Introduction

of fuzzy rough set and Alexandrov $L$-topologies. Algebraic structures of fuzzy rough sets are developed in many directions [3,9,10].

In this paper, we investigate relations between $L$-upper (lower, join meet, meet join) approximation operators and Alexandrov $L$-topologies. We give their examples by various $L$-fuzzy relations.

2. Preliminaries

Definition 2.1. [1,2] An algebra $(L, \land, \lor, \odot, \to, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \lor, \land, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume $(L, \land, \lor, \odot, \to, \bot, \top)$ is a complete residuated lattice with the law of double negation; i.e., $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \to A)(x) = \alpha \to A(x), (\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(y) = \bot$, otherwise.

Lemma 2.2. [1,2] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

1. If $y \leq z$, $(x \odot y) \leq (x \odot z), x \to y \leq x \to z$ and $z \to x \leq y \to x$.
2. $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I}(x \to y_i)$.
3. $(\bigvee_{i \in I} x_i) \to y = \bigwedge_{i \in I}(x_i \to y)$.
4. $\bigwedge_{i \in I} y_i^* = (\bigvee_{i \in I} y_i)^*$ and $\bigvee_{i \in I} y_i^* = (\bigwedge_{i \in I} y_i)^*$.
5. $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
6. $x \odot y = (x \to y^*)^*$.
7. $x \odot (x \to y) \leq y$.
8. $(x \to y) \odot (y \to z) \leq x \to z$.
9. $(x \to y) \to (x \to z) \geq y \to z$ and $(x \to z) \to (y \to z) \geq y \to x$.
10. $x \odot y \to x \odot z \geq y \to z$. 
Definition 2.3. [3,4] (1) A map $H : L^X \rightarrow L^X$ is called an $L$-upper approximation operator iff it satisfies the following conditions

(H1) $A \leq H(A)$,
(H2) $H(\alpha \odot A) = \alpha \odot H(A)$ where $\alpha(x) = \alpha$ for all $x \in X$,
(H3) $H(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} H(A_i)$.

(2) A map $J : L^X \rightarrow L^X$ is called an $L$-lower approximation operator iff it satisfies the following conditions

(J1) $J(A) \leq A$,
(J2) $J(\alpha \rightarrow A) = \alpha \rightarrow J(A)$,
(J3) $J(\bigwedge_{i \in I} A_i) = \bigwedge_{i \in I} J(A_i)$.

(3) A map $K : L^X \rightarrow L^X$ is called an $L$-join meet approximation operator iff it satisfies the following conditions

(K1) $K(A) \leq A^*$,
(K2) $K(\alpha \odot A) = \alpha \rightarrow K(A)$,
(K3) $K(\bigvee_{i \in I} A_i) = \bigwedge_{i \in I} K(A_i)$.

(4) A map $M : L^X \rightarrow L^X$ is called an $L$-meet join approximation operator iff it satisfies the following conditions

(M1) $A^* \leq M(A)$,
(M2) $M(\alpha \rightarrow A) = \alpha \odot M(A)$,
(M3) $M(\bigwedge_{i \in I} A_i) = \bigvee_{i \in I} M(A_i)$.

Definition 2.4. [4,5] A subset $\tau \subset L^X$ is called an Alexandrov $L$-topology if it satisfies:

(T1) $\perp, \top \in \tau$ where $\top_X(x) = \top$ and $\perp_X(x) = \perp$ for $x \in X$.
(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i \in \Gamma} A_i, \bigwedge_{i \in \Gamma} A_i \in \tau$.
(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.
(T4) $\alpha \rightarrow A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Theorem 2.5. [4] (1) $\tau$ is an Alexandrov topology on $X$ iff $\tau^* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on $X$.

(2) If $H$ is an $L$-upper approximation operator, then $\tau_H = \{A \in L^X \mid H(A) = A\}$ is an Alexandrov topology on $X$. 
(3) If \( J \) is an \( L \)-lower approximation operator, then \( \tau_J = \{ A \in L^X \mid J(A) = A \} \) is an Alexandrov topology on \( X \).

(4) If \( K \) is an \( L \)-join meet approximation operator, then \( \tau_K = \{ A \in L^X \mid K(A) = A^* \} \) is an Alexandrov topology on \( X \).

(5) If \( M \) is an \( L \)-meet join operator, then \( \tau_M = \{ A \in L^X \mid M(A) = A^* \} \) is an Alexandrov topology on \( X \).

3. \( L \)-approximation operators and Alexandrov \( L \)-topologies

**Theorem 3.1.** Let \( H : L^X \to L^X \) be an \( L \)-upper approximation operator. Then the following properties hold.

1. For \( A \in L^X \), \( H(A)(y) = \bigvee_{x \in X} (A(x) \odot H(\top_x))(y) \).

2. Define \( J_H(B) = \bigvee \{ A \mid H(A) \leq B \} \). Then \( J_H : L^X \to L^X \) with

\[
J_H(B)(x) = \bigwedge_{y \in X} (H(\top_x)(y) \to B(y))
\]

is an \( L \)-lower approximation operator such that \((H, J_H)\) is a residuated connection; i.e.,

\[
H(A) \leq B \iff A \leq J_H(B).
\]

Moreover, \( \tau_H = \tau_{J_H} \).

3. If \( H(H(A)) = H(A) \) for \( A \in L^X \), then \( J_H(J_H(A)) = J_H(A) \) for \( A \in L^X \) such that \( \tau_H = \tau_{J_H} \) with

\[
\tau_H = \{ H(A) = \bigvee_{x \in X} (A(x) \odot H(\top_x)) \mid A \in L^X \},
\]

\[
\tau_{J_H} = \{ J_H(A)(x) = \bigwedge_{y \in X} (H(\top_x)(y) \to A(y)) \mid A \in L^X \}.
\]

4. If \( H(H^*(A)) = H^*(A) \) for \( A \in L^X \), then \( H(H(A)) = H(A) \) such that

\[
\tau_H = \{ H^*(A) = \bigwedge_{x \in X} (A(x) \to H^*(\top_x)) \mid A \in L^X \} = (\tau_H)_*.
\]

5. Define \( J_s(A) = H(A^*)^* \). Then \( J_s : L^X \to L^X \) with

\[
J_s(B)(x) = \bigwedge_{y \in X} (H(\top_y)(x) \to B(y))
\]
is an $L$-lower approximation operator. Moreover, $\tau_{J_s} = (\tau_H)_s$.

(6) If $H(H(A)) = H(A)$ for $A \in L^X$, then $J_s(J_s(A)) = J_s(A)$ for $A \in L^X$ such that $\tau_{J_s} = (\tau_H)_s$ with

$$\tau_{J_s} = \{J_s(A) = \bigwedge_{y \in X} (H(\top_y) \to A(y)) \mid A \in L^X\}.$$ 

(7) If $H(H^+(A)) = H^+(A)$ for $A \in L^X$, then $J_s(J_s^+(A)) = J_s^+(A)$ such that

$$\tau_{J_s} = \{J_s^+(A) = \bigvee_{y \in X} (H(\top_y) \odot A^+(y)) \mid A \in L^X\} = (\tau_{J_s})_s.$$ 

(8) Define $M_H(A) = H(A^+)$. Then $M_H : L^X \to L^X$ with

$$M_H(A) = \bigvee_{y \in X} (H(\top_y) \odot A^+(y))$$

is an $L$-meet join approximation operator.

(9) If $H(H(A)) = H(A)$ for $A \in L^X$, then $M_H(M_H^+(A)) = M_H^+(A)$ for $A \in L^X$ such that $\tau_{M_H} = (\tau_H)_s$ with

$$\tau_{M_H} = \{M_H^+(A) = \bigwedge_{y \in X} (H(\top_y) \to A(y)) \mid A \in L^X\}. $$

(10) If $H(H^+(A)) = H^+(A)$ for $A \in L^X$, then $M_H(M_H^+(A)) = M_H^+(A)$ such that

$$\tau_{M_H} = \{M_H^+(A) = \bigvee_{y \in X} (H(\top_y) \odot A^+(y)) \mid A \in L^X\} = (\tau_{M_H})_s.$$ 

(11) Define $K_H(A) = (H(A))^+$. Then $K_H : L^X \to L^X$ with

$$K_H(A)(y) = \bigwedge_{x \in X} (A(x) \to H^+(\top_x)(y))$$

is an $L$-join meet approximation operator. Moreover, $\tau_{K_H} = \tau_H$.

(12) If $H(H(A)) = H(A)$ for $A \in L^X$, then $K_H(K_H^+(A)) = K_H(A)$ for $A \in L^X$ such that $\tau_{K_H} = (\tau_H)_s$ with

$$\tau_{K_H} = \{K_H^+(A) = \bigvee_{y \in X} (A(x) \odot H(\top_x)(y)) \mid A \in L^X\}.$$ 

(13) If $H(H^+(A)) = H^+(A)$ for $A \in L^X$, then $K_H(K_H^+(A)) = K_H^+(A)$ such that

$$\tau_{K_H} = \{K_H^+(A) = \bigwedge_{x \in X} (A(x) \to H^+(\top_x)(y)) \mid A \in L^X\} = (\tau_{K_H})_s.$$
(14) Define $M_{J_H}(A) = (J_H(A))^*$. Then $M_{J_H} : L^X \rightarrow L^X$ with

$$M_{J_H}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot H(\top_y)(x))$$

is an $L$-meet join approximation operator. Moreover, $\tau_{M_{J_H}} = \tau_H$.

(15) If $H(H(A)) = H(A)$ for $A \in L^X$, then $M_{J_H}(M_{J_H}^*(A)) = M_{J_H}(A)$ for $A \in L^X$ such that $\tau_{M_{J_H}} = (\tau_H)_*$ with

$$\tau_{M_{J_H}} = \{M_{J_H}^*(A)(y) = \bigwedge_{x \in X} (H(\top_y)(x) \rightarrow A(x)) \mid A \in L^X\}.$$

(16) If $J_H(J_H^*(A)) = J_H^*(A)$ for $A \in L^X$, then $M_{J_H}(M_{J_H}^*(A)) = M_{J_H}^*(A)$ such that

$$\tau_{M_{J_H}} = \{M_{J_H}^*(A)(y) = \bigvee_{x \in X} (A^*(x) \odot H(\top_y)(x)) \mid A \in L^X\} = (\tau_{M_{J_H}})_*.$$

(17) Define $K_{J_H}(A) = J_H(A^*)$. Then $K_{J_H} : L^X \rightarrow L^X$ with

$$K_{J_H}(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow H^*(\top_y)(x))$$

is an $L$-meet join approximation operator. Moreover, $\tau_{K_{J_H}} = (\tau_H)_*$.

(18) If $H(H(A)) = H(A)$ for $A \in L^X$, then $K_{J_H}(K_{J_H}^*(A)) = K_{J_H}(A)$ for $A \in L^X$ such that $\tau_{K_{J_H}} = (\tau_H)_*$ with

$$\tau_{K_{J_H}} = \{K_{J_H}^*(A)(y) = \bigvee_{x \in X} (H(\top)(x) \odot A(x)) \mid A \in L^X\}.$$

(19) If $J_H(J_H^*(A)) = J_H^*(A)$ for $A \in L^X$, then $K_{J_H}(K_{J_H}) = K_{J_H}^*(A)$ such that

$$\tau_{K_{J_H}} = \{K_{J_H}(y) = \bigwedge_{x \in X} (A(x) \rightarrow H^*(\top_y)(x)) \mid A \in L^X\} = (\tau_{K_{J_H}})_*.$$

(20) $(K_{J_H}, K^*_H)$ is a Galois connection; i.e,

$$A \leq K_{J_H}(B) \text{ iff } B \leq K^*_H(A).$$

Moreover, $\tau_{K^*_H} = (\tau_{K_{J_H}})_*$.

(21) $(M^*_H, M_{J_H})$ is a dual Galois connection; i.e,

$$M_{J_H}(A) \leq B \text{ iff } M^*_H(B) \leq A.$$

Moreover, $\tau_{M_{J_H}} = (\tau_{J_{J_H}})_*$. 

Proof. (1) Since \( A = \bigvee_{x \in X} (A(x) \odot \top_x) \), by (H2) and (H3), \( H(A)(y) = \bigvee_{x \in X} (A(x) \odot H(\top_x)(y)) \).

(2) Since \( H(A)(y) = \bigvee_{x \in X} (A(x) \odot H(\top_x)(y)) \leq B(y) \) iff \( A(x) \leq H(\top_x)(y) \rightarrow B(y) \), we have

\[
J_H(B)(x) = \bigwedge_{y \in Y} (H(\top_x)(y) \rightarrow B(y)).
\]

(J1) Since \( H(J_H(B)) \leq B \), we have \( J_H(B) \leq H(J_H(B)) \leq B \).

(J2) Since \( H(a \odot J_H(a \rightarrow B)) = a \odot H(J_H(a \rightarrow B)) \leq a \odot (a \rightarrow B) \leq B \), by the definition of \( J_H \), then \( a \odot J_H(a \rightarrow B) \leq J_H(B) \). We have

\[
J_H(a \rightarrow B) \leq a \rightarrow J_H(B).
\]

Since \( a \odot H(a \rightarrow J_H(B)) = H(a \odot (a \rightarrow J_H(B))) \leq H(J_H(B)) \leq B \), then \( H(a \rightarrow J_H(B)) \leq a \rightarrow B \). By the definition of \( J_H \), we have

\[
a \rightarrow J_H(B) \leq J_H(a \rightarrow B).
\]

(J3) By the definition of \( J_H \), since \( J_H(A) \leq J_H(B) \) for \( B \leq A \), we have

\[
J_H(\bigwedge_{i \in \Gamma} A_i) \leq \bigwedge_{i \in \Gamma} J_H(A_i).
\]

Since \( H(\bigwedge_{i \in \Gamma} J_H(A_i)) \leq H(J_H(A_i)) \leq A_i \), then \( H(\bigwedge_{i \in \Gamma} J_H(A_i)) \leq \bigwedge_{i \in \Gamma} A_i \). Thus

\[
J_H(\bigwedge_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} J_H(A_i).
\]

Thus \( J_H : L^X \rightarrow L^X \) is an \( L \)-lower approximation operator. By the definition of \( J_H \), we have

\[
A \leq J_H(B) \iff B \leq H(A).
\]

Since \( A \leq J_H(A) \) iff \( A \leq H(A) \), we have \( \tau_{J_H} = \tau_H \).

(3) Let \( H(H(A)) = H(A) \) for \( A \in L^X \). Since \( H(B) \leq J_H(A) \) iff \( H(H(B)) = H(B) \leq A \) from the definition of \( J_H \), we have

\[
J_H(J_H(A)) = \bigvee \{ B \mid H(B) \leq J_H(A) \}
\]

\[
= \bigvee \{ B \mid H(H(B)) = H(B) \leq A \}
\]

\[
= J_H(A).
\]
(4) Let $H^*(A) \in \tau_H$. Since $H(H^*(A)) = H^*(A)$, $H(H(A)) = H(H^*(A))) = (H(H^*(A)))^* = H(A)$. Hence $H(A) \in \tau_H$; i.e. $H^*(A) \in (\tau_H)_s$. Thus, $\tau_H \subset (\tau_H)_s$.

Let $A \in (\tau_H)_s$. Then $A^* = H(A^*)$. Since $H(A) = H(H^*(A))) = H^*(A^*) = A$, then $A \in \tau_H$. Thus, $(\tau_H)_s \subset \tau_H$.

(5) (J1) Since $A^* \leq H(A^*)$, $J_s(A) = H(A^*)^* \leq A$.

(J2)

$$J_s(\alpha \rightarrow A) = (H((\alpha \rightarrow A))^* = (H(\alpha \circ A^*))^*$$

$$= (\alpha \circ H(A^*))^* = \alpha \rightarrow H(A^*)$$

$$= \alpha \rightarrow J_s(A).$$

(J3)

$$J_s(\bigwedge_{i \in I} A_i) = (H(\bigwedge_{i \in I} A_i))^* = (H(\bigvee_{i \in I} A_i))^*$$

$$= \bigvee_{i \in I}(H(A_i)^*)^* = \bigwedge_{i \in I}(H(A_i))^*$$

$$= \bigwedge_{i \in I} J_s(A_i).$$

Hence $J_s$ is an $L$-lower approximation operator such that

$$J_s(B)(x) = (H(B^*)(x))^* = \bigwedge_{y \in X} (H(\top_y)(x) \rightarrow B(y)).$$

Moreover, $\tau_{J_s} = (\tau_H)_s$ from:

$$A = J_s(A) \iff A = H(A^*) \iff A^* = H(A^*).$$

(6) Let $H(H(A)) = H(A)$ for $A \in L^X$. Then

$$J_s(J_s(A)) = H^*(J_s^*(A)) = (H(H(A^*)))^*$$

$$= H^*(A^*) = J_s(A).$$

Hence $\tau_{J_s} = \{J_s(A) = \bigwedge_{y \in X}(H(\top_y) \rightarrow A(y)) \mid A \in L^X\}$.

(7) Let $H(H^*(A)) = H^*(A)$ for $A \in L^X$. Then

$$J_s(J_s^*(A)) = H^*(J_s(A)) = (H(H^*(A^*)))^*$$

$$= (H^*(A^*))^* = J_s^*(A).$$

Hence $\tau_{J_s} = \{J_s^*(A) = \bigvee_{y \in X}(H(\top_y) \circ A^*(y)) \mid A \in L^X\}$.

$$J_s(J_s(A)) = J_s(J_s^*(J_s^*(A)))$$

$$= J_s^*(J_s^*(A)) = J_s(A).$$
By a similar method in (4), \( \tau_{J_A} = (\tau_{J_A})_* \).

(8) It is similarly proved as (4).

(9) If \( H(H(A)) = H(A) \) for \( A \in L^X \), then \( M_H(M_H^* (A)) = M_H(A) \)

\[
M_H(M_H^* (A)) = M_H(H^*(A^*)) = H(H^*(A^*)) = H(A^*) = M_H(A).
\]

(10) If \( H(H^*(A)) = H^*(A) \) for \( A \in L^X \), then \( M_H(M_H(A)) = M_H^* (A) \)

\[
M_H(M_H(A)) = H(M_H^* (A)) = H(H^*(A^*)) = H^*(A^*) = M_H^* (A).
\]

Since \( M_H(M_H(A)) = M_H^* (A) \),

\[
M_H(M_H^* (A)) = M_H(M_H(M_H(A))) = M_H^* (M_H(A)) = M_H(A).
\]

Hence \( \tau_{M_H} = \{ M_H(A) \mid A \in L^X \} = (\tau_{M_H})_* \).

(11), (12), (13) and (14) are similarly proved as (5), (9), (10) and (5), respectively.

(15) If \( H(H(A)) = H(A) \) for \( A \in L^X \), then \( J_H(J_H(A)) = J_H(A) \). Thus, \( M_{J_H}(M_{J_H}^*(A)) = M_{J_H}(A) \)

\[
M_{J_H}(M_{J_H}^*(A)) = M_{J_H}(J_H(A)) = (J_H(J_H(A)))^* = (J_H(A))^* = M_{J_H}(A).
\]

Since \( H(A) = A \) iff \( J_H(A) = A \) iff \( M_{J_H}(A) = A^* \), \( \tau_{M_{J_H}} = (\tau_{H})_* \) with

\[
\tau_{M_{J_H}} = \{ M_{J_H}^*(A)(y) = \bigwedge_{x \in X} (H(T_y)(x) \rightarrow A(x)) \mid A \in L^X \}.
\]

(16) If \( J_H(J_H^*(A)) = J_H^*(A) \) for \( A \in L^X \), then \( M_{J_H}(M_{J_H}(A)) = M_{J_H}^*(A) \)

\[
M_{J_H}(M_{J_H}(A)) = M_{J_H}(J_H^*(A)) = J_H^*(J_H^*(A)) = J_H(A) = M_{J_H}^*(A).
\]

(17), (18) and (19) are similarly proved as (14), (15) and (16), respectively.

(20) \((K_{J_H}, K_H)\) is a Galois connection; i.e,

\[
A \leq K_{J_H}(B) \text{ iff } A \leq J_H(B^*)
\]

\[
\text{iff } H(A) \leq B^* \text{ iff } B \leq H^*(A) = K_H(A)
\]
Moreover, since \( A^* \leq K_H(A) \) iff \( A \leq K_{J_H}(A^*) \), \( \tau_{K_H} = (\tau_{K_{J_H}})^* \).

(21) \((M_H, M_{J_H})\) is a dual Galois connection; i.e,

\[
M_{J_H}(A) \leq B \text{ iff } J_H(A) \geq B^*
\]

iff \( H(B^*) \leq A \text{ iff } M_H(B) \leq A. \)

Since \( M_{J_H}(A^*) \leq A \) iff \( M_H(A) \leq A^* \), \( \tau_{M_H} = (\tau_{M_{J_H}})^* \).

**Definition 3.2.** [3,4] Let \( X \) be a set. A function \( R : X \times X \to L \) is called:

(R1) **reflexive** if \( R(x, x) = \top \) for all \( x \in X \).

(R2) **symmetric** if \( R(x, y) = R(y, x) \) for all \( x, y \in X \).

(R3) **transitive** if \( R(x, y) \odot R(y, z) \leq R(x, z) \), for all \( x, y, z \in X \).

(R4) **Euclidean** if \( R(x, z) \odot R(y, z) \leq R(x, y) \), for all \( x, y, z \in X \).

If \( R \) satisfies (R1) and (R3), \( R \) is called an \( L \)-fuzzy preorder.

If \( R \) satisfies (R1), (R2) and (R3), \( R \) is called an \( L \)-fuzzy equivalence relation.

Let \( R \in L^{X \times X} \) be an \( L \)-fuzzy relation. Define operators as follows

\[
\begin{align*}
H_R(A)(y) &= \bigvee_{x \in X} (A(x) \odot R(x, y)), \\
J_R(A)(y) &= \bigwedge_{x \in X} (R(x, y) \to A(x)), \\
K_R(A)(y) &= \bigwedge_{x \in X} (A(x) \to R(x, y)), \\
M_R(A)(y) &= \bigvee_{x \in X} (A^*(x) \odot R(x, y)).
\end{align*}
\]

**Example 3.3.** Let \( R \) be a reflexive \( L \)-fuzzy relation. Define \( H_R : L^X \to L^X \) as follows:

\[
H_R(A)(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)).
\]

(1) \( H_R(A)(y) \geq A(y) \odot R(y, y) = A(y) \). \( H_R \) satisfies the conditions (H1) and (H2).

Hence \( H_R \) is an \( L \)-upper approximation operator.

(2) Define \( J_{H_R}(B) = \bigvee \{ A \mid H_R(A) \leq B \} \). Since \( H_R(A)(y) \leq B(y) \) iff \( A(x) \leq \bigwedge_{y \in X} (R(x, y) \to B(y)) \), then

\[
J_{H_R}(B)(x) = \bigwedge_{y \in X} (R(x, y) \to B(y)) = J_{R^{-1}}(B)(x).
\]
By Theorem 3.1(2), $J_{HR} = J_{R^{-1}}$ is an $L$-lower approximation operator such that $(H_R, J_{HR})$ is a residuated connection; i.e.,

$$H_R(A) \leq B \iff A \leq J_{HR}(B).$$

Moreover, $\tau_{J_{HR}} = \tau_{H_R}$.

(3) If $R$ is an $L$-fuzzy preorder, then $R^{-1}$ is an $L$-fuzzy preorder. Since

$$H_R(H_R(A))(z) = \bigvee_{y \in X} (H_R(A)(y) \circ R(y, z))$$
$$= \bigvee_{y \in X} (\bigvee_{x \in X} (A(x) \circ R(x, y)) \circ R(y, z))$$
$$= \bigvee_{x \in X} (A(x) \circ R(x, z)) = H_R(A)(z).$$

By Theorem 3.1(3), $J_{HR}(J_{HR}(A)) = J_{HR}(A)$. By Theorem 3.1(3), $\tau_{J_{HR}} = \tau_{H_R}$ with

$$\tau_{J_{HR}} = \tau_{J_{R^{-1}}} = \{ J_{R^{-1}}(A) = \bigwedge_{x \in X} (R(-, x) \to A(x)) \mid A \in L^X \},$$

$$\tau_{H_R} = \{ H_R(A) = \bigvee_{x \in X} (A(x) \circ R(x, -)) \mid A \in L^X \}.$$

(4) Let $R$ be a reflexive and Euclidean $L$-fuzzy relation. Since $(R(x, y) \to A(x)) \circ R(y, z) \circ R(x, z) \leq (R(x, y) \to A(x)) \circ R(x, y) \leq A(x)$, then $(R(x, y) \to A(x)) \circ R(y, z) \leq R(x, z) \to A(x)$. Thus, $H_R(H_R^*(A)) = H_R^*(A)$ from:

$$H_R(H_R^*(A))(z) = \bigvee_{y \in X} (H_R^*(A)(y) \circ R(y, z))$$
$$= \bigvee_{y \in X} (\bigwedge_{x \in X} (R(x, y) \to A(x)) \circ R(y, z))$$
$$\leq \bigwedge_{x \in X} (R(x, z) \to A(x)) = H_R^*(A)(z).$$

By Theorem 3.1(4), $H_R(H_R(A)) = H_R(A)$ for $A \in L^X$. Thus, $\tau_{H_R} = (\tau_{H_R})^*$, with

$$\tau_{H_R} = \{ H_R^*(A) = \bigwedge_{x \in X} (R(x, -) \to A(x)) = J_R(A) \mid A \in L^X \}.$$

(5) Define $J_s(A) = H_R(A^*)^*$. By Theorem 3.1(5), $J_s = J_R$ is an $L$-lower approximation operator such that

$$J_s(A)(y) = \bigvee_{x \in X} (A^*(x) \circ R(x, y))^* = \bigwedge_{x \in X} (R(x, y) \to A(x)).$$

Moreover, $\tau_{J_s} = (\tau_{H_R})^* = (\tau_{J_{HR}})^*$. 

(6) If $R$ is an $L$-fuzzy preorder, then $H_R(H_R(A)) = H_R(A)$ for $A \in L^X$. By Theorem 3.1(6), then $J_s(J_s(A)) = J_s(A)$ for $A \in L^X$ such that $\tau_{J_s} = (\tau_{H_R})_* = (\tau_{J_{H_R}})_*$, with

$$\tau_{J_s} = \{J_s(A) = \bigwedge_{y \in X} (R(y, -) \to A(y)) \mid A \in L^X\}.$$

(7) If $R$ is a reflexive and Euclidean $L$-fuzzy relation, then $H_R(H^*_R(A)) = H^*_R(A)$ for $A \in L^X$. By Theorem 3.1(7), $J_s(J^*_s(A)) = J^*_s(A)$ such that

$$\tau_{J_s} = \{J^*_s(A) = \bigvee_{y \in X} (R(y, -) \cap A^*(y)) = M_R(A) \mid A \in L^X\} = (\tau_{J_s})_*.$$

(8) Define $M_{H_R}(A) = H_R(A^*)$. Then $M_{H_R} : L^X \to L^X$ with

$$M_{H_R}(A)(y) = \bigvee_{x \in X} (R(x, y) \circ A^*(x)) = M_R(y)$$

is an $L$-meet join approximation operator. Moreover, $\tau_{M_{H_R}} = (\tau_{H_R})_*$.

(9) $R$ is an $L$-fuzzy preorder, then $H_R(H_R(A)) = H_R(A)$ for $A \in L^X$. By Theorem 3.1(9), $M_{H_R}(M^*_{H_R}(A)) = M_{H_R}(A)$ for $A \in L^X$ such that $\tau_{M_{H_R}} = (\tau_{H_R})_*$, with

$$\tau_{M_{H_R}} = \{M^*_{H_R}(A) = \bigwedge_{x \in X} (R(x, -) \to A(x)) = J_R(A) \mid A \in L^X\}.$$

(10) If $R$ is a reflexive and Euclidean $L$-fuzzy relation, then $H_R(H^*_R(A)) = H^*_R(A)$ for $A \in L^X$. By Theorem 3.1(10), $M_{H_R}(M_{H_R}(A)) = M^*_{H_R}(A)$ such that

$$\tau_{M_{H_R}} = \{M_{H_R}(A) = \bigvee_{x \in X} (R(x, -) \cap A^*(x)) \mid A \in L^X\} = (\tau_{M_{H_R}})_*.$$

(11) Define $K_{H_R}(A) = (H_R(A))^*$. Then $K_{H_R} : L^X \to L^X$ with

$$K_{H_R}(A)(y) = \bigwedge_{x \in X} (A(x) \to R^*(x, y)) = K^*_R(A)(y)$$

is an $L$-join meet approximation operator. Moreover, $\tau_{K_{H_R}} = \tau_{H_R}$.

(12) If $R$ is an $L$-fuzzy preorder, then $H_R(H_R(A)) = H_R(A)$ for $A \in L^X$. By Theorem 3.1(12), $K_{H_R}(K^*_{H_R}(A)) = K_{H_R}(A)$ for $A \in L^X$ such that $\tau_{K_{H_R}} = \tau_{H_R}$ with

$$\tau_{K_{H_R}} = \{K^*_{H_R}(A) = \bigvee_{x \in X} (A(x) \circ R(x, -)) \mid A \in L^X\}.$$
(13) If $R$ is a reflexive and Euclidean $L$-fuzzy relation, then $H_R(H^*_R(A)) = H^*_R(A)$ for $A \in L^X$. By Theorem 3.1(13), $K_{H^*_R}(K_{H_R}(A)) = K^*_R(A)$ such that 

$$
\tau_{K_{H^*_R}} = \{K_{H^*_R}(A) = \bigwedge_{x \in X} (A(x) \rightarrow R(x, -)) \mid A \in L^X \} = (\tau_{K_{H^*_R}})^*.
$$

(14) Define $M_{J_{H^*_R}}(A) = (J_{H^*_R}(A))^*$. Then $M_{J_{H^*_R}} : L^X \rightarrow L^X$ with

$$
M_{J_{H^*_R}}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot R(y, x)) = M_{R^{-1}}(A)(y)
$$

is an $L$-join meet approximation operator. Moreover, $\tau_{M_{R^{-1}}} = \tau_H = \tau_{J_{R^{-1}}}^{-1}$.

(15) If $R$ is an $L$-fuzzy preorder, then $H_R(H_R(A)) = H_R(A)$ for $A \in L^X$. By Theorem 3.1(15), $M_{R^{-1}}(M^*_R(A)) = M_{R^{-1}}(A)$ for $A \in L^X$ such that $\tau_{M_{R^{-1}}} = \tau_H = \tau_{J_{R^{-1}}}$ with

$$
\tau_{M_{R^{-1}}} = \{M^*_R(A)(y) = \bigwedge_{x \in X} (R(y, x) \rightarrow A(x)) \mid A \in L^X \}.
$$

(16) Let $R^{-1}$ be a reflexive and Euclidean $L$-fuzzy relation. Since $(R(y, x) \rightarrow A(x)) \odot R(z, y) \odot R(z, x) \leq R(y, x) \rightarrow A(x) \odot R(y, x) \leq A(x)$, then $(R(y, x) \rightarrow A(x)) \odot R(z, y) \leq R(z, x) \rightarrow A(x)$. Thus,

$$
M_{R^{-1}}(M_{R^{-1}}(A))(z) = \bigvee_{y \in X} (M_{R^{-1}}(A)(y) \odot R(z, y))
= \bigvee_{y \in X} (\bigwedge_{x \in X} (R(y, x) \rightarrow A(x)) \odot R(z, y))
\leq \bigwedge_{x \in X} (R(z, x) \rightarrow A(x)) = M_{R^{-1}}(A)(z).
$$

By (M1), $M_{R^{-1}}(M_{R^{-1}}(A)) = M^*_R(A)$ such that

$$
\tau_{M_{R^{-1}}} = \{M_{R^{-1}}(A) = \bigvee_{x \in X} (A^*(x) \odot R(-, x)) \mid A \in L^X \} = (\tau_{M_{R^{-1}}})^*.
$$

(17) Define $K_{J_{H^*_R}}(A) = J_{H^*_R}(A^*)$. Then $K_{J_{H^*_R}} : L^X \rightarrow L^X$ is an $L$-join meet approximation operator as follows:

$$
K_{J_{H^*_R}}(A)(y) = \bigwedge_{x \in X} (R(y, x) \rightarrow A^*(x))
= \bigwedge_{x \in X} (A(x) \rightarrow R^*(y, x))
= K_{R^{-1}^*}(A)(y).
$$

Moreover, $\tau_{K_{J_{H^*_R}}} = (\tau_H^*)^*$. 

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(18) If $R$ is an $L$-fuzzy preorder, then $H_R(H_R(A)) = H_R(A)$ for $A \in L^X$. By Theorem 3.1(18), $K_{JH_R}(K_{JH_R}^*(A)) = K_{JH_R}(A)$ for $A \in L^X$ such that $\tau_{K_{JH}} = (\tau_H)_*$ with
\[
\tau_{K_{JH_R}} = \{K_{JH_R}^*(A)(y) = \bigvee_{x \in X} (R(y, x) \odot A(x)) = H_R^{-1}(A)(y) \mid A \in L^X\}.
\]

(19) Let $R$ be a reflexive and Euclidean $L$-fuzzy relation. Since $R(z, y) \odot R(z, x) \leq R(y, x)$ iff $R(z, y) \leq R(z, x) \rightarrow R(y, x)$ iff $R(z, x) \odot R^*(y, x) \leq R^*(z, y)$, we have
\[
R(z, x) \odot A(x) \odot (A(x) \rightarrow R^*(y, x)) \leq R(z, x) \odot R^*(y, x) \leq R^*(z, y).
\]
Thus,
\[
K_{R^{-1}*}(K_{R^{-1}*}(A))(z) = \bigwedge_{y \in X} (K_{R^{-1}*}(A)(y) \rightarrow R^*(z, y))
\]
\[
= \bigwedge_{y \in X} \left(\bigwedge_{x \in X} (A(x) \rightarrow R^*(y, x)) \rightarrow R^*(z, y)\right)
\]
\[
\geq \bigvee_{x \in X} (R(z, x) \odot A(x)) = K_{R^{-1}*}(A)(z).
\]

Moreover,
\[
\tau_{K_{JH_R}} = \{K_{JH_R}^*(y) = \bigwedge_{x \in X} (A(x) \rightarrow R(y, x)^* \mid A \in L^X\} = (\tau_{K_{JH_R}})_*.
\]

(20) $(K_{JH_R} = K_{R^{-1}*}, K_{H_R} = K_{R^*})$ is a Galois connection; i.e,
\[
A \leq K_{JH_R}(B) \text{ iff } B \leq K_{H_R}(A).
\]
Moreover, $\tau_{K_{H_R}} = (\tau_{K_{JH_R}})_*$.

(21) $(M_{H_R} = M_{R}, M_{JH_R} = M_{R^{-1}})$ is a dual Galois connection; i.e,
\[
M_{JH_R}(A) \leq B \text{ iff } M_{H_R}(B) \leq A.
\]
Moreover, $\tau_{M_{H_R}} = (\tau_{M_{JH_R}})_*$.

Conflict of Interests
The author declares that there is no conflict of interests.
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