ON $*-n$-PARANORMAL OPERATORS ON BANACH SPACES

MUNEO CHÔ $^1$, KÔTARÔ TANAHASHI$^2$

$^1$Department of Mathematics, Kanagawa University, Hiratsuka City 259-1293, Japan
$^2$Department of Mathematics, Tohoku Pharmaceutical University, Sendai City 981-8558, Japan

Copyright © 2014 Chô and Tanahashi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. Let $T$ be a $*-n$-paranormal operator on a complex Banach space $\mathcal{X}$. In this paper we show that $T$ is isoloid and if $\alpha, \beta$ are distinct eigen-values of $T$, then $\ker(T - \alpha I) \perp \ker(T - \beta I)$. Also we show that if the dual space $\mathcal{X}^*$ is uniformly convex and $(T - \alpha I)x_k \rightarrow 0$ for $(x_k, f_k) \in \Pi(\mathcal{X})$, then $(T - \alpha I)^* f_k \rightarrow 0$.

Keywords: Banach space, $*-n$-paranormal operator, spectrum.

2000 AMS Subject Classification: 47A10, 47B20

1. Introduction


*Corresponding author

Dedicated to Mrs. Aiko Itagaki of first author’s teacher with cordial gratitude

Received November 19, 2013
and K. Tanahashi and M. Chō showed that a polynomially *-paranormal operator $T$ on a complex Hilbert space is isoloid (That is, an isolated point of the spectrum is an eigen-value.) and the spectral mapping theorem holds for the essential approximate point spectrum of $T$. In this paper we study spectral properties of *-$n$-paranormal Banach space operators.

2. Preliminaries

Let $\mathcal{X}$ be a complex Banach space and $T \in B(\mathcal{X})$. Let $\Pi(\mathcal{X})$ be

$$
\Pi(\mathcal{X}) = \{(x,f) \in \mathcal{X} \times \mathcal{X}^* : \|f\| = f(x) = \|x\| = 1\},
$$

where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$. We define the numerical range $V(T)$ of $T$ by

$$
V(T) = \{f(Tx) : (x,f) \in \Pi(\mathcal{X})\}.
$$

It is well known that following inclusion relations hold

$$
\text{co} \sigma(T) \subset V(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\},
$$

where $\text{co} \sigma(T), V(T)$ and $\|T\|$ are the convex hull of the spectrum $\sigma(T)$, the closure of $V(T)$ and the norm of $T$, respectively. See Theorem 19.4 of [3].

**Definition 1.**

(1) $T$ is said to be $n$-paranormal if $\|Tx\|^n \leq \|T^n x\| \cdot \|x\|^{n-1}$ for all $x \in \mathcal{X}$.

(2) $T$ is said to be *-$n$-paranormal if $\|T^* f\|^n \leq \|T^n x\|$ for all $(x,f) \in \Pi(\mathcal{X})$, where $T^*$ is the dual operator of $T$.

2-Paranormal operators are simply called paranormal. We denote the sets of all $n$-paranormal operators and *-$n$-paranormal operators by $\mathcal{P}(n)$ and $\mathcal{S}(n)$, respectively. In [4] M. Chō and S. Ōta proved that $\mathcal{S}(n) \subset \mathcal{P}(n+1)$ for every $n \in \mathbb{N}$ and $\mathcal{P}(2) \subset \bigcap_{n=3}^{\infty} \mathcal{P}(n) = \mathcal{P}(3) \cap \mathcal{P}(4)$.

**Definition 2.** Let $A$ and $B$ be subspaces of $\mathcal{X}$. $A$ is orthogonal to $B$ (denoted $A \perp B$) if

$$
\|a\| \leq \|a+b\| \quad (a \in A, b \in B).
$$
3. Main results

Let \( \ker(T) \) and \( R(T) \) be the kernel and the range of \( T \), respectively. Then it is well known that \( \ker(T) \perp R(T) \) if and only if there exists \( (x, f) \in \Pi(\mathcal{X}) \) such that \( x \in \ker(T) \) and \( f \in \ker(T^*) \). See Lemma 20.3 of [3].

By the definition of (2) it is clear that if \( T \) is \(*-n\)-paranormal and \( Tx = 0 (\|x\| = 1) \), then, for any \( f \in \mathcal{X}^* \) such that \( (x, f) \in \Pi(\mathcal{X}) \), \( T^* f = 0 \).

Hence we have the following result.

**Theorem 1.** Let \( T, S \) be \(*-n\)-paranormal operators. Then \( (\ker(T) \cap \ker(S)) \perp \mathcal{M} \), where \( \mathcal{M} \) is the smallest subspace containing \( R(T) \) and \( R(S) \).

See page 14 of [6].

Next proposition is important in this paper. The paper [7] is for Hilbert space operators. But following results hold for Banach space operators.

**Proposition 1 (Proposition 1 and Theorem 1, [7]).** Let \( T \) be an \( n\)-paranormal operator on \( \mathcal{X} \).

1. \( T \) is normaloid, that is, \( \|T\| = r(T) \) (the spectral radius of \( T \)).
2. If \( T \) is invertible, then \( \|T^{-1}\| \leq r(T^{-1}) \frac{n(n-1)}{2} \cdot r(T) \frac{(n+1)(n-2)}{2} \).

In [1] P. Aiena showed following result for paranormal operators.

**Theorem 2.** Let \( T \) be an \( n\)-paranormal operator on \( \mathcal{X} \). If \( \sigma(T) = \{\alpha\} \), then \( T = \alpha I \).

**Proof.** If \( \alpha = 0 \), then \( \|T\| = r(T) = 0 \). Hence \( T = 0 \). We assume \( \alpha \neq 0 \). Let \( S = \frac{1}{\alpha} T \). Then \( S \) is invertible \( n\)-paranormal, \( \|S\| = 1 \), and \( \sigma(S) = \{1\} \). By Proposition 1, we have \( \|S^{-1}\| = 1 \). Hence \( \|S^n\| \leq \|S\|^n \leq 1 \) and \( \|S^{-n}\| \leq \|S^{-1}\|^n \leq 1 \) for \( n \in \mathbb{N} \). Hence \( S = I \) and \( T = \alpha I \) by Theorem 1 of [8]. This completes the proof.

**Definition 3.** \( T \) is said to be polynomially \( n\)-paranormal (\(*-n\)-paranormal) if there exists a non-constant polynomial \( p(x) \) such that \( p(T) \) is \( n\)-paranormal (\(*-n\)-paranormal).

**Theorem 3.** Let \( T \) be a polynomially \( n\)-paranormal operator on \( \mathcal{X} \). If \( \sigma(T) = \{\alpha\} \), then \( T - \alpha I \) is nilpotent.
Proof. Let \( p(x) \) be a polynomial such that \( p(T) \) is \( n \)-paranormal. Let

\[
p(x) - p(\alpha) = a(x - \alpha)^k \cdot (x - \alpha_1) \cdots (x - \alpha_m),
\]

where \( \alpha_j \neq \alpha \) for all \( j = 1, 2, \ldots, m \) and \( a \neq 0 \). Since \( p(T) \) is \( n \)-paranormal and \( \sigma(p(T)) = \{p(\alpha)\} \), we have \( p(T) = p(\alpha)I \) by Theorem 2. Hence

\[
p(T) - p(\alpha)I = a(T - \alpha I)^k \cdot (T - \alpha_1I) \cdots (T - \alpha_mI) = 0.
\]

Since all \( T - \alpha_I \) \( (j = 1, 2, \ldots, m) \) are invertible, we have \( (T - \alpha I)^k = 0 \). This completes the proof.

If \( T \) is \(*-n\)-paranormal, then \( T \) is \((n + 1)\)-paranormal by Theorem 6 of [4]. Hence we have following corollary.

**Corollary 1.** For \( T \in B(\mathcal{H}) \), let \( \sigma(T) = \{\alpha\} \).

1. If \( T \) be a \(*-n\)-paranormal operator, then \( T = \alpha I \).
2. If \( T \) be a polynomially \(*-n\)-paranormal operator, then \( T - \alpha I \) is nilpotent.

**Theorem 4.** Let \( T \) be a \(*-n\)-paranormal \((n\)-paranormal) operator on \( \mathcal{H} \). If \( \mathcal{M} \) is a closed invariant subspace for \( T \), then \( T \big|_{\mathcal{M}} \) is \(*-n\)-paranormal \((n\)-paranormal).

**Proof.** Let \( T \) be \(*-n\)-paranormal on \( \mathcal{H} \) and \( (x, f) \in \Pi(\mathcal{M}) \). By the Hahn-Banach theorem, there exists \( g \in \mathcal{H}^* \) such that \( g \big|_{\mathcal{M}} = f \) and \( \|g\| = \|f\| \). Hence \( (x, g) \in \Pi(\mathcal{H}) \) and

\[
\left( \left( T \big|_{\mathcal{M}} \right)^* f \right)(x) = f \left( T \big|_{\mathcal{M}} x \right) = f(Tx) = g(Tx) = (T^* g)(x),
\]

and so

\[
\| \left( T \big|_{\mathcal{M}} \right)^* f \|^n \leq \|T^* g\|n \leq \|T^n x\| = \| \left( T \big|_{\mathcal{M}} \right)^n x \|.
\]

It is easy to see that if \( T \) is \( n \)-paranormal, then \( T \big|_{\mathcal{M}} \) is \( n \)-paranormal. This completes the proof.

**Theorem 5.** Let \( T \) be an \( n \)-paranormal operator on \( \mathcal{H} \). Then \( T \) is isoloid.

**Proof.** Let \( \alpha \) be an isolated point of \( \sigma(T) \) and \( P \) be the spectral projection associated with \( \alpha \). Since then \( T \big|_{P(\mathcal{M})} \) is \( n \)-paranormal by Theorem 4 and \( \sigma(T \big|_{P(\mathcal{M})}) = \{\alpha\} \), \( T \big|_{P(\mathcal{M})} = \alpha I \). Hence \( \alpha \) is an eigen-value of \( T \). This completes the proof.
If $T$ is $*-n$-paranormal, then $T$ is $(n+1)$-paranormal. Hence the following corollary is direct from Theorem 5 and Corollary 1.

**Corollary 2.** If $T$ is $*-n$-paranormal, then $T$ is isoloid.

**Theorem 6.** Let $T$ be an operator on $H$ and satisfy one of the following statements. If $\alpha$ is an isolated point of $\sigma(T)$, then $\alpha$ is a pole of the resolvent, that is, $T$ is polaroid.

1. $T$ is $n$-paranormal.
2. $T$ is $*-n$-paranormal.
3. $T$ is polynomially $n$-paranormal.
4. $T$ is polynomially $*-n$-paranormal.

**Proof.** Proof is same with Theorem 1.3 of [1].

An operator $T$ is called hereditarily polaroid if any restriction to an invariant closed subspace is polaroid. Hence, the following result is clear.

**Theorem 7.** Polynomially $n$-paranormal operators on $H$ are hereditarily polaroid.

**Definition 4.** $\alpha \in \sigma(T)$ is said to be a proper boundary point of $\sigma(T)$ if there exists a bounded sequence $\{\alpha_n\} \subset \rho(T)$ (the resolvent set of $T$) such that $\|(\alpha - \alpha_n)(T - \alpha_nI)^{-1}\| \to 1$.

**Proposition 2 (Lemma 1, [5]).** If $\alpha \in \partial V(T) \cap \sigma(T)$, then $\alpha$ is a proper boundary point of $\sigma(T)$, where $\partial V(T)$ is the boundary of $V(T)$.

**Proposition 3 (Proposition 3.7, [6]).** If $0$ is a proper boundary point of $\sigma(T)$ and $Tx = 0$ with $\|x\| = 1$, then $1 \leq \|x + Ty\|$ for every $y \in H$. That is, $\ker(T) \perp R(T)$.

**Theorem 8.** Let $T$ be $n$-paranormal operators on $H$. If $\alpha, \beta$ are distinct eigenvalues of $T$, then $\ker(T - \alpha I) \perp \ker(T - \beta I)$.

For the proof of Theorem 8 we prepare lemmas. For the completeness, we give proofs.

For an eigen-value $\alpha$ of $T$, let $K(\alpha) = \{x \in H : Tx = \alpha x\}$.

**Lemma 1.** Let $T \in B(H)$. Let $\alpha, \beta$ be distinct eigen-values of $T$. Then $K(\alpha) + K(\beta) = \{x + y : x \in K(\alpha), y \in K(\beta)\}$ is a closed subspace.
Proof. Let \( \mathcal{M} = K(\alpha) + K(\beta) \). Then it is easy \( \mathcal{M} \) is a subspace. We show \( \mathcal{M} \) is closed. Let \( x_n + y_n \to z \), where \( x_n \in K(\alpha), y_n \in K(\beta) \). Then
\[
(T - \alpha I)(x_n + y_n) = (\beta - \alpha) y_n \to (T - \alpha I)z.
\]
Since \( K(\beta) \) is closed and \((\beta - \alpha) y_n \in K(\beta)\), this implies \((T - \alpha I)z \in K(\beta)\). Similarly \((T - \beta I)z \in K(\alpha)\). Thus
\[
z = \frac{(T - \beta I)z}{\alpha - \beta} - \frac{(T - \alpha I)z}{\alpha - \beta} \in K(\alpha) + K(\beta).
\]
Hence \( \mathcal{M} \) is closed. This completes the proof.

Lemma 2. Let \( T \in B(\mathcal{H}) \) and \( \alpha, \beta \) be distinct eigen-values of \( T \). If \( \mathcal{M} = K(\alpha) + K(\beta) \), then
\[
\sigma(T_{\lceil \mathcal{M} \rceil}) = \{ \alpha, \beta \}.
\]

Proof. By Lemma 1, \( \mathcal{M} \) is a closed invariant subspace for \( T \) and it is obvious that
\[
\alpha, \beta \in \sigma_p(T_{\lceil \mathcal{M} \rceil}) \subset \sigma(T_{\lceil \mathcal{M} \rceil}).
\]
We show \( T_{\lceil \mathcal{M} \rceil} - \lambda I \) is bijective if \( \lambda \neq \alpha, \beta \). Let \( (T_{\lceil \mathcal{M} \rceil} - \lambda I)(x + y) = 0 \) where \( x \in K(\alpha) \) and \( y \in K(\beta) \). Then \((\alpha - \lambda)x + (\beta - \lambda)y = 0\).

Since \( K(\alpha) \) and \( K(\beta) \) are linear independent, \( x = 0 \) and \( y = 0 \). Hence \( T_{\lceil \mathcal{M} \rceil} - \lambda \) is injective.

Let \( x \in K(\alpha) \) and \( y \in K(\beta) \). Then \( \frac{x}{\alpha - \lambda} \in K(\alpha) \) and \( \frac{y}{\beta - \lambda} \in K(\beta) \). Since
\[
(T_{\lceil \mathcal{M} \rceil} - \lambda I) \left( \frac{x}{\alpha - \lambda} + \frac{y}{\beta - \lambda} \right) = x + y,
\]
\( T_{\lceil \mathcal{M} \rceil} - \lambda I \) is surjective. Hence \( \sigma(T_{\lceil \mathcal{M} \rceil}) = \{ \alpha, \beta \} \). This completes the proof.

Proof of Theorem 8. We may assume that \( |\alpha| \geq |\beta| \). Let \( \mathcal{M} = K(\alpha) + K(\beta) \). Then \( \mathcal{M} \) is a closed subspace and invariant for \( T \) by Lemma 1. Hence it holds \( \sigma(T_{\lceil \mathcal{M} \rceil}) = \{ \alpha, \beta \} \) by Lemma 2. Since \( T_{\lceil \mathcal{M} \rceil} \) is \(*\)-\(n\)-paranormal by Theorem 4, \( T_{\lceil \mathcal{M} \rceil} \) is normaloid by Proposition 1. Hence
\[
\|T_{\lceil \mathcal{M} \rceil}\| = |\alpha| \text{ and }
\]
\[
\alpha \in \sigma(T_{\lceil \mathcal{M} \rceil}) \subset \mathcal{V}(T_{\lceil \mathcal{M} \rceil}) \subset \{ z \in \mathbb{C} : |z| \leq |\alpha| \}.
\]
Therefore, \( \alpha \in \partial \mathcal{V}(T_{\|\cdot\|}) \cap \sigma(T_{\|\cdot\|}) \). So we have \( \ker(T - \alpha I) \perp R(T - \alpha I) \) by Proposition 3.

Let \( x \in \ker(T - \alpha I) \) and \( y \in \ker(T - \beta I) \) such that \( \|x\| = 1 \). Then

\[
1 \leq \|x + (\beta - \alpha)^{-1}(T - \alpha I)y\| = \|x + y + (\beta - \alpha)^{-1}(T - \beta I)y\| = \|x + y\|.
\]

Therefore, \( \ker(T - \alpha I) \perp \ker(T - \beta I) \). This completes the proof.

Since a \(*\)-\(n\)-paranormal operator \( T \) is \((n + 1)\)-paranormal, we have following corollary.

**Corollary 3.** Let \( T \) be \(*\)-\(n\)-paranormal operators on \( \mathcal{X} \). If \( \alpha, \beta \) are distinct eigen-values of \( T \), then \( \ker(T - \alpha I) \perp \ker(T - \beta I) \).

In [4] M. Chô and Ôta proved that if \( \mathcal{X}^* \) is strictly convex and \( Tx = \alpha x \) for some \((x, f) \in \Pi(\mathcal{X})\), then \( T^*f = \alpha f \) (Theorem 15, [4]). Finally we extend this result for an approximate point spectrum of \( T \) on a uniformly convex space.

**Definition 5.** A Banach space \( \mathcal{X} \) is said to be uniformly convex if and only if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \|x\| = \|y\| = 1 \) and \( \|x - y\| \geq \varepsilon \), then

\[
\|x + y\| \leq 2(1 - \delta).
\]

By the definition of uniformly convexity, it holds that if \( \lim \|x_k\| = \lim \|y_k\| = 1 \) and \( \lim (\|x_k\| + \|y_k\|) = 2 \), then \( \lim (\|x_k - y_k\|) = 0 \).

**Theorem 9.** Let the dual space \( \mathcal{X}^* \) be uniformly convex and \( T \) be \(*\)-\(n\)-paranormal on \( \mathcal{X} \).

If \((T - \alpha I)x_k \to 0 \) for \((x_k, f_k) \in \Pi(\mathcal{X})\), then \((T - \alpha I)^*f_k \to 0 \).

**Proof.** If \( \alpha = 0 \), \( \|T^*f_k\|^n \leq \|T^n x_k\| \) and hence \( \lim_{k \to \infty} \|T^*f_k\| = 0 \). So we may show the theorem for \( \alpha \neq 0 \). Since \( \frac{1}{\alpha} T \) is \(*\)-\(n\)-paranormal, we may assume \( \alpha = 1 \). Hence we show that if \((T - I)x_k \to 0 \), then \((T - I)^*f_k \to 0 \). Since \( T \) is \(*\)-\(n\)-paranormal, it holds

\[
\|T^*f_k\|^n \leq \|T^n x_k\| \to 1.
\]

Hence \( \lim \sup \|T^*f_k\| \leq 1 \). Since \( f_k(T x_k) \to 1 \), it holds

\[
2 \geq \lim \sup (\|T^*f_k\| + \|f_k\|) \geq \lim \inf (\|T^*f_k\| + \|f_k\|)
\]

\[
\geq \lim \inf (\|T^*f_k + f_k\|) \geq \lim \inf \|T^*f_k + f_k\|(x_k) \to 2.
\]
Hence \( \lim (\|T^*f_k\| + \|f_k\|) = 2 \) and \( \lim \|T^*f_k\| = 1 \).

Since
\[
2 \geq \lim \sup (\|T^*f_k\| + \|f_k\|) \geq \lim \sup (\|T^*f_k + f_k\|)
\]
\[
\geq \lim \inf (\|T^*f_k + f_k\|) \geq \lim \inf |(T^*f_k + f_k)(x_k)| \rightarrow 2,
\]
we have \( \lim \|T^*f_k + f_k\| = 2 \).

Since it is clear that \( \lim \|T^*f_k\| = \lim \|f_k\| = 1 \), by uniformly convexity it holds
\[
\lim \|T^*f_k - f_k\| = 0, \text{ i.e., } (T - I)^*f_k \rightarrow 0.
\]
This completes the proof.

Since uniformly convex space is reflexive, following corollary is clear.

**Corollary 4.** Let \( \mathcal{X} \) be uniformly convex and \( T \in B(\mathcal{X}) \). If \( T^* \) is \( \ast\)-\( n \)-paranormal on \( \mathcal{X}^* \) and \( (T - \alpha I)^*f_k \rightarrow 0 \) for \( (x_k, f_k) \in \Pi(\mathcal{X}) \), then \( (T - \alpha I)x_k \rightarrow 0 \).

**Conflict of Interests**
The authors declare that there is no conflict of interests.

**Acknowledgements**
This is partially supported by Grant-in-Aid Scientific Research No. 24540195.

**References**


