\
x_{n+1} & = \Pi_{C_{n+1}}x_0.
\end{aligned}
$$

(3.26)
Suppose that \( \{\alpha_n\} \) and \( \{\delta_n\} \) are two sequences in \([0,1]\) satisfying the restrictions:

(a) \( \lim_{n \to \infty} \alpha_n (1 - \alpha_n) > 0 \), \( \lim_{n \to \infty} \delta_n (1 - \delta_n) > 0 \);

(b) \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \).

Then \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in F \), where \( z = \Pi_F x_0 \).

**Remark 3.8** From Corollary 3.4, 3.5, 3.6 and Corollary 3.7, we see Theorem 3.1 improve and extend the recent ones announced by W. Takahashi and K. Zembayashi[2], Qin et al.[3], K. Wattanawitoon and P. Kumam [4] and S. S. Chang[5].

**References**


