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ASYMPTOTIC BEHAVIOR IN NEUTRAL DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS AND MORE THAN ONE DELAY ARGUMENTS

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Abstract. In this paper, we study the asymptotic behavior of the solutions of a neutral type difference equation of the form

$$\Delta\left[x(n)+\sum_{j=1}^{w} q_j(n)x(\tau_j(n))\right]+p(n)x(\sigma(n))=0, \quad n\geq 0$$

where $(p(n))_{n\geq 0}$ is a sequence of positive real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, $\tau_j(n)$, $j=1,\ldots,w$ are general retarded arguments, $\sigma(n)$ is a general deviated argument (retarded or advanced), $(q_j(n))_{n\geq 0}$, $j=1,\ldots,w$ are sequences of real numbers, and Δ denotes the forward difference operator $\Delta x(n)=x(n+1)-x(n)$.

Keywords: Neutral difference equations, retarded argument, deviated argument, oscillatory solutions, nonoscillatory solutions, bounded solutions, unbounded solutions.

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1. Introduction

A neutral difference equation is a difference equation in which the higher order difference of the unknown sequence appears in the equation both with and without delays or advances. See, for example, [1−3, 9] and the references cited therein. We should note that, the theory of neutral difference equations presents complexities, and results which are true for non-neutral difference equations may not be true for neutral equations [16].

The study of the asymptotic and oscillatory behavior of the solutions of neutral difference equations has a strong theoretical interest. Moreover, results on those equations can be applied in several disciplines/fields of science and mathematics, including circuit theory, bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. As a result of the wide range of applications, neutral difference equations have attracted a great interest in the literature.

Consider the neutral difference equation in which the difference of the unknown sequence appears in the equation both with and without more than one delays

$$\text{(E)} \qquad \Delta\left[x(n)+\sum_{j=1}^{w} q_j(n)x(\tau_j(n))\right]+p(n)x(\sigma(n))=0, \qquad n\geq 0,$$

where $(p(n))_{n\geq 0}$ is a sequence of positive real numbers such that $p(n) \geq p$, $p \in \mathbb{R}_+$, $(q_j(n))_{n\geq 0}$, $j=1,...,w$ are sequences of real numbers, $(\tau_j(n))_{n\geq 0}$, $j=1,...,w$ are increasing sequences of integers that satisfy

$$\text{(1.1)} \qquad \begin{gathered} \tau_j(n)\leq n-1,\ j=1,...,w \qquad \forall n\geq 0, \qquad \lim_{n\to\infty}\tau_j(n)=+\infty \\ \text{and} \\ \tau_\ell(n)<\tau_m(n+1), \qquad \forall \ell,m\in[1,w]\cap\mathbb{N} \end{gathered}$$

and $(\sigma(n))_{n\geq 0}$ is an increasing sequence of integers such that

$$\text{(1.2)} \qquad \begin{gathered} \sigma(n)\leq n-1 \qquad \forall n\geq 0, \qquad \lim_{n\to\infty}\sigma(n)=+\infty, \\ \text{or} \\ \sigma(n)\geq n+1 \qquad \forall n\geq 0. \end{gathered}$$

Define

$$k_1 = -\min_{\substack{n \geq 0 \\ 1 \leq j \leq w}} \tau_j(n) \ , \quad k_2 = -\min_{n \geq 0} \sigma(n)$$

and

$$k = \max\{k_1, k_2\}\,.$$

(Clearly, k is a positive integer.)

By a *solution* of the neutral difference equation (E), we mean a sequence of real numbers $(x(n))_{n \geq -k}$ which satisfies (E) for all $n \geq 0$. It is clear that, for each choice of real numbers c_{-k}, c_{-k+1}, ..., c_{-1}, c_0, there exists a unique solution $(x(n))_{n \geq -k}$ of (E) which satisfies the initial conditions $x(-k) = c_{-k}$, $x(-k+1) = c_{-k+1}$, ..., $x(-1) = c_{-1}$, $x(0) = c_0$.

A solution $(x(n))_{n \geq -k}$ of the neutral difference equation (E) is called *oscillatory* if for every positive integer n_0 there exist $n_1, n_2 \geq n$ such that $x(n_1)x(n_2) \leq 0$. In other words, a solution $(x(n))_{n \geq -k}$ is *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the special case where $\tau_j(n) = n - a_j$ and $\sigma(n) = n \pm b$, $a_j, b \in \mathbb{N}$, equation (E) takes the form

$$\text{(E}'\text{)} \qquad \Delta\left[x(n) + \sum_{j=1}^{w} q_j(n)x(n - a_j)\right] + p(n)x(n \pm b) = 0, \quad n \geq 0.$$

In the last few decades, our insight in the asymptotic behavior of neutral difference equations has been significantly advanced. A large number of papers have contributed to the research on this subject. See [4−8, 10−24] and the references cited therein.

The objective in this paper is to investigate the asymptotic behavior of the solutions of Eq. (E). Equation (E) formally describes an extended neutral difference equation, involving several retarded arguments $\tau_j(n)$, $j = 1, 2, ..., w$. In the following sections, we (first) establish some preliminary results that will serve as a useful tool in examining the asymptotic behavior of the solutions of Eq. (E), depending on sequences of real numbers $(q_j(n))$, $j = 1, 2, ..., w$. Then we postulate and prove a theorem setting convergence and divergence conditions on the solutions of Eq. (E).

2. Some Preliminaries

Assume that $(x(n))_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -k}$ is also a solution of (E), we can restrict ourselves to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$, $\forall n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x(\sigma(n)) > 0,\, x(\tau_j(n)) > 0, \quad j = 1, 2, ..., w \qquad \forall n \geq n_0.$$

Set

$$z(n) = x(n) + \sum_{j=1}^{w} q_j(n)x(\tau_j(n)). \tag{2.1}$$

In view of (2.1), Eq. (E) becomes

$$\Delta z(n) + p(n)x(\sigma(n)) = 0. \tag{2.2}$$

Taking into account that $p(n) \geq p > 0$, we have

$$\Delta z(n) = -p(n)x(\sigma(n)) \leq -px(\sigma(n)) < 0 \qquad \forall n \geq n_0,$$

which means that the sequence $(z(n))$ is eventually strictly decreasing, regardless of the values of the terms $q_j(n)$.

Let the domain of τ_j be the set $D(\tau_j) = \mathbb{N}_{n_j^*} = \left\{n_j^*, n_j^* + 1, n_j^* + 2, ...\right\}$, where n_j^* is the smallest natural number that τ_j is defined with. Set

$$n_* = \max_{1\leq j\leq w} n_j^*.$$

Then τ_j, $j = 1, 2, ..., w$ are defined in the set $\mathbb{N}_{n_*} = \{n_*, n_* + 1, n_* + 2, ...\}$.

Let the subsequences

$$\begin{gathered} x(\tau_{\rho(n)}(n)) = \max\left\{x(\tau_1(n)), x(\tau_2(n)), ..., x(\tau_w(n))\right\} \\ \text{and} \\ x(\tau_{\varphi(n)}(n)) = \min\left\{x(\tau_1(n)), x(\tau_2(n)), ..., x(\tau_w(n))\right\}, \end{gathered} \tag{2.3}$$

where $\rho(n), \varphi(n)$ are sequences that take values in the set $\{1, 2, ..., w\}$. Clearly, condition (1.1) guarantees that $\left(x(\tau_{\rho(n)}(n))\right)$ and $\left(x(\tau_{\varphi(n)}(n))\right)$ are subsequences of $(x(n))$.

Notice that

$$\tau_{j_1}(\tau_{j_2}(\cdots\tau_{j_\ell}(n))) = \tau_{j_1}(n_s) \qquad \text{where} \qquad n_s = \tau_{j_2}(\cdots\tau_{j_\ell}(n)). \tag{2.4}$$

The following lemma provides us with some useful tools for establishing the main results:

Lemma 2.1. *Assume that the sequence* $(x(n))_{n\geq -k}$ *is a positive solution of* (E). *Then the following statements hold:*

(i) *If*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty,$$

then

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}. \tag{2.5}$$

(ii) *If*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty,$$

then

$$z(n) < 0, \qquad \textit{eventually.} \tag{2.6}$$

Proof. Summing up (2.2) from n_0 to n, $n \geq n_0$, we obtain

$$z(n+1) - z(n_0) + \sum_{i=n_0}^{n} p(i)x(\sigma(i)) = 0,$$

or

$$z(n+1) = z(n_0) - \sum_{i=n_0}^{n} p(i)x(\sigma(i)). \tag{2.7}$$

For the above relation, exactly one of the following can be true:

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty, \tag{2.7.a}$$

or

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty. \tag{2.7.b}$$

Assume that (2.7.a) holds. Since $p(n) \geq p > 0$, we have

$$+\infty > S_0 = \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) \geq p \sum_{i=n_0}^{\infty} x(\sigma(i)).$$

The last inequality guarantees that

$$\sum_{i=n_0}^{\infty} x(\sigma(i)) < +\infty$$

and, consequently

$$\lim_{n\to\infty} x(\sigma(n)) = 0. \tag{2.8}$$

Also, (2.7.a) guarantess that $\lim_{n\to\infty} z(n)$ exists as a real number. Set

$$\lim_{n\to\infty} z(n) = A \in \mathbb{R}.$$

Since $(z(\sigma(n)))$ is a subsequence of $(z(n))$, we have

$$\lim_{n\to\infty} z(\sigma(n)) = A,$$

or

$$\lim_{n\to\infty} \left[x(\sigma(n)) + \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) \right] = A.$$

Using (2.8), we obtain

$$\lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) = A.$$

Thus

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))).$$

The proof of Part (i) of the lemma is complete.

Assume that (2.7.b) holds. Then, by taking limits on both sides of (2.7) we obtain

$$\lim_{n\to\infty} z(n) = -\infty,$$

which in conjunction with the fact that the sequence $(z(n))$ is eventually strictly decreasing, means that $z(n) < 0$ eventually.

The proof of Part (ii) of the lemma is complete.

The proof of the lemma is complete.

3. Main results

Throughout this section, we are going to use the following conditions:

$$(\mathrm{C}_1) \qquad \left\{ \begin{array}{c} c := \limsup q(n) < -1, \\ and \\ (q(n)) \quad \textit{is bounded} \end{array} \right\}$$

$$(\mathrm{C}_2) \qquad \left\{ \begin{array}{c} q(n) < -1, \qquad \lim_{n\to\infty} q(n) = -1 \\ \\ and \\ \\ \lim_{n\to\infty} \prod_{j=0}^{n} (-q(\mu(j)) = B < +\infty \end{array} \right\}$$

$$(\mathrm{C}_3) \qquad \left\{ \begin{array}{c} q(n) > -1, \qquad \lim_{n\to\infty} q(n) = -1 \\ \\ and \\ \\ \lim_{n\to\infty} \prod_{j=0}^{n} (-q(\mu(j))) = B > 0 \end{array} \right\}$$

$$(\mathrm{C}_4) \qquad -1 < q(n) \le 0 \quad \text{and} \quad \liminf q(n) > -1,$$

where

$$q(n) = \sum_{j=1}^{w} q_j(n) \tag{3.1}$$

and $(q(\mu(j)))$ is a subsequence of $(q(n))$.

The asymptotic behavior of the solutions of the neutral difference equation (E) is described by the following theorem:

Theorem 3.1. *For Eq.* (E) *the following statements hold:*

(I) *Every nonoscillatory solution does not converge in* $\mathbb{R}$*, if the terms* $q_j(n)$ *are all nonpositive and condition* (C_1) *holds.*

(II) *Every solution oscillates if the terms* $q_j(n)$ *are all nonpositive and condition* (C_2) *or condition* (C_3) *holds.*

(III) *Every nonoscillatory solution tends to zero if the terms* $q_j(n)$ *are all nonpositive and condition* (C_4) *holds.*

(IV) *Every nonoscillatory solution is bounded if the terms* $q_j(n)$ *are all nonnegative. Furtheremore, if* $0 \leq q(n) < 1$ *and* $\limsup q(n) < 1$ *then every nonoscillatory solution tends to zero.*

Proof. Assume that $(x(n))_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -k}$ is also a solution of (E), we can restrict ourselves to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$, $\forall n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x(\sigma(n)) > 0,\ x(\tau_j(n)) > 0, \quad j = 1, 2, ..., w \qquad \forall n \geq n_0.$$

Set

$$z(n) = x(n) + \sum_{j=1}^{w} q_j(n) x(\tau_j(n)). \tag{2.1}$$

In view of (2.1), Eq. (E) becomes

$$\Delta z(n) + p(n) x(\sigma(n)) = 0. \tag{2.2}$$

Taking into account that $p(n) \geq p > 0$, we have

$$\Delta z(n) = -p(n)x(\sigma(n)) \leq -px(\sigma(n)) < 0 \qquad \forall n \geq n_0,$$

which means that the sequence $(z(n))$ is eventually strictly decreasing, regardless of the values of the terms $q_j(n)$.

Assume that the terms $q_j(n)$ are all nonpositive and (C_1) holds. If (2.7.a) holds then, in view of Part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n)) x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}$$

which guarantees that $A \leq 0$.

If $A = 0$, then

$$\lim_{n\to\infty} z(n) = 0.$$

Taking into account that the sequence $(z(n))$ is eventually strictly decreasing, it follows that eventually, $z(n) > 0$. Thus, from (2.1), (2.3), (3.1) and (2.4), we obtain

$$\begin{aligned} x(n) &> -\sum_{j=1}^{w} q_j(n)x(\tau_j(n)) \geq \left(-\sum_{j=1}^{w} q_j(n)\right) x(\tau_{\varphi_1(n)}(n)) \\ &= -q(n)x(\tau_{\varphi_1(n)}(n)) > -cx(\tau_{\varphi_1(n)}(n)) > \cdots > (-c)^{m(n)}\, x(\tau_{\varphi_{m(n)}}(n_*)), \end{aligned}$$

or

$$x(n) > (-c)^{m(n)}\, x(\tau_{\varphi_{m(n)}}(n_*)).$$

Consequently,

$$\lim_{n\to\infty} x(n) \geq \lim_{n\to\infty} \left[(-c)^{m(n)}\, x(\tau_{\varphi_{m(n)}}(n_*))\right] = +\infty,$$

which contradicts (2.8). Therefore $A < 0$. Thus, since $(z(\sigma(n)))$ is a subsequence of $(z(n))$, we have

$$\lim_{n\to\infty} z(\sigma(n)) = A < 0,$$

or

$$\lim_{n\to\infty} \left[x(\sigma(n)) + \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n)))\right] = A < 0$$

or

$$\lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) = A < 0.$$

If the only accumulation point of $(x(n))$ is zero or else $\lim_{n\to\infty} x(n) = 0$, then $\lim_{n\to\infty} x(\tau_j(\sigma(n))) = 0$. Combined with the fact that (C_1) holds or else $(q_j(n))$, $j = 1, 2, ..., w$ are all bounded, we will have

$$\lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) = 0,$$

which contradicts the previous inequality. Therefore $(x(n))$ has more than one accumulation points, and consequently $(x(n))$ does not converge in $\mathbb{R}$.

Assume that (2.7.b) holds. Then, by taking limits on both sides of (2.7) we obtain

$$\lim_{n\to\infty} z(n) = -\infty,$$

or

$$\lim_{n\to\infty} \left[x(n) + \sum_{j=1}^{w} q_j(n)x(\tau_j(n)) \right] = -\infty.$$

Thus

$$\lim_{n\to\infty} \left[x(n) + \left(\sum_{j=1}^{w} q_j(n) \right) x(\tau_{\rho(n)}(n)) \right] = -\infty,$$

or

$$\lim_{n\to\infty} \left[x(n) + q(n)x(\tau_{\rho(n)}(n)) \right] = -\infty.$$

Since $(q(n))$ is bounded, the last relation guarantees that

$$\lim_{n\to\infty} x(\tau_{\rho(n)}(n)) = +\infty,$$

which means that $(x(n))$ is unbounded. Therefore, $(x(n))$ does not converge in $\mathbb{R}$. The proof of Part (I) of the theorem is complete.

Assume that the terms $q_j(n)$ are all nonpositive and condition (C_2) holds. Supose, for the sake of contradiction, that $(x(n))_{n\geq -k}$ is an eventually positive solution of (E).

If (2.7.b) holds then, from Part (ii) of Lemma 2.1, (2.6) holds. Consequently, for all large n, we have

$$x(n) + \sum_{j=1}^{w} q_j(n)x(\tau_j(n)) < 0.$$

Using (2.3), (3.1) and (2.4), the last inequality becomes

$$\begin{aligned} x(n) &< -\sum_{j=1}^{w} q_j(n)x(\tau_j(n)) < \left(-\sum_{j=1}^{w} q_j(n)\right) x(\tau_{\rho_1(n)}(n)) \\ &= -q(n)x(\tau_{\rho_1(n)}(n)) < (-q(n))[-q(\tau_{\rho_1(n)}(n))x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n)))] \\ &< \cdots < (-q(n))\left(-q(\tau_{\rho_1(n)}(n))\right)\cdots\left(-q(\tau_{\rho_{m(n)}}(n))\right)x(\tau_{\rho_{m(n)}}(n_*)) \\ &= x(\tau_{\rho_{m(n)}}(n_*))\prod_{j=0}^{m(n)}\left(-q(\tau_{\rho_j(n)}(n))\right) \\ &\leq x(\tau_{\rho_{m(n)}}(n_*))\lim_{n\to\infty}\prod_{j=0}^{m(n)}\left(-q(\tau_{\rho_j(n)}(n))\right) = Bx(\tau_{\rho_{m(n)}}(n_*)) \end{aligned}$$

or

$$x(n) < Bx(\tau(n_*)). \tag{3.2}$$

The above inequality means that the sequence $(x(n))$ is bounded and therefore $(z(n))$ is bounded. This contradicts (2.7.b), and therefore $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < +\infty$. Thus, from Part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty}\sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}$$

which guarantees that $A \leq 0$.

Assume that $A < 0$. Taking into account that the sequence $(z(n))$ is eventually strictly decreasing, it follows that $z(n) < 0$, eventually. As in the previous case, (3.2) is satisfied and therefore $(x(n))$ is bounded.

Let

$$M = \limsup x(n). \tag{3.3}$$

Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that

$$\lim_{n\to\infty} x(\theta(n)) = M.$$

Since $(z(\theta(n)))$ is a subsequence of $(z(n))$ we have

$$\lim_{n\to\infty}\left[x(\theta(n)) + \sum_{j=1}^{w} q_j(\theta(n))x(\tau_j(\theta(n)))\right] = A,$$

or

$$-\lim_{n\to\infty}\sum_{j=1}^{w} q_j(\theta(n))x(\tau_j(\theta(n))) = M - A.$$

Therefore, for every ε with $0 < \varepsilon < -A$, there exists $n_2 \in \mathbb{N}$ such that for every $n \geq n_2$

$$-\sum_{j=1}^{w} q_j(\theta(n))x(\tau_j(\theta(n))) + \varepsilon \geq M - A.$$

Thus

$$\left(-\sum_{j=1}^{w} q_j(\theta(n))\right) x(\tau_{\rho(\theta(n))}(\theta(n))) + \varepsilon \geq M - A,$$

or

$$-q(\theta(n))x(\tau_{\rho(\theta(n))}(\theta(n))) + \varepsilon \geq M - A.$$

Consequently,

$$\limsup\left[-q(\theta(n))x(\tau_{\rho(\theta(n))}(\theta(n))) + \varepsilon\right] \geq M - A,$$

or

$$M + \varepsilon \geq \limsup x(\tau_{\rho(\theta(n))}(\theta(n))) + \varepsilon > M - A$$

or

$$\varepsilon \geq -A.$$

This result contradicts that $\varepsilon < -A$ and therefore $A = 0$. Furthermore, taking into account that the sequence $(z(n))$ is eventually strictly decreasing, we conclude that $z(n) > 0$, eventually. Thus, for all large n, we have

$$x(n) + \sum_{j=1}^{w} q_j(n)x(\tau_j(n)) > 0.$$

Using (2.3), (3.1) and (2.4), the last inequality becomes

$$\begin{aligned} x(n) &> -\sum_{j=1}^{w} q_j(n)x(\tau_j(n)) \geq \left(-\sum_{j=1}^{w} q_j(n)\right) x(\tau_{\varphi_1(n)}(n)) \\ &= -q(n)x(\tau_{\varphi_1(n)}(n)) > (-q(n))[-q(\tau_{\varphi_1(n)}(n))x(\tau_{\varphi_2(n)}(\tau_{\varphi_1(n)}(n)))] \\ &> \cdots > (-q(n))\left(-q(\tau_{\varphi_1(n)}(n))\right)\cdots\left(-q(\tau_{\varphi_{m(n)}}(n))\right) x(\tau_{\varphi_{m(n)}}(n_*)) \\ &= x(\tau_{\varphi_{m(n)}}(n_*)) \prod_{j=0}^{m(n)} \left(-q(\tau_{\varphi_j(n)}(n))\right). \end{aligned}$$

Since $(x(n))$ has a lower bound greater than zero, it cannot have any subsequence that tends to zero. Thus $\lim_{n\to\infty} x(\sigma(n)) = 0$ is not valid, and therefore $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = \infty$. This result comes to contradiction with our previous assumptions, leading us to conclude that $(x(n))$ oscillates.

Assume that the terms $q_j(n)$ are all nonpositive and condition (C_3) holds. Supose, for the sake of contradiction, that $(x(n))_{n\geq -k}$ is an eventually positive solution of (E). If (2.7.b) holds then, in view of Part (ii) of Lemma 2.1, (2.6) is satisfied. By a similar procedure, as in the previous case we conclude that (3.2) holds, which means that the sequence $(x(n))$ is bounded and consequently $(z(n))$ is bounded. This contradicts $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty$.

Thus $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < +\infty$. By a similar procedure, as in the previous case we eventually have

$$x(n) > x(\tau_{\varphi_{m(n)}}(n_*)) \prod_{j=0}^{m(n)} \left(-q(\tau_{\varphi_j(n)}(n))\right).$$

Since the sequence $(x(n))$ has a lower bound greater than zero, it cannot have any subsequence that tends to zero. Thus $\lim_{n\to\infty} x(\sigma(n)) = 0$ is not valid, and therefore $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = \infty$. This result comes to contradiction with our previous assumptions, leading us to conclude that $(x(n))$ oscillates. The proof of Part (II) of the theorem is complete.

Assume that the terms $q_j(n)$ are all nonpositive and condition (C_4) holds. Let

$$\liminf q(n) = d > -1. \tag{3.4}$$

This ensures that $q(n) > d$ or $-q(n) < -d < 1$ eventually.

Supose that (2.7.b) holds. Then, in view of Part (ii) of Lemma 2.1, (2.6) is satisfied. Using (3.1), (2.3), (3.4) and (2.4), we obtain

$$\begin{aligned}
x(n) &< -\sum_{j=1}^{w} q_j(n)x(\tau_j(n)) < \left(-\sum_{j=1}^{w} q_j(n)\right) x(\tau_{\rho_1(n)}(n)) \\
&= -q(n)x(\tau_{\rho_1(n)}(n)) < -dx(\tau_{\rho_1(n)}(n)) \\
&< (-d)[-q(\tau_{\rho_1(n)}(n))x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n)))] \\
&< (-d)[-dx(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n)))] < \cdots < (-d)\,(-d)\cdots(-d)\,x(\tau_{\rho_{m(n)}}(n_*)) \\
&= x(\tau_{\rho_{m(n)}}(n_*))\prod_{j=0}^{m(n)}(-d) = x(\tau_{\rho_{m(n)}}(n_*))(-d)^{m(n)+1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This result means that the sequence $(x(n))$ tends to zero as $n \to \infty$, and therefore $(z(n))$ is bounded. This contradicts (2.7.b).

Thus, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < +\infty$ which, in view of Part (i) of Lemma 2.1, means that

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}$$

which guarantees that $A \le 0$.

Assume that $A < 0$. Taking into account that the sequence $(z(n))$ is eventually strictly decreasing, it follows that $z(n) < 0$ eventually. Following a similar procedure as in the previous case, we eventually have

$$x(n) < x(\tau_{\rho_{m(n)}}(n_*))\prod_{j=0}^{m(n)}(-d) = x(\tau_{\rho_{m(n)}}(n_*))(-d)^{m(n)+1} \to 0 \text{ as } n \to \infty$$

which means that $\lim_{n\to\infty} x(n) = 0$, and consequently $\lim_{n\to\infty} z(n) = 0$. This contradicts $A < 0$. Therefore $A = 0$.

Since the sequence $(z(n))$ is eventually strictly decreasing, it is obvious that $z(n) > 0$ eventually. Therefore, there exists $\varepsilon > 0$ such that

$$z(n) < \varepsilon \qquad \text{and} \qquad q(n) > d - \varepsilon > -1 \quad \text{for sufficiently large } n.$$

Using (3.1), (2.3), (3.4) and (2.4), we obtain

$$x(n) + \sum_{j=1}^{w} q_j(n) x(\tau_j(n)) < \varepsilon,$$

or

$$\begin{aligned}
x(n) &< -\sum_{j=1}^{w} q_j(n) x(\tau_j(n)) + \varepsilon \leq \left(-\sum_{j=1}^{w} q_j(n)\right) x(\tau_{\rho_{1(n)}}(n)) + \varepsilon \\
&= -q(n) x(\tau_{\rho_{1(n)}}(n)) + \varepsilon < (-d+\varepsilon)\, x(\tau_{\rho_{1(n)}}(n)) + \varepsilon \\
&< (-d+\varepsilon)\left[(-d+\varepsilon)\, x(\tau_{\rho_{2(n)}}(\tau_{\rho_{1(n)}}(n)))] + \varepsilon\right] + \varepsilon \\
&= (-d+\varepsilon)^2\, x(\tau_{\rho_{2(n)}}(\tau_{\rho_{1(n)}}(n))) - (-d+\varepsilon)\,\varepsilon + \varepsilon \\
&< \cdots < (-d+\varepsilon)^m\, x(\tau_{\rho_{m(n)}(n)}(\tau_{\rho_{m(n)-1}(n)}(n))) \\
&\quad +\varepsilon + (-d+\varepsilon)\,\varepsilon + \cdots + (-d+\varepsilon)^m\,\varepsilon.
\end{aligned}$$

As $n \to \infty$, clearly $m \to \infty$, and therefore

$$\begin{aligned}\lim_{n\to\infty} x(n) &\leq \lim_{m\to\infty}\left[(-d+\varepsilon)^m x(\tau_{\rho_{m(n)}(n)}(\tau_{\rho_{m(n)-1}(n)}(n)))\right] \\ &\quad + \lim_{m\to\infty}\left[\varepsilon + (-d+\varepsilon)\varepsilon + \cdots + (-d+\varepsilon)^m \varepsilon\right] \\ &= 0 + \lim_{m\to\infty}\left[\varepsilon + (-d+\varepsilon)\varepsilon + \cdots + (-d+\varepsilon)^m \varepsilon\right] = \frac{\varepsilon}{1+d-\varepsilon}.\end{aligned}$$

Since ε is an arbitrary real positive number, and taking into account that $x(n) > 0$, it is apparent that

$$\lim_{n\to\infty} x(n) = 0.$$

The proof of Part (III) of the theorem is complete.

Assume that the terms $q_j(n)$ are all nonnegative. Clearly, $q(n) \geq 0$. If (2.7.b) holds then, in view of Part (ii) of Lemma 2.1, (2.6) is satisfied. This contradicts $z(n) = x(n) + \sum_{j=1}^{w} q_j(n)x(\tau_j(n) > 0$. Therefore (2.7.b) is not valid and consequently (2.7.a) holds. This result, combined with (2.7), implies that the sequence $(z(n))$ is bounded and therefore $(x(n))$ is bounded.

Assume that $0 \leq \limsup q(n) = q < 1$. If

$$M = \limsup x(n),$$

then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that

$$\lim_{n\to\infty} x(\theta(n)) = M.$$

Since $z(n) > 0$ we have $\lim_{n\to\infty} z(n) = A \geq 0$.

Let $A > 0$. Then

$$\lim_{n\to\infty}\left[x(\theta(n)) + \sum_{j=1}^{w} q_j(\theta(n))x(\tau_j(\theta(n)))\right] = A,$$

or

$$\lim_{n\to\infty}\sum_{j=1}^{w} q_j(\theta(n))x(\tau_j(\theta(n))) = A - M \geq 0.$$

Therefore

$$A \geq M.$$

On the other hand

$$\lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) = A$$

or

$$\limsup \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) = A.$$

But

$$\begin{aligned} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) &\leq \left(\sum_{j=1}^{w} q_j(\sigma(n))\right) x(\tau_{\rho(\sigma(n))}(\sigma(n))) \\ &= q(\sigma(n))x(\tau_{\rho(\sigma(n))}(\sigma(n))). \end{aligned}$$

Consequently

$$\begin{aligned} \limsup \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))) &\leq \limsup \left[q(\sigma(n))x(\tau_{\rho(\sigma(n))}(\sigma(n)))\right] \\ &\leq \limsup q(\sigma(n)) \limsup x(\tau_{\rho(\sigma(n))}(\sigma(n))) \\ &\leq qM. \end{aligned}$$

Therefore,

$$M > qM \geq A,$$

which contratICTS that $M \leq A$. Thus $A = 0$, and consequently $\lim_{n\to\infty} x(n) = 0$. The proof of Part (IV) of the theorem is complete.

The proof of the theorem is complete.

Remark 3.2. As a consequence of Theorem 3.1, we derive the following corollary for Eq. (E′).

Corollary 3.3. *For Eq.* (E′) *the following statements hold:*

(i) *Every nonoscillatory solution is unbounded if the terms* $q_j(n)$ *are all nonpositive and condition* (C_1) *holds.*

(ii) *Every solution oscillates if the terms* $q_j(n)$ *are all nonpositive and condition* (C_2) *or condition* (C_3) *holds.*

(iii) *Every nonoscillatory solution tends to zero either the terms* $q_j(n)$ *are all nonpositive and condition* (C_4) *holds, or the terms* $q_j(n)$ *are all nonnegative.*

Proof. Assume that the terms $q_j(n)$ are all nonpositive and (C_1) holds. If (2.7.a) holds then, in view of Part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(n \pm b)x(n - a_j \pm b),$$

which guarantees that $A \leq 0$.

Let $A < 0$. Since (2.8) is satisfied we have $\lim_{n\to\infty} x(n \pm b) = 0$. Then $\lim_{n\to\infty} x(n) = 0$ and consequently $\lim_{n\to\infty} x(n - a_j \pm b) = 0$. Taking into account that (C_1) holds, we conclude that $\lim_{n\to\infty} z(n) = 0$ which contradicts $A < 0$. Thus $A = 0$ and since the sequence $(z(n))$ is eventually strictly decreasing , it is obvious that $z(n) > 0$ eventually, or

$$\begin{aligned} x(n) &> -\sum_{j=1}^{w} q_j(n)x(n-a_j) \geq \left(-\sum_{j=1}^{w} q_j(n)\right) x(n - a_{\varphi_1(n)}) \\ &= -q(n)x(n - a_{\varphi_1(n)}) > (-c)x(n - a_{\varphi_1(n)}) > \cdots > (-c)^{m(n)} x(n_* - a_{\varphi_{m(n)}}). \end{aligned}$$

Consequently

$$\lim_{n\to\infty} x(n) \geq \lim_{n\to\infty} \left[(-c)^{m(n)} x(n_* - a_{\varphi_{m(n)}})\right] = +\infty,$$

which contradicts (2.7.a). Therefore $\sum_{i=n_0}^{\infty} p(i)x(i \pm b) = +\infty$.

Summing up (2.2) from n_0 to n, $n \geq n_0$, we obtain

$$z(n+1) = z(n_0) - \sum_{i=n_0}^{n} p(i)x(i \pm b)$$

which implies that

$$\lim_{n\to\infty} z(n) = -\infty.$$

Thus

$$\lim_{n\to\infty}\left[x(n)+\sum_{j=1}^{w} q_j(n)x(n-a_j)\right]=-\infty,$$

or

$$\lim_{n\to\infty}\left[x(n)+\left(\sum_{j=1}^{w} q_j(n)\right)x(n-a_{\rho(n)})\right]=-\infty,$$

or

$$\lim_{n\to\infty}\left[x(n)+q(n)x(n-a_{\rho(n)})\right]=-\infty.$$

Since $(q(n))$ is bounded, the last relation guarantees that

$$\lim_{n\to\infty} x(n-a_{\rho(n)})=+\infty,$$

which means that $(x(n))$ is unbounded. The proof of Part (i) of the corollary is complete.

Part (ii) follows directly from Part (II) of Theorem 3.1.

As we have shown in Parts (III) and (IV) of Theorem 3.1, it is true that

$$\lim_{n\to\infty} x(\sigma(n))=0$$

and consequently

$$\lim_{n\to\infty} x(n\pm b)=0,$$

which means that

$$\lim_{n\to\infty} x(n)=0.$$

The proof of Part (iii) of the corollary is complete.

The proof of the corollary is completed.

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