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RECURSION RELATIONS FOR THE CANONICAL SOLUTION OF STURM-LIOUVILLE PROBLEM WITH TURNING POINTS

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Abstract. The paper studies the differential equation $y'' + (\rho^2\phi^2(x) - q(x))y = 0$, on a finite interval I, say $I = [0, 1]$, where I contains m turning points, $x_1, x_2, \ldots, x_m$, that is, zeros of ϕ. We show that if one obtains the canonical solution of an initial value problem generated by the equation in the interval $[0, x_1)$, the canonical solutions in the remaining intervals can be obtained recursively.

Keywords: Sturm-Liouville problem, turning point, canonical solution.

2010 AMS Subject Classification: 35G25.

1. Introduction

Sturm-Liouville systems are not just objects of interest to mathematicians. They are the one-dimensional models of a large number important physical processes in more than one dimension by such methods as the separation of variables. Also, Sturm-Liouville equations with turning points play an important role in various areas of mathematics as well as in applications. For example, turning points connected with physical situations in which zeros correspond to the limit of motion of a wave mechanical particle bound by a potential field. Turning points appear

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also in elasticity, optics, geophysics and other branches of natural sciences; see [1]-[5] and the references therein. It is obvious that the exact solutions of the nonlinear differential equations can help us to know the described process. So an important issue of the nonlinear differential equations is to find their new exact solutions. Various methods for obtaining exact solutions to nonlinear differential equations have been proposed. But, for most differential equations no exact solutions is known and in some cases, it is not even clear whether a unique solution exist. On the other hand, often the obtained exact solution is not proper to use. So, approximation methods have been developed. One of the most useful mathematical methods of achieving this, is representing the solution by an asymptotic form. The importance of asymptotic analysis in the theory of differential equations was clearly recognized in the second half of the nineteenth century. Starting from the work of Birkohoff [6], others developed an asymptotic theory for linear differential equations, based on transformations to a first order system, when various diagonal transformations were applied. The importance of asymptotic analysis in obtaining information on the solution of a Sturm-Liouville equation with multiple turning points was realized by Leung [7], Olver [8]-[10], Heading [11], Eberhard, Freiling & Schneider in [4] and Marasi, Jodayree [12, 13]. But the weakness of asymptotic methods is that one cannot generally express the exact solution in closed form. Indeed, in methods connected with dual equations, the closed form of the solution is needed. Halvorsen [14], proved that for $0 < c < x < 1$ if $\int_c^x |\phi(t)|dt \neq 0$ the solution $y(x,\lambda)$ of $y'' + (\rho^2\phi^2(x) - q(x))y = 0$,, determined by fixed values of y, y' at c, is an entire function of λ of order $\frac{1}{2}$. Thus it follows from the classical Hadamard factorization theorem; see [15], such solutions are expressible as an infinite product, that is, a closed form of solution.

2. Preliminaries

Consider the real second order differential equation

$$y'' + (\rho^2\phi^2(x) - q(x))y = 0, \qquad x \in I = [0,1], \tag{1}$$

where $\rho^2 = \lambda$ is the spectral parameter, q is bounded and integrable on I and

$$\phi^2(x) = \phi_0(x) \prod_{v=1}^{m} (x - x_v)^{\ell_v}, \tag{2}$$

where $0 = x_0 < x_1 < x_2 < ... < x_m < 1 = x_{m+1}$, $\phi_0(x) > 0$ for $x \in I = [0,1]$ and ϕ_0 is twice continuously differentiable on I. In other words, $\phi^2(x)$ has in I m zeros x_v, $v = 1,...,m$ of order l_v, so called turning points.

We distinguish four different types of turnings points: For $1 \leq v \leq m$

$$T_v = \begin{cases} I, & \text{if } l_v \text{ is even and } \phi^2(x)(x-x_v)^{-l_v} < 0, \\ II, & \text{if } l_v \text{ is even and } \phi^2(x)(x-x_v)^{-l_v} > 0, \\ III, & \text{if } l_v \text{ is odd and } \phi^2(x)(x-x_v)^{-l_v} < 0, \\ IV, & \text{if } l_v \text{ is odd and } \phi^2(x)(x-x_v)^{-l_v} > 0, \end{cases}$$

is called type of x_v. Specially in this paper, we suppose that $\ell_i = 4k+2$ for $1 \leq i \leq s$, $\ell_{s+1} = 4k+1$ and $\ell_j = 4k$ for $s+2 \leq j \leq m$. In other words, $x_1,...,x_s$ are of type I , x_{s+1} is of type IV and $x_{s+2},...,x_m$ are of type II.

In order to represent the solution as an infinite product one can use a suitable fundamental system of solutions (FSS) for equation (1) as provided in [4]. Let $\varepsilon > 0$ be fixed, sufficiently small and $D_{0\varepsilon=[0,x_1-\varepsilon]}$, $D_{v\varepsilon=[x_v+\varepsilon,x_{v+1}-\varepsilon]}$ for $1 \leq v \leq m-1$, $D_{m\varepsilon} = [x_m+\varepsilon,1]$, $D_\varepsilon = \cup_{v=0}^{m} D_{v\varepsilon}$ and $I_{v\varepsilon} = D_{v-1,\varepsilon} \cup [x_v - \varepsilon, x_v + \varepsilon] \cup D_{v\varepsilon}$. Further we set for $1 \leq v \leq m$

$$\mu_v = \frac{1}{2+l_v},$$

$$\theta_v = \begin{cases} 1, & \mu_v > \frac{1}{4}, \\ 1-\delta_0, & \mu_v = \frac{1}{4}, \\ 4\mu_v, & \mu_v < \frac{1}{4}, \end{cases}$$

and $0 < \theta_0 = min\{\theta_v \mid 1 \leq v \leq m\}$. We also denote

$$I_+ = \{x \mid \phi^2(x) > 0\},\ I_- = \{x \mid \phi^2(x) < 0\},$$

$$\xi(x) = \begin{cases} 0, & x \in I_+, \\ 1, & x \in I_-, \end{cases}$$

$$\gamma_v = \begin{cases} 2\sin\frac{\pi\mu_v}{2}, & T_v = II, IV, \\ \sin\pi\mu_v, & T_v = I, II, \end{cases}$$

$$K_\pm(x) = (\prod_{x_v\in(0,x)} \gamma_v^{-1})e^{\pm\iota\frac{\pi}{4}(\xi(x)-\xi(0))},$$

$$K_\pm^* = (\prod_{x_v\in(0,x)} \gamma_v^{-1})e^{\pm\iota\frac{\pi}{4}(\xi(x)+\xi(0))},$$

$$\phi_+^2(x) = max(0,\phi^2(x)),\ \phi_-^2(x) = max(0,-\phi^2(x)).$$

Clearly,

$$K_\pm(x)K_\pm^* = e^{\pm\iota\frac{\pi}{2}\xi(x)} = \begin{cases} 1, & x\in I_+, \\ \pm\iota, & x\in I_-, \end{cases}$$

Let

$$S_k = \{\rho : arg\rho \in [\frac{k\pi}{4}, \frac{(k+1)\pi}{4}]\},$$

$$\sigma_s^\delta = \{\rho : arg\rho \in [\frac{s\pi}{2}-\delta, \frac{s\pi}{2}+\delta]\},\ \delta > 0,$$

$$\sigma^\delta = \cup_s\sigma_s^\delta,\ S_k^\delta = S_k - \sigma^\delta\ S^\delta = \cup_{k=-2}^{1} S_k^\delta.$$

It is sufficient to consider the sectors S_k and S_k^δ for $k = -2,-1,0,1$ only. It is shown in [4] that for each fixed sector $S_k(k = -2,-1,0,1)$ there exist a fundamental system of solutions of (1) $\{Z_1(x,\rho), Z_2(x,\rho)\}, x\in I, \rho\in S_k$ such that the functions

$$(x,\rho)\mapsto Z_j^s(x,\rho),\ (j = 1,2; s = 0,1)$$

are continues for $x\in I, \rho\in S_k$ and holomorphic for each fixed $x\in I$ with respect to $\rho\in S_k$. Moreover for $|\rho|\to\infty, \rho\in S_k, x\in D_\varepsilon, j = 1,2,$

$$Z_1^{(j)}(x,\rho) = (\pm\iota\rho)^j \mid \phi(x) \mid^{j-\frac{1}{2}} (exp(\mp\iota\frac{\pi}{2}\xi(x))^j exp(\rho\int_o^x \mid \phi_-(t) \mid dt)\times$$

(3)

$$exp(\pm\int_0^x \mid \phi_+(t) \mid dt)K_\pm(x)\kappa(x,\rho),$$

$$Z_2^{(j)}(x,\rho) = (\mp\iota\rho)^j \mid \phi(x) \mid^{j-\frac{1}{2}} (exp(\mp\iota\frac{\pi}{2}\xi(x))^j exp(-\rho\int_o^x \mid \phi_-(t) \mid dt)\times$$

(4)

$$exp(\mp\int_0^x \mid \phi_+(t) \mid dt)K_\pm^*(x)\kappa(x,\rho),$$

$$W(\rho) = W(Z_1(x,\rho), Z_2(x,\rho)) = \mp(2\iota\rho)[1],$$

where $W(f(x),g(x)) := f(x)g'(x) - f'(x)g(x)$ is the Wronskian of f and g. Note that, the choice of the root $\phi(x)$ of $\phi^2(x)$ depends on the interval and has to determined carefully. For example, in this paper, due to the type of turning points x_1 and $x_v, v = 2,3,...m$ we have for $\rho \in S_0$

$$\phi(x) = \begin{cases} |\phi(x)|, & x > x_1, \\ |\phi(x)|e^{i\frac{\pi}{2}}, & x < x_1. \end{cases}$$

So, For $\rho \in S_{-1}$, from (3) and (4) fundamental system of solutions of (1), $\{Z^{T_v}_{v,1}(x,\rho), Z^{T_v}_{v,2}(x,\rho)\}$, for a turning point of type I, II, III or IV are of the form:

Turning point of type I:

$$Z^I_{v,1}(x,\rho) = \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} e^{\rho \int_{x_v}^x |\phi(t)|dt}[1], & x_{v-1} < x < x_v, \\ \mid \phi(x) \mid^{-\frac{1}{2}} \csc \pi\mu_v e^{\rho \int_{x_v}^x |\phi(t)|dt}[1], & x_v < x < x_{v+1}, \end{cases} \tag{5}$$

$$Z^I_{v,2}(x,\rho) = \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} e^{-\rho \int_{x_v}^x |\phi(t)|dt}[1], & x_{v-1} < x < x_v, \\ \mid \phi(x) \mid^{-\frac{1}{2}} \sin \pi\mu_v e^{-\rho \int_{x_v}^x |\phi(t)|dt}[1], & x_v < x < x_{v+1}, \end{cases} \tag{6}$$

$$Z^I_{v,1}(x_v,\rho) = \frac{\sqrt{2\pi}}{2}(\imath\rho)^{\frac{1}{2}-\mu_v} \csc \pi\mu_v e^{\imath\pi(-\frac{1}{4}+\frac{\mu_v}{2})} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{7}$$

$$Z^I_{v,2}(x_v,\rho) = \frac{\sqrt{2\pi}}{2}(\imath\rho)^{\frac{1}{2}-\mu_v} e^{\imath\pi(-\frac{1}{4}+\frac{\mu_v}{2})} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{8}$$

where

$$\psi(x_1) = \lim_{x\to x_1} \phi^{-\frac{1}{2}}(x) \left\{ \int_{x_1}^x \phi(t)dt \right\}^{\frac{1}{2}-\mu_1}.$$

It follows that

$$W(\rho) = W(Z^I_{v,1}(x,\rho), Z^I_{v,2}(x,\rho)) = -2\rho[1]. \tag{9}$$

Turning point of type II:

(10)

$$Z^{II}_{v,1}(x,\rho) = \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} e^{\imath\rho \int_{x_v}^x |\phi(t)|dt}[1], & x_{v-1} < x < x_v, \\ \mid \phi(x) \mid^{-\frac{1}{2}} \csc \pi\mu_v \{e^{\imath\rho \int_{x_v}^x |\phi(t)|dt}[1] + \imath \cos \pi\mu_v e^{-\imath\rho \int_{x_v}^x |\phi(t)|dt}[1]\}, & x_v < x < x_{v+1}, \end{cases}$$

$$Z^{II}_{v,2}(x,\rho) = \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} \{e^{-\imath\rho \int_{x_v}^x |\phi(t)|dt}[1] + \imath \cos \pi\mu_v e^{\imath\rho \int_{x_v}^x |\phi(t)|dt}[1]\}, & x_{v-1} < x < x_v, \\ \mid \phi(x) \mid^{-\frac{1}{2}} \sin \pi\mu_v e^{-\imath\rho \int_{x_v}^x |\phi(t)|dt}[1], & x_v < x < x_{v+1}, \end{cases} \tag{11}$$

$$Z^{II}_{\nu,1}(x_\nu,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_\nu}\csc\pi\mu_\nu e^{\iota\pi(\frac{1}{4}-\frac{\mu_\nu}{2})}\frac{2^{\mu_\nu}\psi(x_\nu)}{\Gamma(1-\mu_\nu)}[1], \tag{12}$$

$$Z^{II}_{\nu,2}(x_\nu,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_\nu}e^{\iota\pi(\frac{1}{4}-\frac{\mu_\nu}{2})}\frac{2^{\mu_\nu}\psi(x_\nu)}{\Gamma(1-\mu_\nu)}[1], \tag{13}$$

$$W(\rho)=W(Z^{II}_{\nu,1}(x,\rho),Z^{II}_{\nu,2}(x,\rho))=-2\iota\rho[1]. \tag{14}$$

Turning point of type III:

$$Z^{III}_{\nu,1}(x,\rho)=\begin{cases}\mid\phi(x)\mid^{-\frac{1}{2}}\ e^{\iota\rho\int_{x_\nu}^{x}|\phi(t)|dt}[1], & x_{\nu-1}<x<x_\nu,\\ \frac{1}{2}\mid\phi(x)\mid^{-\frac{1}{2}}\ \csc\frac{\pi\mu_\nu}{2}e^{\rho\int_{x_\nu}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}}[1], & x_\nu<x<x_{\nu+1},\end{cases} \tag{15}$$

$$Z^{III}_{\nu,2}(x,\rho)=\begin{cases}\mid\phi(x)\mid^{-\frac{1}{2}}\ \{e^{-\iota\rho\int_{x_\nu}^{x}|\phi(t)|dt}[1]+\iota e^{\iota\rho\int_{x_\nu}^{x}|\phi(t)|dt}[1]\}, & x_{\nu-1}<x<x_\nu\\ 2\mid\phi(x)\mid^{-\frac{1}{2}}\ \sin\frac{\pi\mu_\nu}{2}e^{-\rho\int_{x_\nu}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}}[1], & x_\nu<x<x_{\nu+1}\end{cases} \tag{16}$$

$$Z^{III}_{\nu,1}(x_\nu,\rho)=\frac{\sqrt{2\pi}}{2}(\iota\rho)^{\frac{1}{2}-\mu_\nu}\csc\pi\mu_\nu\frac{2^{\mu_\nu}\psi(x_\nu)}{\Gamma(1-\mu_\nu)}[1], \tag{17}$$

$$Z^{III}_{\nu,2}(x_\nu,\rho)=\frac{\sqrt{2\pi}}{2}(\iota\rho)^{\frac{1}{2}-\mu_\nu}e^{\iota\frac{\pi\mu_\nu}{2}}\sec(\frac{\pi\mu_\nu}{2})\frac{2^{\mu_\nu}\psi(x_\nu)}{\Gamma(1-\mu_\nu)}[1], \tag{18}$$

$$W(\rho)=W(Z^{III}_{\nu,1}(x,\rho),Z^{III}_{\nu,2}(x,\rho))=-2\iota\rho[1]. \tag{19}$$

Turning point of type IV:

(20)

$$Z^{IV}_{\nu,1}(x,\rho)=\begin{cases}\mid\phi(x)\mid^{-\frac{1}{2}}\ e^{\rho\int_{x_\nu}^{x}|\phi(t)|dt}[1], & x_{\nu-1}<x<x_\nu,\\ \frac{1}{2}csc\frac{\pi\mu_\nu}{2}\mid\phi(x)\mid^{-\frac{1}{2}}\ \{e^{\iota\rho\int_{x_\nu}^{x}|\phi(t)|dt-\iota\frac{\pi}{4}}[1]+e^{-\iota\rho\int_{x_\nu}^{x}|\phi(t)|dt+\iota\frac{\pi}{4}}[1]\}, & x_\nu<x<x_{\nu+1},\end{cases}$$

$$Z^{IV}_{\nu,2}(x,\rho)=\begin{cases}\mid\phi(x)\mid^{-\frac{1}{2}}\ e^{-\rho\int_{x_\nu}^{x}|\phi(t)|dt}[1], & x_{\nu-1}<x<x_\nu,\\ 2\sin\frac{\pi\mu_\nu}{2}\mid\phi(x)\mid^{-\frac{1}{2}}\ \{e^{-\iota\rho\int_{x_\nu}^{x}|\phi(t)|dt-\iota\frac{\pi}{4}}[1]\}, & x_\nu<x<x_{\nu+1},\end{cases} \tag{21}$$

$$Z^{IV}_{\nu,1}(x_\nu,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_\nu}\csc\pi\mu_\nu\frac{2^{\mu_\nu}\psi(x_\nu)}{\Gamma(1-\mu_\nu)}[1], \tag{22}$$

$$Z^{IV}_{\nu,2}(x_\nu,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_\nu}e^{-\iota\frac{\pi\mu_\nu}{2}}\sec(\frac{\pi\mu_\nu}{2})\frac{2^{\mu_\nu}\psi(x_\nu)}{\Gamma(1-\mu_\nu)}[1], \tag{23}$$

$$W(\rho) = W(Z^{IV}_{\nu,1}(x,\rho), Z^{IV}_{\nu,2}(x,\rho)) = -2\rho[1]. \tag{24}$$

3. Results

Let $y(x,\rho)$ be the solution of (1) corresponding to the initial conditions

$$y(0,\lambda) = 0, \qquad y'(0,\lambda) = 1.$$

Since, x_1 is a turning point of type I, using the FSS, $\{Z^I_{1,1}(x,\rho), Z^I_{1,2}(x,\rho)\}$, we obtain

$$y(x,\rho) = \frac{1}{-2\rho}(Z^I_{1,1}(0,\rho)Z^I_{1,2}(x,\rho) - Z^I_{1,1}(x,\rho)Z^I_{1,2}(0,\rho)), \qquad x \in (0,x_1). \tag{25}$$

Then from (5) it follows that

$$y(x,\rho) = \frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\{e^{\rho\int_0^x|\phi(t)dt}[1] - e^{-\rho\int_0^x|\phi(t)dt}[1]\}, \qquad x \in (0,x_1), \tag{26}$$

or

$$y(x,\rho) = \frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}e^{\rho\int_0^x|\phi(t)dt}E_k(x,\rho), \qquad x \in (0,x_1). \tag{27}$$

If the asymptotic solution in the interval$(x_\nu, x_{\nu+1})$ is of the form $y_{(x_\nu,x_{\nu+1})} = A_{(x_\nu,x_{\nu+1})}E_k(x,\rho)$, where

$$E_k(x,\rho) = [1] + \sum_{n=1}^{\nu(x)} e^{\rho\alpha_k\beta_{kn}(x)}[b_{kn}(x)],$$

and $\alpha_{-2} = \alpha_1 = -1, \alpha_0 = -\alpha_{-1} = \imath, \beta_{k\nu(x)}(x) \neq 0, 0 < \delta \leq \beta_{k1}(x) < \beta_{k2}(x) < \ldots < \beta_{k\nu(x)}(x) \leq 2max\{R_+(1), R_-(1)\}$, the integer-valued functions ν and b_{kn} are constant in every interval $[0, x_1-\varepsilon]$ and $[x_1+\varepsilon, x_2-\varepsilon]$ for ε sufficiently small and

$$R_+(x) = \int_0^x \sqrt{max\{0,\phi^2(t)\}}dt, \qquad R_-(x) = \int_0^x \sqrt{max\{0,-\phi^2(t)\}}dt. \tag{28}$$

It is proved in [16] that

i: If x_ν is a turning point of type I, then

$$A_{(x_\nu,x_{\nu+1})}(x,\rho) = A_{(x_{\nu-1},x_\nu)}(x_\nu,\rho)\csc\pi\mu_\nu e^{\rho\int_{x_\nu}^x|\phi(t)|dt}.$$

ii: If x_ν is a turning point of type II, then

$$A_{(x_\nu,x_{\nu+1})}(x,\rho) = A_{(x_{\nu-1},x_\nu)}(x_\nu,\rho)\csc\pi\mu_\nu e^{\imath\rho\int_{x_\nu}^x|\phi(t)|dt}.$$

iii: If x_v is a turning point of type III, then

$$A_{(x_v,x_{v+1})}(x,\rho) = \frac{1}{2}A_{(x_{v-1},x_v)}(x_v,\rho)\csc\frac{\pi\mu_v}{2}e^{\rho\int_{x_v}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}}.$$

iv: If x_v is a turning point of type IV, then

$$A_{(x_v,x_{v+1})}(x,\rho) = \frac{1}{2}A_{(x_{v-1},x_v)}(x_v,\rho)\csc\frac{\pi\mu_v}{2}e^{\iota\rho\int_{x_v}^{x}|\phi(t)|dt-\frac{\iota\pi}{4}}.$$

Now, one can use this recursive relations to obtain the asymptotic solution of the problem in the remaining intervals. Therefore, Since $x_1,\ldots,x_s$ are I type of turning points we have

(29)
$$y(x,\rho) = \frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\csc\pi\mu_1\csc\pi\mu_2\ldots\csc\pi\mu_t e^{\rho\int_0^x|\phi(t)dt}E_k(x,\rho),\ x\in(x_t,x_{t+1}), 1\le t\le s.$$

By Halvorsen's result, $y(x,\lambda)$ is an entire function of order $\frac{1}{2}$ for each fixed $x\in(0,1)$; therefore, by using Hadamard's theorem [15], $y(x,\lambda)$ can be represented in the form

$$y(x,\lambda) = s(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{w_n(x)}),$$

where $s(x)$ is a function independent of λ but may depend on x. The sequence of $\{w_n(x)\}_1^\infty$ is a zero set of $y(x,\lambda)$ for each x, so that $y(x,w_n(x)) = 0$, which corresponds to eigenvalues of the boundary value problem $L(\phi^2(x),q(x),x)$ involves the second-order differential equation (1) with the boundary conditions

$$y(0,\lambda) = 0,\ y(0,\lambda) = 1,\ y(x,\lambda) = 0.$$

We see that each $w_n(x), n = 1,2,\ldots$ for each fixed x appears in the denominator and must be nonzero. By adding the extra condition $q(x)\ge 0$, we will have $w_n(x)\ne 0$ for any x by Sturm's comparison theorem.

The boundary value problem $L(\phi^2(x),q(x),x)$ for $x\in[0,x_{s+1})$, has an infinite number of negative eigenvalues, say $\{\lambda_n^-(x)\}$(note that in this case $w_n(x)=\lambda_n^-(x)$). From (29) the asymptotic representation of each $\lambda_n^-(x)$ is of the form

$$\sqrt{-\lambda_n^-(x)} = \frac{n\pi}{\int_0^x|\phi(t)|dt}+O(\frac{1}{n}). \tag{30}$$

By Hadamard's theorem the solution $y(x,\lambda)$, for fixed x, can be represented in the

$$y(x,\lambda) = h(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{\lambda_n^-(x)}),$$

where $h(x)$ is a function independent of λ but may depend on x and the infinite number of negative eigenvalues, $\{\lambda_n^-(x)\}_{n=1}^{\infty}$, form the zero set of $y(x,\lambda)$ for each x. We rewrite the infinite product as

$$y(x,\lambda) = h(x)\prod_{n\geq 1}(1-\frac{\lambda}{\lambda_n^-(x)}) = h_1(x)\prod_{n\geq 1}\frac{\lambda-\lambda_n^-(x)}{z_n^2},$$

with

$$h_1(x) := h(x)\prod_{n\geq 1}\frac{-z_n^2}{\lambda_n^-(x)},$$

where $z_n = \frac{n\pi}{R_-(x)}$. We can easily deduce the following results:

Theorem 3.1. *Let $y(x,\lambda)$ be the solution of (1) satisfying the initial conditions $y(0,\lambda) = 0, y'(0,\lambda) = 1$. Then for $0 \leq x < x_1$*

$$y(x,\lambda) = |\phi(x)\phi(0)|^{-\frac{1}{2}}R_-(x)\prod_{m\geq 1}\frac{\lambda-\lambda_m(x)}{z_m^2} = B(x),$$

where $z_m = \frac{m\pi}{R_-(x)}$, $R_-(x) = \int_0^x \sqrt{max\{0,-\phi^2(t)\}}dt$ and the sequence $\lambda_m^-(x), m \geq 1$, represents the sequence of negative eigenvalues of the boundary value problem L on $[0,x]$. Also for $x_t < x < x_{t+1},\ 1 \leq t \leq s$ we have

$$y(x,\lambda) = \csc\pi\mu_1\csc\pi\mu_2...\csc\pi\mu_t B(x).$$

Similarly for each $x_v < x < x_{v+1},\ v = s+1,...m,\ x_{m+1} = 1$, since x_{s+1} is a turning point of type IV and $x_{s+2},...,x_m$ are of type II, from ([16] Theorem 3.1) the asymptotic form of the solution of initial value problem is of the form

$$y(x,\rho) = \frac{1}{4\rho}|\phi(x)\phi(0)|^{-\frac{1}{2}}\csc\pi\mu_1\csc\pi\mu_2...\csc\pi\mu_s\csc\frac{\pi\mu_{s+1}}{2}\csc\pi\mu_{s+2}...\csc\pi\mu_v \\ e^{\rho\int_0^{x_{s+1}}|\phi(t)|dt+\iota\rho\int_{x_{s+1}}^{x}|\phi(t)|dt-\frac{\iota\pi}{4}}E_k(x,\rho). \tag{31}$$

The spectrum$\{\lambda_n\}$ of boundary value problem L for $x_v < x < x_{v+1}, v = s+1,...,m$ consist of two sequences of negative and positive eigenvalues: $\{\lambda_n(x)\} = \{\lambda_n^+(x)\}\cup\{\lambda_n^-(x)\}, n \in N$, such that

$$(32)\qquad \begin{aligned} \sqrt{\lambda_n^+(x)} &= \frac{n\pi-\frac{\pi}{4}}{\int_{x_{s+1}}^{x}|\phi(t)|dt}+O(\tfrac{1}{n}),\\ \sqrt{-\lambda_n^-(x)} &= -\frac{n\pi-\frac{\pi}{4}}{\int_0^{x_{s+1}}|\phi(t)|dt}+O(\tfrac{1}{n}). \end{aligned}$$

By Hadamard's Theorem, the solution for $x_v < x < x_{v+1}$ is of the form

$$y(x,\lambda) = g(x)\prod_{n\geq 1}(1-\frac{\lambda}{\lambda_n^-(x)})(1-\frac{\lambda}{\lambda_n^+(x)}).$$

Since, the infinite products $\prod_{n\geq 1}\frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)}$ and $\prod_{n\geq 1}\frac{-\tilde{j}_n^2}{R_-^2(x)\lambda_n^-(x)}$ are absolutely convergent for each $x\in(x_v,x_{v+1})$. Therefore we may write

$$(33)\qquad y(x,\lambda) = g_v(x)\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(x))R_-^2(x_{s+1})}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(x)-\lambda)R_+^2(x)}{\tilde{j}_n^2},$$

with

$$g_v(x) = g(x)\prod_{n\geq 1}\frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)}\prod_{n\geq 1}\frac{-\tilde{j}_n^2}{R_-^2(x_{s+1})\lambda_n^-(x)}.$$

Theorem 3.2. *For $x_{s+1} < x < x_{s+2}$*

$$(34)\qquad \begin{aligned} y(x,\lambda) = \tfrac{\pi}{8}|\varphi(x)\varphi(0)|^{\frac{-1}{2}}(R_-(x_s)R_+(x))^{\frac{1}{2}}\csc\pi\mu_1\csc\pi\mu_2\ldots\csc\pi\mu_s\csc\frac{\pi\mu_{s+1}}{2}\\ \prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(x))R_-^2(x_s)}{\tilde{j}_n^2}\prod_{n\geq 1}\frac{(\lambda_n^+(x)-\lambda)R_+^2(x)}{\tilde{j}_n^2} = C(x), \end{aligned}$$

where $R_+(x) = \int_0^x\sqrt{max\{0,\phi^2(t)\}}dt$, $R_-(x) = \int_0^x\sqrt{max\{0,-\phi^2(t)\}}dt$ and the sequence $\{\lambda_n^+(x)\}$ represents the sequence of positive eigenvalues and $\{\lambda_n^-(x)\}$ the sequence of negative eigenvalues of the boundary value problem L on [0,x]. Also for $x_v \leq x \leq x_{v+1},\ v = s+2,\ldots,m,\ x_{m+1} = 1$ we have

$$y(x,\lambda) = \csc\pi\mu_{s+2}\ldots\csc\pi\mu_v C(x).$$

Conflict of Interests

The author declares that there is no conflict of interests.

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