ZWEIER DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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Abstract. In this article, we introduce the sequence spaces \( \mathcal{Z}_0(F, \Delta) \) and \( \mathcal{Z}_\infty(F, \Delta) \) for the sequence of moduli \( F = (f_k) \) and give some inclusion relations.

Keywords: difference sequence spaces; sequence of moduli; matrices; limitation method.

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1. Introduction

Let \( N, R \) and \( C \) be the sets of all natural, real and complex numbers respectively. We write

\[ \omega = \{ x = (x_k) : x_k \in R \text{ or } C \}, \]

the space of all real or complex sequences. Let \( l_\infty, c \) and \( c_0 \) be the linear spaces of bounded, convergent and null sequences respectively, normed by

\[ ||x||_\infty = \sup_k |x_k|, \text{ where } k \in N. \]

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Each linear subspace of $\omega$, for example, $\lambda, \mu \subset \omega$ is called a sequence space. A sequence space $\lambda$ with linear topology is called a K-space provided each of maps $p_i : C \rightarrow C$ defined by $p_i(x) = x_i$ is continuous for all $i \in N$. A K-space $\lambda$ is called an FK-space provided $\lambda$ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space. Let $\lambda$ and $\mu$ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers $(a_{nk})$, where $n, k \in N$. Then we say that $A$ defines a matrix mapping from $\lambda$ to $\mu$, and we denote it by writing $A : \lambda \rightarrow \mu$. If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of $x$ is in $\mu$, where

$$(Ax)_n = \sum_k a_{nk}x_k, \ (n \in N).$$

By $(\lambda : \mu)$, we denote the class of matrices $A$ such that $A : \lambda \rightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1.1) converges for each $n \in N$ and every $x \in \lambda$. The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen [1], Başar and Altay [2], Malkowsky [8], Ng and Lee [10], and Wang [15]. Şengönül [14] defined the sequence $y = (y_i)$ which is frequently used as the $Z^p$ transform of the sequence $x = (x_i)$, i.e,

$$y_i = px_i + (1 - p)x_{i-1},$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and $Z^p$ denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in N), \\ 0, \text{otherwise}. \end{cases}$$

Following Başar and Altay [2], Şengönül [14] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^px \in c\},$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^px \in c_0\}.$$
Theorem 1.1. [14, Theorem 2.1.] The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the coordinate wise addition and scalar multiplication which are the BK-spaces with the norm

$$||x||_{\mathcal{Z}} = ||x||_{\mathcal{Z}_0} = ||Z^p x||_c.$$ 

Theorem 1.2. [14, Theorem 2.2.] The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces $c$ and $c_0$ respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$.

Theorem 1.3. [14, Theorem 2.3.] The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.

The idea of difference sequence spaces was introduced by Kizmaz [5]. In 1981, Kizmaz [5] defined the sequence spaces

$$l_\infty(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in l_\infty\},$$

$$c(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c\},$$

and

$$c_0(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c_0\},$$

where $\Delta x = (x_k - x_{k+1})$ and $\Delta^0 x = (x_k)$. These are Banach spaces with the norm

$$||x||_{\Delta} = |x_1| + ||\Delta x||_\infty.$$ 

The idea of modulus was structured in 1953 by Nakano; see [9] and the references therein. A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

1. $f(t) = 0$ if and only if $t = 0$,
2. $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
3. $f$ is increasing, and
4. $f$ is continuous from the right at zero.

Let $X$ be a sequence space. Ruckle [11-13] defined the sequence space $X(f)$ as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}.$$
for a modulus $f$. Kolk [6-7] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$, that is

$$X(F) = \{ x = (x_k) : (f_k(|x_k|)) \in X \}.$$ 

After then Gaur and Mursaleen[4] defined the following sequence spaces

$$l_\infty(F, \Delta) = \{ x = (x_k) : \Delta x \in l_\infty(F) \},$$
$$c_0(F, \Delta) = \{ x = (x_k) : \Delta x \in c_0(F) \}$$

for a sequence of moduli $F = (f_k)$ and gave the necessary and sufficient conditions for the inclusion relations between $X(\Delta)$ and $Y(F, \Delta)$, where $X, Y = l_\infty$ or $c_0$.

**Lemma 1.4.** [3, Lemma 1.2.] The condition $\sup_k f_k(t) < \infty$, $t > 0$ holds if and only if there is a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$.

**Lemma 1.5.** [3, Lemma 1.3.] The condition $\inf_k f_k(t) > 0$ holds if and only if there exists a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$.

**2. Main results**

In this section, we introduce the following classes of sequence spaces.

$$\mathcal{Z}_\infty(F, \Delta) = \{ x = (x_k) \in \omega : \Delta x \in \mathcal{Z}_\infty(F) \},$$
$$\mathcal{Z}_0(F, \Delta) = \{ x = (x_k) \in \omega : \Delta x \in \mathcal{Z}_0(F) \}.$$ 

**Theorem 2.1.** For a sequence $F = (f_k)$ of moduli, the following statements are equivalent:

(a) $\mathcal{Z}_\infty(\Delta) \subseteq \mathcal{Z}_\infty(F, \Delta)$,

(b) $\mathcal{Z}_0(\Delta) \subseteq \mathcal{Z}_\infty(F, \Delta)$,

(c) $\sup_k f_k(t) < \infty$, $(t > 0)$.

**Proof.** (a) implies (b) is obvious.

(b) implies (c). Let $\mathcal{Z}_0(\Delta) \subset \mathcal{Z}_\infty(F, \Delta)$. 


Suppose that (c) is not true. Then by Lemma 1.4 \( \sup_k f_k(t) = \infty \) for all \( t > 0 \), and, therefore there is a sequence \( (k_i) \) of positive integers such that

\[
f_{k_i} \left( \frac{1}{i} \right) > i, \text{ for } i = 1, 2, 3, \ldots
\]

(2.1)

Define \( x = (x_k) \) as follows

\[
x_k = \begin{cases} 
\frac{1}{i}, & \text{if } k = k_i, i = 1, 2, 3, \ldots ; \\
0, & \text{otherwise.}
\end{cases}
\]

Then \( x \in \mathcal{Z}_0(\triangle) \) but by (2.1), \( x \notin \mathcal{Z}_\infty(F, \triangle) \) which contradicts (b). Hence (c) must hold.

(c) implies (a). Let (c) be satisfied and \( x \in \mathcal{Z}_\infty(\triangle) \). If we suppose that \( x \notin \mathcal{Z}_\infty(F, \triangle) \), then \( \sup_k f_k(|\triangle x_k|) = \infty \) for \( \triangle x \in \mathcal{Z}_\infty \). If we take \( t = |\triangle x| \), then \( \sup_k f_k(t) = \infty \) which contradicts (c).

Hence \( \mathcal{Z}_\infty(\triangle) \subseteq \mathcal{Z}_\infty(F, \triangle) \).

**Theorem 2.2.** If \( F = (f_k) \) is a sequence of moduli, then the following statements are equivalent:

(a) \( \mathcal{Z}_0(F, \triangle) \subseteq \mathcal{Z}_0(\triangle) \),

(b) \( \mathcal{Z}_0(F, \triangle) \subset \mathcal{Z}_\infty(\triangle) \),

(c) \( \inf_k f_k(t) > 0, \ (t > 0) \).

**Proof.** (a) implies (b) is obvious.

(b) implies (c). Let \( \mathcal{Z}_0(F, \triangle) \subset \mathcal{Z}_\infty(\triangle) \).

Suppose that (c) does not hold. Then, by lemma 1.5,

\[
\inf_k f_k(t) = 0, \ (t > 0),
\]

(2.2)

and therefore there is a sequence \( (k_i) \) of positive integers such that

\[
f_{k_i} \left( \frac{i^2}{t} \right) < \frac{1}{i} \text{ for } i = 1, 2, \ldots
\]
Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t^2, & \text{if } k = k_i, \ i = 1, 2, 3 \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

By (2.2) $x \in \mathcal{Z}_0(F, \triangle)$ but $x \notin \mathcal{Z}_\infty(\triangle)$ which contradicts (b). Hence (c) must hold.

(c) implies (a). Let (c) holds and $x \in \mathcal{Z}_0(F, \triangle)$, that is, $\lim_{k} f_k(|\triangle x_k|) = 0$. Suppose that $x \notin \mathcal{Z}_0(\triangle)$. Then for some number $\varepsilon_0 > 0$ and positive integer $k_0$ we have $|\triangle x_k| \geq \varepsilon_0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq f_k(|\triangle x_k|)$ for $k \geq k_0$ and hence $\lim_{k} f_k(\varepsilon_0) = 0$ which contradicts (c). Thus $\mathcal{Z}_0(F, \triangle) \subseteq \mathcal{Z}_0(\triangle)$.

**Theorem 2.3.** The inclusion $\mathcal{Z}_\infty(F, \triangle) \subseteq \mathcal{Z}_0(\triangle)$ holds if and only if

$$\lim_{k} f_k(t) = \infty \text{ for } t > 0. \tag{2.3}$$

**Proof.** Let $\mathcal{Z}_\infty(F, \triangle) \subseteq \mathcal{Z}_0(\triangle)$ such that $\lim_{k} f_k(t) = \infty$ for $t > 0$ does not hold. Then there is a number $t_0 > 0$ and a sequence $(k_i)$ of positive integers such that

$$f_{k_i}(t_0) \leq M < \infty \tag{2.4}$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, & \text{if } k = k_i, \ i = 1, 2, 3 \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $x \in \mathcal{Z}_\infty(F, \triangle)$, by (2.4). But $x \notin \mathcal{Z}_0(\triangle)$, so that (2.3) must hold. If $\mathcal{Z}_\infty(F, \triangle) \subseteq \mathcal{Z}_0(\triangle)$. Conversely, let (2.3) hold. If $x \in \mathcal{Z}_\infty(F, \triangle)$, then $f_k(|\triangle x_k|) \leq M < \infty$ for $k = 1, 2, 3 \ldots$. Suppose that $x \notin \mathcal{Z}_0(\triangle)$. Then for some number $\varepsilon_0 > 0$ and positive integer $k_0$ we have $|\triangle x_k| \geq \varepsilon_0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq f_k(|\triangle x_k|) \leq M$ for $k \geq k_0$ which contradicts (2.3). Hence $x \in \mathcal{Z}_0(\triangle)$.

**Theorem 2.4.** The inclusion $\mathcal{Z}_\infty(\triangle) \subseteq \mathcal{Z}_0(F, \triangle)$ holds, if and only if

$$\lim_{k} f_k(t) = 0 \text{ for } t > 0. \tag{2.5}$$
Proof. Suppose that \( \mathcal{Z}_\infty(\triangle) \subseteq \mathcal{Z}_0(F, \triangle) \) but (2.5) does not hold.

Then

\[
\lim_k f_k(t_0) = l \neq 0 \tag{2.6}
\]

for some \( t_0 > 0 \). Define the sequence \( x = (x_k) \) by

\[
x_k = t_0 \sum_{v=0}^{k-1} (-1)^v \frac{k-v}{k-v}
\]

for \( k = 1, 2, 3, \ldots \). Then \( x \notin \mathcal{Z}_0(F, \triangle) \), by (2.6). Hence (2.5) holds. Conversely, let \( x \in \mathcal{Z}_\infty(\triangle) \) and suppose that (2.5) holds. Then \( |\triangle x_k| \leq M < \infty \) for \( k = 1, 2, 3, \ldots \). Therefore \( f_k(|\triangle x_k|) \leq f_k(M) \) for \( k = 1, 2, 3, \ldots \) and \( \lim_k f_k(|\triangle x_k|) \leq \lim_k f_k(M) = 0 \), by (2.5).

Hence \( x \in \mathcal{Z}_0(F, \triangle) \).

Conflict of Interests
The author declares that there is no conflict of interests.

References


