(1, 2) – DOMINATION IN GRAPHS

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Abstract. In this paper we discuss (1,2) – domination in different graphs and results obtained are compared with the domination in graphs.

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1. Introduction

Domination in a graph along with its many variations provide an extremely rich area of study. Berge [2] and Ore[6] were the first to define dominating sets. A new type of dominating set, (1,2) – dominating set is introduced by Steve Hedetniemi and Sandee Hedetniemi [5]. In this paper we present some basic theorems on these sets and the relation between the usual domination and (1,2)- domination.

2. Preliminaries

Let G = (V,E) be a simple graph. A subset D of V is a dominating set of G if every vertex of V- D is adjacent to a vertex of D. The domination number of G, denoted by γ (G),is the minimum cardinality of a dominating set of G.
A \((1,2)\)–dominating set in a graph \(G = (V,E)\) is a set \(S\) having the property that for every vertex \(v\) in \(V - S\) there is at least one vertex in \(S\) at distance 1 from \(v\) and a second vertex in \(S\) at distance at most 2 from \(v\). The order of the smallest \((1,2)\)-dominating set of \(G\) is called the \((1,2)\)–domination number of \(G\) and we denote it by \(\gamma_{(1,2)}\).

From the definition of \((1,2)\)–dominating sets, we see that a \((1,2)\)–dominating set contains at least 2 vertices, \((1,2)\)–domination number of a graph will be always \(\geq 2\) and \((1,2)\)–dominating sets occur in graphs of order at least 3.

A graph is said to be complete if each of its vertices is adjacent to every other vertex. A graph is said to be regular if each of its vertices has the same degree. A graph is said to be cubic graph if each of its vertices is of degree three. A bipartite graph is a graph in which vertices can be divided into two disjoint sets \(A\) and \(B\) such that every edge connects a vertex in \(A\) to one in \(B\).

For each vertex \(x\) in a graph \(G\), we introduce a new vertex \(x'\) and join \(x\) and \(x'\) by an edge. The resulting graph is called the \textit{corona} of \(G\).

\section*{3. Main results}

\textbf{Theorem 3.1}

All \((1,2)\)–dominating sets are dominating sets.

Proof:

The result is trivial from the definition of \((1,2)\)-dominating sets.

But the converse need not be true.

Example:

Consider the graph
For this, \{1,4\} is a dominating set.
But it is not a \((1,2)\) – dominating set.
\{2,3,4\} is a \((1,2)\) – dominating set. And it is a dominating set also.

3.1. 2. \((1,2)\) – domination in Complete Graphs

Theorem 3.2

\((1,2)\) – domination is not possible in complete graphs.

Proof: In a complete graph, each vertex is adjacent to every other vertices. So we cannot find a
\((1,2)\) – dominating set. No vertex can be found at a distance atmost 2 from any other vertex.

Let \(G\) be a complete graph with \(n\) vertices. Then it will have \(\frac{n(n-1)}{2}\) edges and each vertex is of degree \(n - 1\). The minimum number of edges to be deleted so as to become the resulting graph \((1,2)\) – dominating is \(n - 2\). If we delete \(n - 2\) edges from a complete graph, then in the resulting graph, we can find a \((1,2)\) – dominating set.

Lemma 3.3

If a graph \(G\) with \(n\) vertices, has a vertex of degree \(n - 1\), we cannot find a
\((1,2)\) – dominating set.
Example:

In this graph, we cannot find a (1,2) – dominating set since each vertex is adjacent to all other vertices.

3.1.3. Relation between domination number and (1,2) – domination number

In this section we consider different types of graphs and find out their domination number and (1,2)- domination number and check the relation between them.

Consider the following graphs

Here \{2\} is a dominating set. \(\gamma(G) = 1\).

\{2,3\} is a (1,2) – dominating set. \(\gamma_{(1,2)} = 2\).

That is, \(\gamma < \gamma_{(1,2)}\)
Here \{1,3\}, \{1,4\}, \{2,4\}, \{2,3\} are all dominating sets. \(\gamma(G) = 2\).

\{1,4\} is a \((1,2)\) – dominating set. \(\gamma_{(1,2)} = 2\).

\[\gamma = \gamma_{(1,2)}\]

![Diagram 1](image1)

Here \{1,4\}, \{2,3\}, \{1,3\}, \{2,4\} are dominating sets. \(\gamma(G) = 2\).

\{2,3\} is a \((1,2)\) – dominating set. \(\gamma_{(1,2)} = 2\).

\[\gamma = \gamma_{(1,2)}\]

![Diagram 2](image2)

Here \{1,4\}, \{1,3\}, \{2,4\}, \{3,5\} are dominating sets. \(\gamma(G) = 2\).

\{2,3\} is a \((1,2)\) – dominating set. \(\gamma_{(1,2)} = 3\).

\[\gamma \leq \gamma_{(1,2)}\]

Consider some regular graphs with even number of vertices
Here \( \{1,4\}, \{2,3\}, \{1,3\}, \{2,4\} \) are dominating sets. \( \gamma(G) = 2 \).

\( \{2,3\} \) is a \( (1,2) \)–dominating set. \( \gamma_{(1,2)} = 2 \).

\( \gamma = \gamma_{(1,2)} \)

Here \( \{1,3,5\}, \{2,4,6\} \) are dominating sets. \( \gamma(G) = 3 \).

\( \{1,4,6\} \) is a \( (1,2) \)–dominating set. \( \gamma_{(1,2)} = 3 \).

\( \gamma = \gamma_{(1,2)} \)

Here \( \{1,2,3,4\}, \{5,6,7,8\} \) are dominating. \( \gamma(G) = 4 \).
\{1,2,3,4\} is a \((1,2)\) – dominating set. \(\gamma_{(1,2)} = 4\).
\[\gamma = \gamma_{(1,2)}\]

Consider the bipartite graph given below

\[\{1,2\} \text{ is a dominating set. } \gamma(G) = 2.\]
\[\{1,4,5\} \text{ is a } (1,2) \text{ – dominating set. } \gamma_{(1,2)} = 3.\]
\[\gamma < \gamma_{(1,2)}\]

Consider the cubic bipartite graphs

For \(n = 6\),

\[\{1,5\}, \{2,6\} \text{ are dominating sets. } \gamma(G) = 2.\]
\[\{1,5\} \text{ is a } (1,2) \text{ – dominating set. } \gamma_{(1,2)} = 2.\]
\( \gamma = \gamma_{(1,2)} \)

For \( n = 8 \),

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
8 & 7 & 6 & 5 \\
\end{array}
\]

\{1,6\} is a dominating set. \( \gamma(G) = 2 \).

\{1,6\} is a \((1,2)\)–dominating set. \( \gamma_{(1,2)} = 2 \).

\( \gamma = \gamma_{(1,2)} \)

For \( n = 10 \),

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
10 & 9 & 8 & 7 \\
5 & 6 & & \\
\end{array}
\]

\{2,5,9\} is a dominating set. \( \gamma(G) = 3 \).

\{2,4,6,8,10\} is a \((1,2)\)–dominating set. \( \gamma_{(1,2)} = 5 \).
\( \gamma < \gamma_{(1,2)} \)

In all the above cases, domination number is less than or equal to \((1,2)\)-domination number.

From the above examples we have the following theorem.

**Theorem 3.4**

In a graph \( G \), domination number is less than or equal to \((1,2)\) – domination number.

**Proof:**

Let \( G \) be a graph and \( D \) be its dominating set. Then every vertex in \( V - D \) is adjacent to a vertex in \( D \). That is, in \( D \), for every vertex \( u \), there is a vertex which is at distance 1 from \( u \). But it is not necessary that there is a second vertex at distance at most 2 from \( u \). So if we find a \((1,2)\) – dominating set, it will contain more vertices or at least equal number of vertices than the dominating set. So the domination number is less than or equal to \((1,2)\) – domination number.

**Theorem 3.5**

If \( G \) is a 2-regular graph, then the \((1,2)\) – domination number of the corona of \( G \) is equal to the number of vertices of \( G \).

**Proof:** Let \( G \) be a 2-regular graph. Then each of its vertices will be of degree 2. In the corona of \( G \), for each vertex \( x \), we introduce a new vertex and join them. Consequently, an edge is added to each of its vertices. By the definition of \((1,2)\) – dominating set each vertex \( v \) in \( V - S \) has at least one vertex in \( S \) at distance 1 from \( v \) and a second vertex in \( S \) at distance at most 2 from \( v \). Hence \((1,2)\) – dominating set of the corona of \( G \) will consist of all the vertices of \( G \).

**Theorem 3.6**

If in a graph \( G \), an edge \( e \) is added, \( \gamma_{(1,2)}(G+e) \geq \gamma_{(1,2)}(G) \)

**Proof:** Let \( G \) be a graph. Let \( S \) be the \((1,2)\) – dominating set of \( G \).

If we add an edge to a vertex in \( S \), that will not affect the cardinality of \( S \).
If we add an edge to a vertex in $V - S$, the cardinality of $(1,2)$ – dominating set will increase. 
Therefore, $\gamma_{(1,2)}(G+ e) \geq \gamma_{(1,2)}(G)$.

**Theorem 3.7**

If $G$ is a complete bipartite graph, then the $(1,2)$ – domination number $\gamma_{(1,2)}$ is 2.

**Proof:** Let $G$ be a complete bipartite graph. Then $V(G)$ can be partitioned into 2 disjoint sets $X$ and $Y$ and each edge has one end in $X$ and other end in $Y$. Since $G$ is complete bipartite, each vertex of $X$ is joined to every vertex in $Y$. A set of 2 vertices, one from $X$ and another from $Y$ will constitute a $(1,2)$ – dominating set.
Therefore, $\gamma_{(1,2)} = 2$.

**4. Conclusion**

We found $(1,2)$ – domination number of some graphs and compared them with the domination number. Also some preliminary theorems on $(1,2)$- dominating sets are proved.

**REFERENCES**

[5] Steve Hedetniemi , Sandee Hedetniemi , (1,2)- Domination in Graphs