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COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE WITH SPECIAL REFERENCE TO OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS

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Abstract: The present paper deals with common fixed point theorem in Fuzzy metric space by employing the notion of occasionally weakly compatible mappings. Our result generalizes the recent result of Singh et. al. [17].

Keywords: common fixed points, fuzzy metric space, compatible maps, occasionally weakly compatible mappings and weak compatible mappings.

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1. Introduction

In 1965, Zadeh [18] introduced the concept of Fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a Fuzzy metric space is more suitable. We can divide them into following two

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groups: The first group involves those results in which a Fuzzy metric on a set X is treated as a map where X represents the totality of all Fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between Fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is Fuzzy and the objects themselves may or may not be Fuzzy. In this paper, we deal with the Fuzzy metric space defined by Kramosil and Michalek [10] and modified by George and Veeramani [4]. Recently, Grabiec [5] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [14] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [8] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [2, 3] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in Fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type (β), Singh et. al. [15, 16] proved fixed point theorems in a Fuzzy metric space. Recently in 2012, Jain et. al. [6, 7] and Sharma et. al. [12] proved various fixed point theorems using the concepts of semi-compatible mappings, property (E.A.) and absorbing mappings.

In this paper, a fixed point theorem for six self maps has been established using the concept of occasionally weak compatible maps which generalizes the result of Singh et. al. [17].

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. Preliminaries

Definition 2.1. [11] A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *t-norm* if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [11] The 3-tuple (X, M, *) is said to be a *Fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1.[11] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and

$M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, *) is a Fuzzy metric space.

It is called the Fuzzy metric space induced by d.

Definition 2.3.[11] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \ge n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \ge n_0$.

A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4.[14] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.5.[15] Two self maps A and B of a Fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points, i.e. Ax = Bx implies ABx = BAx.

Definition 2.6. Self maps A and S of a Fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Proposition 2.1. [16] In a Fuzzy metric space (X, M, *) limit of a sequence is unique.

Proposition 2.2.[14] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and let $\{x_n\}$ be a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X. Then $STx_n \to Tu$ provided T is continuous.

Proposition 2.3. [14] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and Su = Tu for some u in X then

$$STu = TSu = SSu = TTu.$$

Lemma 2.1. [5] Let (X, M, *) be a Fuzzy metric space. Then for all $x, y \in X$, M(x, y, .) is a non-decreasing function.

Lemma 2.2. [1] Let (X, M, *) be a Fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t)\ \forall\ t > 0$, then x = y.

Lemma 2.3. [16] Let $\{x_n\}$ be a sequence in a Fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that $M(x_{n+2},x_{n+1},kt) \geq M(x_{n+1},x_n, t)\ \forall\ t > 0$ and $n \in N$.

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4.[9] The only t-norm * satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

3. Main Result.

We prove the following.

Theorem 3.1. Let A, B, S and T be self mappings of a complete Fuzzy metric space (X, M, *). Suppose that they satisfy the following conditions :

(3.1.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(3.1.2) the pairs (A, S) and (B, T) are occasionally weakly compatible;

(3.1.3) There exists $k \in (0, 1)$ such that $\forall$ x, y $\in$ X and t > 0,

$$M(Ax, By, kt) \geq \text{Min}\{M(By, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\}.$$

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point.

As $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.Then there exists x_1, $x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, we can construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1},\ y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0, 1, 2 \ldots$$

We first show that $\{y_n\}$ is a Cauchy sequence in X.

Now, by (3.1.3) with $x = x_{2n}$, $y = x_{2n+1}$, we obtain that

$$\begin{aligned}
M(Ax_{2n}, Bx_{2n+1}, kt) = M(y_{2n+1}, y_{2n+2}, kt) \\
&\geq \text{Min}\{M(Bx_{2n+1},Tx_{2n+2},t), M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t)\} \\
&\geq \text{Min}\{M(y_{2n+1},y_{2n+2},t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\} \\
&\geq \text{Min}\{M(y_{2n+1},y_{2n+2},t), M(y_{2n}, y_{2n+1}, t). \qquad \text{(i)}
\end{aligned}$$

Thus we have,

$$M(y_{2n+1},y_{2n+2},t) \geq \text{Min}\{M(y_{2n+1},y_{2n+2},t/k), M(y_{2n},y_{2n+1},t/k). \qquad \text{(ii)}$$

By putting (ii) in (i), we have,

$$\begin{aligned}
M(y_{2n+1},y_{2n+2},kt) &\geq \text{Min}\{M(y_{2n+1},y_{2n+2},t/k), M(y_{2n},y_{2n+1},t/k), M(y_{2n},y_{2n+1},t)\} \\
&= \text{Min}\{M(y_{2n+1},y_{2n+2},t/k), M(y_{2n},y_{2n+1},t)\} \\
&\geq \text{Min}\{M(y_{2n+1}, y_{2n+2},t/k^2), M(y_{2n},y_{2n+1},t/k^2), M(y_{2n},y_{2n+1},t) \\
&= \text{Min}\{M(y_{2n+1},y_{2n+2},t/k^2), M(y_{2n},y_{2n+1},t)\} \\
&\geq \ldots\ldots\ldots\ldots\ldots. \\
&\geq \text{Min}\{M(y_{2n+1}, y_{2n+2},t/k^m), M(y_{2n},y_{2n+1},t)\}.
\end{aligned}$$

Taking limit as $m \to \infty$, we have

$$M(y_{2n+1},y_{2n+2},kt) \geq M(y_{2n},y_{2n+1},t)\},\ \forall\ t > 0.$$

Similarly, we also have

$$M(y_{2n+2},y_{2n+3},kt) \geq M(y_{2n+1},y_{2n+2},t)\},\ \forall\ t > 0.$$

Thus, for all n, and $t > 0$

$$M(y_n, y_{n+1}, kt) \geq M(y_n, y_{n-1}, t).$$

Therefore,

$$M(y_n,y_{n+1},t) \geq M(y_{n-1},y_n,t/k) \geq M(y_{n-2},y_{n-1},t/k^2) \geq \ldots \geq M(y_0,y_1,t/k^n).$$

Hence, $\lim_{n\to\infty} M(y_n,y_{n+1},t) = 1 \;\; \forall\, t > 0$.

Now, for any integer p, we have

$$M(y_n,y_{n+p},t) \geq M(y_n,y_{n+1},t/p) * M(y_{n+1},y_{n+2},t/p) * \ldots * \ldots * M(y_{n+p-1},y_{n+p},t/p).$$

Therefore, $\lim_{n\to\infty} M(y_n,y_{n+p},t) = 1*1*1*\ldots*1=1$

$$\lim_{n\to\infty} M(y_n,y_{n+p},t) = 1.$$

This shows that $\{y_n\}$ is a Cauchy sequence in X, which is complete.

Therefore, $\{y_n\}$ converges to $z \in X$.

We have the following subsequences;

$$\{Ax_{2n}\}\to z, \quad \{Bx_{2n+1}\}\to z \tag{1}$$

$$\{Sx_{2n}\}\to z, \quad \{Tx_{2n+1}\}\to z. \tag{2}$$

Since $A(X) \subseteq T(X)$ $\exists$ for $p \in X$ such that $p = T^{-1} z$ i.e. $Tp = z$

By (3.1.3) we have (at $x = x_{2n}$, $y = p$)

$$M(Ax_{2n},Bp, kt) \geq \text{Min}\,\{M(Bp,Tp, t), M(Sx_{2n},Tp, t), M(Ax_{2n},Sx_{2n}, t)\}$$

$$M(Ax_{2n}, Bp, kt) \geq \text{Min}\,\{M(Bp, z, t), M(Sx_{2n},z, t), M(Ax_{2n},Sx_{2n}, t)\}$$

Taking the limit $n\to\infty$ and using (i) and (ii) we have,

$$M(z, Bp, kt) \geq \text{Min}\,\{M(Bp,z, t), M(z,z, t), M(z,z, t)\}$$

$$M(z, Bp, kt) \geq M(Bp, z, t).$$

Therefore by lemma (2.2) we have.

$z = Bp$. Since $z = Tp$ therefore $z = Bp = Tp$.

i. e. p is a coincidence point of B and T.

Similarly, since $B(X) \subseteq S(X)$; $\exists$ $q \in X$ such that $q = S^{-1}z$ i.e. $Sq = z$.

By (3.1.3) we have (at $x = q$, $y = x_{2n+1}$)

$$M(Aq,Bx_{2n+1}, kt) \geq \text{Min}\,\{M(Bx_{2n+1},Tx_{2n+1}, t), M(Sq,Tx_{2n+1}, t), M(Aq,Sq, t)\}.$$

$M(Aq,Bx_{2n+1}, kt) \geq \text{Min}\{M(Bx_{2n+1},Tx_{2n+1}, t), M(z,Tx_{2n+1}, t), M(Aq,z, t)\}$.

Taking the limit $n\to\infty$ and using (i) and (ii) we have;

$M(Aq,z, kt) \geq \text{Min}\{M(Bz,Tz, t), M(z,z, t), M(Aq,z, t)\}$.

$M(Aq,z, kt) \geq M(Aq,z, t)$.

Therefore by lemma 2.2, we have.

$Aq = z$. Since $Sq = z$, therefore, $z = Aq = Sq$. i.e. q is a coincidence point of A and S.

Since $\{A, S\}$ is occasionally weakly compatible. Therefore, we have

$ASq = SAq$ or $Az = Sz$.

Similarly $\{B, T\}$ is occasionally weakly compatible, therefore, we have

$BTp = TBp$ or $Bz = Tz$.

Now by (3.1.3) we have (at $x = z$, $y = x_{2n+1}$)

$M(Az,Bx_{2n+1}, kt) \geq \text{Min}\{M(Bx_{2n+1},Tx_{2n+1}, t), M(Sz,Tx_{2n+1}, t), M(Az,Sz, t)\}$.

$M(Az,Bx_{2n+1}, kt) \geq \text{Min}\{M(Bx_{2n+1},Tx_{2n+1}, t), M(Az,Tx_{2n+1}, t), M(Az,Sz, t)\}$.

Taking the limit $n \to \infty$, we have,

$M(Az,z, kt) \geq \text{Min}\{M(z,z, t), M(Az,z, t), 1\}$.

$M(Az,z, kt) \geq M(Az,z, t)$.

Therefore by lemma 2.2, we have

$Az = z$. Since $Az = Sz$, therefore $z = Az = Sz$.

Again by (3.1.3), we have (at $x = x_{2n}$, $y = z$)

$M(Ax_{2n}, Bz, kt) \geq \text{Min}\{M(Bz, Tz, t), M(Sx_{2n},Tz, t), M(Ax_{2n},Sx_{2n}, t)\}$.

$M(Ax_{2n}, Bz, kt) \geq \text{Min}\{M(Bz,Bz, t), M(Sx_{2n},Bz, t), M(Ax_{2n},Sx_{2n}, t)\}$.

Taking the limit $n \to\infty$, we have

$M(z,Bz, kt) \geq \text{Min}\{1, M(z,Bz, t), M(z,z, t)\}$

$M(z,Bz, kt) \geq \text{Min}\{1, M(z,Bz, t), 1\}$

$M(z,Bz, kt) \geq M(z,Bz, t)$.

Therefore by lemma 2.2, we have

$z = Bz$. Since $Bz = Tz$, therefore $z = Bz = Tz$.

Thus we have, $z = Az = Sz = Bz = Tz$.

Hence z is a common fixed point of A, B, S and T.

Uniqueness - Let z and z' be two common fixed points of the maps A, B, S and T.

Then

$$z = Az = Sz = Bz = Tz = \text{ and } z' = Az' = Sz' = Bz' = Tz'$$

Now by (3.1.3), we have (at $x = z$, $y = z'$)

$$M(Az,Bz', kt) \geq \text{Min} \{M (Bz',Tz', t), M (Sz,Tz', t), M (Az,Sz, t)\}$$

$$M(z,z', kt) \geq \text{Min} \{M (z',z', t), M (z,z', t), M (z,z, t)\}$$

$$M(z,z', kt) \geq \text{Min} \{1, M (z',z', t),1\}$$

$$M(z,z', kt) \geq M (z',z', t)$$

Therefore by lemma 2.2, we have $z = z'$.

Hence z is the unique common fixed point of the four self maps A, B, S and T.

This completes the proof.

If we take $B = A$ in theorem 3.1, we get the following corollary for three self maps.

Corollary 3.2. Let A, S and T be self mappings of a complete Fuzzy metric space (X, M, *); satisfying ;

(3.2.1) $A(X) \subseteq S(X) \cap T(X)$

(3.2.2) Pairs (A, S) and (A, T) are occasionally weak compatible,

(3.2.3) $M(Ax, Ay, kt) \geq \text{Min} \{M (Ay, Ty, t) M (Sx, Ty, t), M (Ax, Sx, t)\}$

for all $x, y \in X$, $t > 0$ and $0 < k < 1$.

Then A, S and T have a unique common fixed point in X.

Proof. The proof is similar to the proof of theorem 3.1.

If we take $S = T = I$, the identity maps on X in corollary 3.2, then the conditions (3.2.1) and (3.2.2) are trivially satisfied.

4. An Application.

Theorem 4.1. Let A be a self map on a complete Fuzzy metric space (X, M, *) such that for some $k \in (0, 1)$.

$M(Ax, Ay, kt) \geq M(x, y, t)$ for all $x, y \in X$, $t > 0$.

Then A has a unique common fixed point in X.

Proof. On taking only one factor in R.H.S. of the contraction (3.2.3), we obtain the desired result.

Conclusion. Theorem 3.1 is a generalization of the result of Singh et. al. [17] in the sense that condition of compatibility and weak compatibility of the pairs of self maps has been restricted to occasionally weakly compatible self maps and the requirement of continuity is completely removed.

Conflict of Interests

The author declares that there is no conflict of interests.

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