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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF MULTIVALENT FUNCTIONS OF COMPLEX ORDER DEFINED BY MULTIPLIER OPERATOR

AMJAD S. BARHAM*

Department of Mathematics and Computer Science, Palestine Polytechnic University, Hebron, Palestine

Abstract. In this paper, we first define the multiplier operator $\mathcal{J}^{m,\delta}_{c,p,\lambda}$ in terms of Komatu integral operator. Then we define new classes of $p-$valent starlike and convex functions with complex order. The main object is to obtain coefficient inequalities for functions belonging to the newly defined classes. Also we get coefficient inequalities for functions in certain subclass satisfying a nonhomogeneous Cauchy-Euler differential equation. Several particular results (known or new) of the main theorems are mentioned.

Keywords: Multivalent functions, Komatu integral operator, nonhomogeneous Cauchy-Euler differential equation, coefficient inequalities.

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1. Introduction

Let $\mathcal{A}(p,n)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \ (p, n \in \mathbb{N}) \tag{1}$$

*Corresponding author

E-mail address: amjad@ppu.edu (A. S. Barham)

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that are analytic and $p-$valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}, \mid z \mid < 1\}$

The generalized Komatu integral operatot $\mathcal{K}^{\delta}_{c,p} : \mathcal{A}(p,n) \to \mathcal{A}(p,n)$ is defined for $\delta > 0$ and $c > -p$ as

$$\mathcal{K}^{\delta}_{c,p} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta) z^{c}} \int_{0}^{z} t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t)dt, \tag{2}$$

and $\mathcal{K}^{0}_{c,p} f(z) = f(z)$.

For $f(z) \in \mathcal{A}(p,n)$, it can be easily verified that

$$\mathcal{K}^{\delta}_{c,p} f(z) = z^{p} + \sum_{k=p+n}^{\infty} \left(\frac{c+p}{c+k}\right)^{\delta} a_k z^k \tag{3}$$

Now, in terms of $\mathcal{K}^{\delta}_{c,p}$, we introduce the linear multiplier operator $\mathcal{J}^{m,\delta}_{c,p,\lambda} : \mathcal{A}(p,n) \to \mathcal{A}(p,n)$ as follows:

$$\begin{aligned}
\mathcal{J}^{0,0}_{c,p,\lambda} f(z) &= f(z) \\
\mathcal{J}^{1,\delta}_{c,p,\lambda} f(z) &= (1-\lambda)\mathcal{K}^{\delta}_{c,p} f(z) + \frac{\lambda z}{p}(\mathcal{K}^{\delta}_{c,p} f(z))' = \mathcal{J}^{\delta}_{c,p,\lambda} f(z) \\
\mathcal{J}^{2,\delta}_{c,p,\lambda} f(z) &= \mathcal{J}^{\delta}_{c,p,\lambda}(\mathcal{J}^{1,\delta}_{c,p,\lambda} f(z)) \\
&\cdot \\
&\cdot \\
&\cdot \\
\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z) &= \mathcal{J}^{\delta}_{c,p,\lambda}(\mathcal{J}^{m-1,\delta}_{c,p,\lambda} f(z))
\end{aligned} \tag{4}$$

for $\delta > 0$, $c > -p$, $\lambda \geq 0$ and $m \in \mathbb{N}$.

If $f \in \mathcal{A}(p,n)$ is given by (1), then making use of (3) and (4) we conclude that

$$\mathcal{J}^{m,\delta}_{c,p,\lambda} f(z) = z^{p} + \sum_{k=p+n}^{\infty} B_{k,m}(c,p,\lambda,\delta) a_k z^k, \tag{5}$$

where

$$B_{k,m}(c,p,\lambda,\delta) = \left[\left(\frac{c+p}{c+k}\right)^{\delta} (1 + \frac{\lambda}{p}(k-p))\right]^{m} \tag{6}$$

Now if $f(z), g(z) \in \mathcal{A}(p,n)$, where $g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k$, then the Hadamard product (or convolution) is defined as

$$f(z) * g(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k \tag{7}$$

It can be easily verified that

(i) $z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)\right)' = (c+p)\mathcal{J}_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z) - c\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)$

(ii) $\mathcal{J}_{c,p,\lambda}^{m+1,\delta} f(z) = \mathcal{J}_{c,p,\lambda}^{m,\delta} f(z) * \mathcal{J}_{c,p,\lambda}^{\delta} f(z)$,

(iii)$\mathcal{J}_{c,p,\lambda}^{m,\delta+1} f(z) = \mathcal{J}_{c,p,\lambda}^{m,\delta} f(z) * \mathcal{K}_{c,p}^{m} f(z)$

and

(iv) $\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z) = \underbrace{[\varphi(z) * \ldots * \varphi(z)]}_{m-times} * \mathcal{K}_{c,p}^{\delta m} f(z)$

where

$$\varphi(z) = z^p \left[1 + \left(1 + \frac{\lambda}{p}(n-1)\right)\frac{z^n}{1-z} + \frac{\lambda}{p}\frac{z^n}{(1-z)^2}\right]$$

Remark 1:

(i) $\mathcal{J}_{c,p,0}^{1,\delta} \equiv \mathcal{K}_{c,p}^{\delta}$ which is the generalized Komatu integral operator [8]

(ii) $\mathcal{J}_{c,1,0}^{1,\delta} \equiv \mathcal{P}_{c}^{\delta}$ which is the integral operator studied by Komatu [9] and Raina and Bapna [11]

(iii) $\mathcal{J}_{1,p,0}^{1,\delta} \equiv \mathcal{I}_{p}^{\delta}$ which is the integral operator studied by Shams *et al.* [13] and Ebadian *et al.* [6]

(iv) $\mathcal{J}_{c,1,0}^{1,1} \equiv \mathcal{L}_{c}$ which is the Bernardi-Libra-Livingston integral operator [3]

(v) $\mathcal{J}_{1,1,0}^{1,\delta} \equiv \mathcal{I}^{\delta}$ which is the integral operator studied by Ebadian and Najafzadeh [5].

(vi) $\mathcal{J}_{c,1,\lambda}^{m,0} \equiv \mathcal{D}_{\lambda}^{m}$ which is the generalized Sălăgean operator studied by Al-Oboudi [1].

(vii) $\mathcal{J}_{c,1,0}^{m,0} \equiv \mathcal{D}^{m}$ which is the Sălăgean operator [12].

Now, we define the class $\mathcal{S}_{c,\lambda}^{m,\delta}(\beta, b, p, n)$ for functions $f \in \mathcal{A}(p,n)$ satisfying

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{1}{p}\frac{z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)\right)'}{\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)} - 1\right)\right\} > \beta \tag{8}$$

and the class $\mathcal{V}_{c,\lambda}^{m,\delta}(\beta,b,p,n)$ for functions $f\in\mathcal{A}(p,n)$ satisfying

$$\operatorname{Re}\left\{1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right)''}{\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right)'}\right)\right\}>\beta \tag{9}$$

where $z\in\mathcal{U}$, $b\in\mathbb{C}\backslash\{0\}$, $0\leq\beta<1$ and $\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)$ is the linear multiplier operator defined by (4).

Note that $f\in\mathcal{V}_{c,\lambda}^{m,\delta}(\beta,b,p,n)$ if and only if $\frac{1}{p}zf'(z)\in\mathcal{S}_{c,\lambda}^{m,\delta}(\beta,b,p,n)$.

Now, let us introduce the class $\mathcal{SV}_{c,\lambda}^{m,\delta}(\beta,b,p,n,\mu)$ of functions $f\in\mathcal{A}(p,n)$ satisfying the inequality

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{1}{p}\frac{z\left(\frac{1}{1+\mu(p-1)}\left[\mu z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right)'+(1-\mu)\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right]\right)'}{\frac{1}{1+\mu(p-1)}\left[\mu z\left(\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right)'+(1-\mu)\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)\right]}-1\right)\right\}>\beta \tag{10}$$

where $z\in\mathcal{U}$ and $\mathcal{J}_{c,p,\lambda}^{m,\delta}f(z)$ is the linear multiplier operator defined by (4) and where (and throughout the paper unless otherwise stated), the parameters $m,n,p,b,c,\delta,\lambda,\beta$ and μ are constrained as follows:

$m,n,p\in\mathbb{N},\ \ c>-p,\ \ b\in\mathbb{C}\backslash\{0\},\ \ \delta>0,\ \ \lambda\geq 0,\ \ 0\leq\beta<1$ and $0\leq\mu<1$.

Remark 2:

(i) $\mathcal{SV}_{c,\lambda}^{m,\delta}(\beta,b,p,n,0)=\mathcal{S}_{c,\lambda}^{m,\delta}(\beta,b,p,n)$

(ii) $\mathcal{SV}_{c,\lambda}^{m,\delta}(\beta,b,p,n,1)=\mathcal{V}_{c,\lambda}^{m,\delta}(\beta,b,p,n)$

(iii) $\mathcal{SV}_{c,\lambda}^{0,0}(\beta,b,1,1,\mu)=\mathcal{SV}_{c,0}^{1,0}(\beta,b,1,1,\mu)\equiv\mathcal{SC}(b,\mu,\beta)$ which is the class defined and studied by Altintaş *et al.* [2].

(iv) $\mathcal{SV}_{c,\lambda}^{0,0}(0,b,1,1,0)\equiv\mathcal{S}(b)$ and $\mathcal{SV}_{c,\lambda}^{0,0}(0,b,1,1,1)\equiv\mathcal{C}(b)$ which are the classes studied by Nasr and Aouf [10].

Now we define the class $\mathcal{E}_{c,\lambda}^{m,\delta}(\beta,b,p,n,\mu,\eta)$ for functions $f\in\mathcal{A}(p,n)$ satisfying the non-homogeneous Cauchy-Euler differential equation

$$z^2\frac{d^2\omega}{dz^2}+2(1+\eta)z\frac{d\omega}{dz}+\eta(1+\eta)\omega=(p+\eta)(p+1+\eta)g(z) \tag{11}$$

where $\omega = f(z)$, $g(z) \in \mathcal{SV}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu)$, $\eta > -p$ $(\eta \in \mathbb{R})$ and the other parameters are constrained as above.

The purpose of the present investigation is to obtain coefficient bounds for the classes $\mathcal{SV}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu)$ and $\mathcal{E}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu, \eta)$, from which we can also get coefficient bounds for the other mentioned classes.

2. Coefficient Inequalities for the Class $\mathcal{SV}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu)$

Theorem 2.1. *Let the function $f \in \mathcal{A}(p, n)$ be defined by (1). If the function f belongs to the class $\mathcal{SV}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu)$, then*

$$|a_{p+n}| \leq \frac{[1+\mu(p-1)]\, 2p^{m+1}\, |b|\, (1-\beta)(c+p+n)^{\delta m}}{[1+\mu(p+n-1)]\, n(c+p)^{\delta m}(p+\lambda n)^m}, \tag{12}$$

$$\begin{aligned} |a_k| \ \leq\ & \frac{[1+\mu(p-1)]\, 2p^{m+1}\, |b|\, (1-\beta)(c+k)^{\delta m}(n-1)!}{[1+\mu(k-1)]\, (c+p)^{\delta m}\, [p+\lambda(k-p)]^m\, (k-p)!} \\ & \times \prod_{j=0}^{k-p-n-1} [(n+j) + 2p\, |b|\, (1-\beta)\ ], \end{aligned} \tag{13}$$

for $k \geq p+n+1$

Proof. Define the function $F(z)$ by

$$F(z) = \frac{1}{1+\mu(p-1)} \left[\mu z \left(\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)\right)' + (1-\mu)\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)\right], \tag{14}$$

then for $f \in \mathcal{A}(p, n)$, we have

$$F(z) = z^p + \sum_{k=p+n}^{\infty} A_k z^k \tag{15}$$

where

$$\begin{aligned} A_k \ &=\ B_{k,m}(c, p, \lambda, \delta) \left[\frac{1+\mu(k-1)}{1+\mu(p-1)}\right] a_k \\ &=\ \frac{[1+\mu(k-1)]\, (c+p)^{\delta m}\, [p+\lambda(k-p)]^m}{[1+\mu(p-1)]\, p^m (c+k)^{\delta m}} a_k \end{aligned} \tag{16}$$

Thus, from (10) and (14), we get

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{1}{p}\frac{zF'(z)}{F(z)} - 1\right)\right\} > \beta, \qquad (z \in \mathcal{U})$$

Define the function $q(z)$ by

$$q(z)=\frac{1+\frac{1}{b}\left(\frac{1}{p}\frac{zF'(z)}{F(z)}-1\right)-\beta}{1-\beta}, \qquad (z\in\mathcal{U})$$

Hence, $q(z)$ is an analytic function in $\mathcal{U}$ with $q(0)=1$ and $\mathrm{Re}(q(z))>0$. Let $q(z)=1+q_1z+q_2z^2+\ldots$, then we get

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zF'(z)}{F(z)}-1\right)=(1-\beta)\left(q_1z+q_2z^2++\ldots\right) \tag{17}$$

which is equivalent to

$$zF'(z)-pF(z)=pb(1-\beta)F(z)$$

Expressing $F(z)$ in its series form as in (15), and the equating coefficients of $z^k(k\geq p+n)$, thus (17) implies

$$(k-p)A_k=pb(1-\beta)\left(q_{k-p}+\sum_{e=p+n}^{k-1}A_eq_{k-e}\right)$$

Setting $k=p+n+r \quad (r\in\mathbb{N}_0)$, then

$$(n+r)A_{p+n+r}=pb(1-\beta)\left(q_{n+r}+\sum_{e=p+n}^{p+n+r-1}A_eq_{p+n+r-e}\right)$$

Since $q(z)$ is a Carathéodory function, then $|q_k|\leq 2 \quad (k\in\mathbb{N})$ [4], then we obtain

$$|A_{p+n+r}|\leq\frac{2p\,|b|\,(1-\beta)}{n+r}\left(1+|A_{p+n}|+\ldots+|A_{p+n+r-1}|\right)$$

which for $r=0,1,2$ implies

$$|A_{p+n}|\leq\frac{2p\,|b|\,(1-\beta)}{n},$$

$$|A_{p+n+1}|\leq\frac{2p\,|b|\,(1-\beta)}{n+1}\left(1+|A_{p+n}|\right)\leq\frac{2p\,|b|\,(1-\beta)\,(n+2p\,|b|\,(1-\beta))}{n(n+1)},$$

and

$$\begin{aligned}|A_{p+n+2}| &\leq \frac{2p\,|b|\,(1-\beta)}{n+1}\left(1+|A_{p+n}|+|A_{p+n+1}|\right)\\ &\leq \frac{2p\,|b|\,(1-\beta)\,(n+2p\,|b|\,(1-\beta))\,((n+1)+2p\,|b|\,(1-\beta))}{n(n+1)(n+2)}\end{aligned}$$

respectively. Making use of mathematical induction yields

$$\begin{aligned} |A_{p+n+r}| &\leq \frac{2p\,|b|\,(1-\beta)}{n(n+1)\cdots(n+r)} \prod_{j=0}^{r-1} [(n+j)+2p\,|b|\,(1-\beta)] \\ &= \frac{2p\,|b|\,(1-\beta)(n-1)!}{(n+r)!} \prod_{j=0}^{r-1} [(n+j)+2p\,|b|\,(1-\beta)] \end{aligned}$$

for $r \geq 1$. So, we have

$$|A_{p+n}| \leq \frac{2p\,|b|\,(1-\beta)}{n}, \tag{18}$$

and

$$|A_k| \leq \frac{2p\,|b|\,(1-\beta)(n-1)!}{(k-p)!} \prod_{j=0}^{k-p-n-1} [(n+j)+2p\,|b|\,(1-\beta)] \tag{19}$$

for $k \geq n+p+1$. Hence making use of (16) , we readily get the inequalities (12) and (13), and the proof is complete.

By choosing suitable values of the parameters m, δ, c, λ, β, b, p, n and μ in Theorem 2.1, we deduce particular results. Some of these special cases are mentioned in the corollaries below.

Setting $\mu = 0$ and $\mu = 1$, respectively, in Theorem 2.1, we get

Corollary 2.2. *If a function $f(z) \in \mathcal{A}(p,n)$ is in the class $\mathcal{S}_{c,\lambda}^{m,\delta}(\beta, b, p, n)$, then*

$$|a_{p+n}| \leq \frac{2p^{m+1}\,|b|\,(1-\beta)(c+p+n)^{\delta m}}{n(c+p)^{\delta m}(p+\lambda n)^m},$$

$$|a_k| \leq \frac{2p^{m+1}\,|b|\,(1-\beta)(c+k)^{\delta m}(n-1)!}{(c+p)^{\delta m}\,[p+\lambda(k-p)]^m\,(k-p)!} \prod_{j=0}^{k-p-n-1} [(n+j)+2p\,|b|\,(1-\beta)\ ],$$

for $k \geq p+n+1$.

Corollary 2.3. *If a function $f(z) \in \mathcal{A}(p,n)$ is in the class $\mathcal{V}_{c,\lambda}^{m,\delta}(\beta, b, p, n)$, then*

$$|a_{p+n}| \leq \frac{2p^{m+2}\,|b|\,(1-\beta)(c+p+n)^{\delta m}}{(p+n)(c+p)^{\delta m}(p+\lambda n)^m},$$

$$|a_k| \leq \frac{2p^{m+2}\,|b|\,(1-\beta)(c+k)^{\delta m}(n-1)!}{k(c+p)^{\delta m}\,[p+\lambda(k-p)]^m\,(k-p)!} \prod_{j=0}^{k-p-n-1} [(n+j)+2p\,|b|\,(1-\beta)\ ],$$

for $k \geq p+n+1$.

For $m=1$ and $\lambda=0$, then $\mathcal{J}_{c,p,0}^{1,\delta} \equiv \mathcal{K}_{c,p}^{\delta}$ and $\mathcal{SV}_{c,0}^{1,\delta}(\beta,b,p,n,\mu) \equiv \mathcal{SV}_{c}^{\delta}(\beta,b,p,n,\mu)$ which is the class of all functions $f(z) \in \mathcal{A}(p,n)$ satisfying the inequality defined by replacing $\mathcal{K}_{c,p}^{\delta} f(z)$ instead of $\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)$ in (10).

Corollary 2.4. *If a function* $f(z) \in \mathcal{A}(p,n)$ *is in the class* $\mathcal{SV}_{c}^{\delta}(\beta,b,p,n,\mu)$, *then*

$$|a_{p+n}| \leq \frac{[1+\mu(p-1)]\,2p\,|b|\,(1-\beta)(c+p+n)^{\delta}}{[1+\mu(p+n-1)]\,n(c+p)^{\delta}},$$

$$|a_k| \leq \frac{[1+\mu(p-1)]\,2p\,|b|\,(1-\beta)(c+k)^{\delta}(n-1)!}{[1+\mu(k-1)]\,(c+p)^{\delta}(k-p)!} \prod_{j=0}^{k-p-n-1} [(n+j)+2p\,|b|\,(1-\beta)\ ]\,,$$

for $k \geq p+n+1$

Also, for $m=1$ and $\lambda=1$, then $\mathcal{J}_{c,p,1}^{1,\delta} f(z) \equiv \frac{z}{p}\left(\mathcal{K}_{c,p}^{\delta} f(z)\right)'$ and $\mathcal{SV}_{c,1}^{1,\delta}(\beta,b,p,n,\mu) \equiv \mathcal{SW}_{c}^{\delta}(\beta,b,p,n,\mu)$ which is the class of all functions $f(z) \in \mathcal{A}(p,n)$ satisfying the inequality defined by replacing $\frac{z}{p}\left(\mathcal{K}_{c,p}^{\delta} f(z)\right)'$ instead of $\mathcal{J}_{c,p,\lambda}^{m,\delta} f(z)$ in (10).

Corollary 2.5. *If a function* $f(z) \in \mathcal{A}(p,n)$ *is in the class* $\mathcal{SW}_{c}^{\delta}(\beta,b,p,n,\mu)$, *then*

$$|a_{p+n}| \leq \frac{[1+\mu(p-1)]\,2p^2\,|b|\,(1-\beta)(c+p+n)^{\delta}}{[1+\mu(p+n-1)]\,n(c+p)^{\delta}(p+n)},$$

$$|a_k| \leq \frac{[1+\mu(p-1)]\,2p^2\,|b|\,(1-\beta)(c+k)^{\delta}(n-1)!}{[1+\mu(k-1)]\,(c+p)^{\delta}k(k-p)!} \prod_{j=0}^{k-p-n-1} [(n+j)+2p\,|b|\,(1-\beta)\ ]\,,$$

for $k \geq p+n+1$.

Also if $\mathcal{A} \equiv \mathcal{A}(1,1)$ which is the class of all functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ that are analytic and univalent in $\mathcal{U}$. So, we have the following known coefficient bounds.

Corollary 2.6. (cf, e.g. Altintaş *et al.* [2]) *If a function* $f(z) \in \mathcal{A}$ *is in the class* $\mathcal{SC}(b,\mu,\beta)$, *then*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2} [j+2p\,|b|\,(1-\beta)\ ]}{[1+\mu(k-1)]\,(k-1)!},$$

for $k \geq 2$.

Corollary 2.7. (cf, e.g. Nasr and Aouf [10]) *If a function* $f(z) \in \mathcal{A}$ *is in the class* $\mathcal{S}(b)$, *then*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2\,|b|)}{(k-1)!},$$

for $k \geq 2$.

Corollary 2.8. (cf, e.g. Nasr and Aouf [10]) *If a function* $f(z) \in \mathcal{A}$ *is in the class* $\mathcal{C}(b)$, *then*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2\,|b|)}{k!},$$

for $k \geq 2$.

3. Coefficient Inequalities for the Class $\mathcal{E}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu, \eta)$

Theorem 3.1. *Let the function* $f \in \mathcal{A}(p,n)$ *be defined by (1). If the function* f *belongs to the class* $\mathcal{E}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu, \eta)$, *then*

$$|a_{p+n}| \leq \frac{(p+\eta)(p+\eta+1)\,[1+\mu(p-1)]\,2p^{m+1}\,|b|\,(1-\beta)(c+p+n)^{\delta m}}{(p+n+\eta)(p+n+\eta+1)\,[1+\mu(p+n-1)]\,n(c+p)^{\delta m}(p+\lambda n)^{m}}, \tag{20}$$

$$\begin{aligned} |a_k| \;\; \leq \;\; & \frac{(p+\eta)(p+\eta+1)\,[1+\mu(p-1)]\,2p^{m+1}\,|b|\,(1-\beta)(c+k)^{\delta m}(n-1)!}{(k+\eta)(k+\eta+1)\,[1+\mu(k-1)]\,(c+p)^{\delta m}\,[p+\lambda(k-p)]^{m}\,(k-p)!} \\ & \times \prod_{j=0}^{k-p-n-1} \left[(n+j) + 2p\,|b|\,(1-\beta)\,\right], \end{aligned} \tag{21}$$

for $k \geq p+n+1$ *and* $\eta > -p \;\; (\eta \in \mathbb{R})$.

Proof. $f(z) \in A(p,n)$ given by (1) and $g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \in \mathcal{SV}_{c,\lambda}^{m,\delta}(\beta, b, p, n, \mu)$, then by the Cauchy-Euler differential equation (11), we get

$$a_k = \frac{(p+\eta)(p+\eta+1)}{(k+\eta)(k+\eta+1)} b_k$$

where for$k \geq p+n+1$ and $\eta > -p \quad (\eta \in \mathbb{R})$. Since b_k satisfies the inequalities (12) and (13) in Theorem (2.1), then we get readily the inequalities (20) and (21) which completes the proof of the theorem.

Corollary 3.2. *Let the function $f \in \mathcal{A}(p,n)$ satisfy the Cauchy-Euler differential equation given by (11) such that $g(z) \in \mathcal{S}_{c,\lambda}^{m,\delta}(\beta, b, p, n)$, then*

$$|a_{p+n}| \leq \frac{(p+\eta)(p+\eta+1)2p^{m+1}\,|b|\,(1-\beta)(c+p+n)^{\delta m}}{(p+n+\eta)(p+n+\eta+1)n(c+p)^{\delta m}(p+\lambda n)^{m}},$$

$$\begin{aligned} |a_k| \quad &\leq \quad \frac{(p+\eta)(p+\eta+1)2p^{m+1}\,|b|\,(1-\beta)(c+k)^{\delta m}(n-1)!}{(k+\eta)(k+\eta+1)(c+p)^{\delta m}\,[p+\lambda(k-p)]^{m}\,(k-p)!} \\ &\times \prod_{j=0}^{k-p-n-1}\left[(n+j)+2p\,|b|\,(1-\beta)\ \right], \end{aligned}$$

for $k \geq p+n+1$ and $\eta > -p \quad (\eta \in \mathbb{R})$.

Corollary 3.3. *Let the function $f \in \mathcal{A}$ satisfy the Cauchy-Euler differential equation given by (11) such that $g(z) \in \mathcal{V}_{c,\lambda}^{m,\delta}(\beta, b, p, n)$, then*

$$|a_{p+n}| \leq \frac{(p+\eta)(p+\eta+1)2p^{m+2}\,|b|\,(1-\beta)(c+p+n)^{\delta m}}{(p+n+\eta)(p+n+\eta+1)(p+n)n(c+p)^{\delta m}(p+\lambda n)^{m}},$$

$$\begin{aligned} |a_k| \quad &\leq \quad \frac{(p+\eta)(p+\eta+1)2p^{m+2}\,|b|\,(1-\beta)(c+k)^{\delta m}(n-1)!}{(k+\eta)(k+\eta+1)k(c+p)^{\delta m}\,[p+\lambda(k-p)]^{m}\,(k-p)!} \\ &\times \prod_{j=0}^{k-p-n-1}\left[(n+j)+2p\,|b|\,(1-\beta)\ \right], \end{aligned}$$

for $k \geq p+n+1$ and $\eta > -p \quad (p \in \mathbb{R})$.

Corollary 3.4. *Let the function $f \in \mathcal{A}$ satisfy the Cauchy-Euler differential equation*

$$z^2\frac{d^2\omega}{dz^2} + 2(1+\eta)z\frac{d\omega}{dz} + \eta(1+\eta)\omega = (1+\eta)(2+1+\eta)g(z)$$

where $\omega = f(z)$ and $g(z) \in \mathcal{SC}(b, \mu, \beta)$, then

$$|a_k| \leq \frac{(1+\eta)(2+\eta)\prod_{j=0}^{k-2}\left[j+2p\,|b|\,(1-\beta)\ \right]}{(k+\eta)(k+\eta+1)\left[1+\mu(k-1)\right](k-p)!}$$

for $k \geq 2$ *and* $\eta > -1 \quad (\eta \in \mathbb{R})$.

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