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A NOTE ON DISTRIBUTIONAL LAPLACE-HARDY INTEGRAL TRANSFORMATIONS

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Abstract. In this note, certain testing function spaces are constructed and classical Laplace-Hardy integral transformation is extended to generalized functions. Our work corrects the errors contained in a paper of Ahirrao and More (1987).

Keywords: testing function space; generalized function; integral transform.

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1. Introduction

Given a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, the classical Laplace transform of a conventional function ϕ on a $(-\infty, \infty)$ is defined by

$$F(s) = \int_{-\infty}^{\infty} e^{-st} \phi(t)\, dt, \tag{1.1}$$

where s is given complex number. Similarly the classical Hardy transformation of a conventional function ϕ on $(0, \infty)$ is defined as

$$F(y) = \int_0^{\infty} F_\nu(ty) t\, dt \int_0^{\infty} C_\nu(tx) x \phi(x)\, dx. \tag{1.2}$$

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The details of these transformations and the functions involved in them may be found in Watson [5] and Zemannian [6]. Ahirao and More [1] extended the classical Laplace-Hardy transformation viz.,

$$F(s,y) = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-st} C_{\nu}(xy) t x \phi(t,x)\, dx\, dt \tag{1.3}$$

of a conventional function ϕ on Ω,

$$\Omega = \left\{ (t,x) - \infty < t < \infty, 0 < x < \infty \right\}$$

to generalized functions. But the paper of Ahirrao and More[1] contains several errors [3] and under the given conditions the conventional Laplace-Hardy transformation does not exist as a against the claim made by the authors. Consequently almost all the result of the paper of Ahirrao and More [1] are not correct and need appropriate changes and corrections. This led to give a set of sufficient conditions for the existence of classical and generalized transformation of the function in the appropriate function spaces. We follow the definitions and notations of Ahirrao and More [1] and the details may be found in the literature cited at the end of the paper.

2. Testing functions spaces

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{C}$ the set of complex numbers. Denote by $\mathbb{N}$ the set of natural numbers Let $a, b, c, d, t \in \mathbb{R}$ and $s \in \mathbb{C}$ and let $\kappa_{a,b}(t,x)$ be a function defined by

$$\kappa_{a,b}(t,x) = \begin{cases} e^{at} x^{\alpha}, & 0 \leq \infty, 0 < x < 1, \\ \\ e^{bt} x^{\alpha+2}, & -\infty < t < 0, x > 1, \end{cases} \tag{2.1}$$

where α is fixed positive real number satisfying $|\nu| \leq \alpha \leq \frac{1}{2}$. Now for each number $\kappa = (k_1, k_2) \in \mathbb{N} \times \mathbb{N}$. We define a space $LG_{\alpha}(\Omega)$ consisting of all infinitely differentiable functions $\phi(t,x)$ defined over the domain Ω given by

$$\Omega = \left\{ (t,x) | -\infty < t < \infty, 0 < x < \infty \right\} \tag{2.2}$$

satisfying

$$\gamma_{a,b,k}(\phi(t,x)) = \sup_{\substack{-\alpha<t<\alpha \\ 0<x<\alpha}} \left| \kappa_{a,b}(t,x) D_t^{k_1} \Delta_x^{k_2} \phi(t,x) \right| < \infty, \tag{2.3}$$

where

$$D_t = \frac{\partial}{\partial t},\ D_x = \frac{\partial}{\partial x}$$

and Δ_x is the Bessel's differential operation defined by

$$\Delta_x = D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2}. \tag{2.4}$$

It can be proved by using the arguments similar to Ahirrao and More[1] that $LG_\alpha(\Omega)$ is a countably multi normed, Frechet and testing function space. Let $LH_\alpha(\Omega)$ denotes the space of all infinitely differentiable functions ϕ over the domain Ω such that

$$\phi(t,x) \in LH_\alpha(\Omega)$$

if and only if $[m'(t,x)]^{-1}\phi(t,x) \in LG_\alpha(\Omega)$, where $m'(t,x) = tx$. The topology of $LG_\alpha(\Omega)$ is defined by the collection of the semi-norms $\{\beta_k\}_{k=0}^{\infty}$ given by

$$\beta_{a,b,k}^{\alpha}(\phi(t,x)) = \gamma_{a,b,k}^{\alpha}\left(\frac{\phi(t,x)}{m'(t,x)}\right) \tag{2.5}$$

for all $\phi(t,x) \in LH_\alpha(\Omega)$ and for all numbers $k = (k_1,k_2) \in \mathbb{N} \times \mathbb{N}$. The members of the space $LH_\alpha(\Omega)$ are called the generalized functions. For $f \in LH_\alpha^1(\Omega)$, we define $m'f \in LG_\alpha(\Omega)$ by the relation

$$\langle m'f, \phi \rangle = \langle f, m'\phi \rangle, \phi \in LG_\alpha(\Omega). \tag{2.6}$$

We note that if

$$f(t,x) \in LG_\alpha^1(\Omega) m'(t,x)\phi(t,x) \in LG_\alpha^1(\Omega).$$

We need the following result in the sequel.

Theorem 2.1. *If $|\nu| \le \alpha \le \frac{1}{2}$ and $a < \Re(s) < b$, then for fixed $s \in \mathbb{C}$ and $y \in \mathbb{R}$, $y > 0$, $e^{-st} C_\nu(xy) \in LG_\alpha(\Omega)$.*

Proof. It can be easily seen that

$$\sup_{0<y<\infty} |y^\alpha C_\nu(y)| < \infty, \tag{2.7}$$

where $|\nu| \le \alpha \le \frac{1}{2}$ (see [4], page 251).

Now, we consider

$$
\begin{aligned}
\gamma^{\alpha}_{a,b,k}(e^{-st}e_{\nu}(xy)) &= \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left|\kappa_{a,b}(t,x)D_t^{k_1}\delta_x^{k_2}e^{-st}C_{\nu}(xy)\right| \\
&= \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left|\kappa_{a,b}(t,x)(-1)^{k_1}(-1)^{k_2}y^{2k_2}e^{-st}C_{\nu}(xy)\right| \\
&= \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left|\kappa_{a,b}(t,x)(s)^{k_1}y^{2k_2}e^{-st}C_{\nu}(xy)\right| \\
&= \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left|\kappa_{a,b}(t,x)(s)^{k_1}x^{-\alpha}y^{2k_2-\alpha}e^{-st}(xy)^{\alpha}C_{\nu}(xy)\right| \\
&\leq \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left|e^{at}(s)^{k_1}y^{2k_2-\alpha}e^{-st}(xy)^{\alpha}C_{\nu}(xy)\right| \\
&\quad + \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left|e^{bt}x^{\alpha+2}(s)^{k_1}y^{2k_2-\alpha}e^{-st}(xy)^{\alpha}C_{\nu}(xy)\right| \\
&= \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left\{\left|e^{(a-st)t}s^{k_1}y^{2k_2-\alpha}\right|\left|(xy)^{\alpha}C_{\nu}(xy)\right|\right\} \\
&\quad + \sup_{\substack{-\infty<t<\infty\\ 0<x<\infty}} \left\{\left|e^{(b-st)^t}x^2s^{k_1}y^{2k_2-\alpha}\right|\left|(xy)^{\alpha}C_{\nu}(xy)\right|\right\} \\
&< \infty \quad [\because a<\Re(s)<b].
\end{aligned}
$$

Therefore, $e^{-st}C_{\nu}(xy) \in LG_{\alpha}(\Omega)$ and the proof of the theorem is complete.

3. Generalized Laplace-Hardy transformation

The conventional or classical Laplace-Hardy integral transformation F is a mapping $\mathscr{F}$: $\mathscr{L}H_{\alpha} \to \mathscr{L}\mathscr{H}_{\alpha}(\Omega)$ defined by

$$
\begin{aligned}
\mathscr{F}(s,y) &= \mathscr{L}\mathscr{H}(\phi(t,x)) \\
&= \int_{-\infty}^{\infty} e^{-st}t\,dt \int_0^{\infty} C_{\nu}(xy)x\phi(t,x)\,dx \\
&= \int_{-\infty}^{\infty} e^{-st}C_{\nu}(xy)tx\phi(t,x)\,dx\,dt.
\end{aligned}
\tag{3.1}
$$

Theorem 3.1. *Let $\phi \in \mathscr{L}\mathscr{G}_{\alpha}(\Omega)$, then the Laplace-Hardy transformation (3.1) exists for $a+\Re(s)>0$ and $b+\Re(s)<0$ and $|\nu| \leq \alpha \leq \frac{1}{2}$.*

Proof. It can be readily seen that for an appropriate $M>0$,

$$|C_\nu(xy)| \leq M(xy)^{-\frac{1}{2}} \tag{3.2}$$

for $x>0, y>0$ (see [5], page 251). Therefore, one has

$$\begin{aligned}
\Big|F(s,y)\Big| &= \Big|\int_{-\infty}^{\infty}\int_0^{\infty} e^{-st}C_\nu(xy)tx\phi(t,x)\,dx\,dt\Big| \\
&\leq \int_{-\infty}^{\infty}\int_0^{\infty}\Big|e^{-st}C_\nu(xy)tx\phi(t,x)\Big|\,dx\,dt \\
&\leq \frac{M}{\sqrt{y}}\int_{-\infty}^{\infty}\int_0^{\infty}\Big|e^{-st}tx^{\frac{1}{2}}\phi(t,x)\Big|\,dx\,dt \\
&\leq \frac{M}{\sqrt{y}}\int_0^{\infty}\int_0^{1}\Big|e^{-st}\gamma^{\alpha}_{a,b,0}(\phi(t,x))\frac{tx^{\frac{1}{2}}}{e^{at}x^{\alpha}}|\Big|\,dx\,dt \\
&+\frac{M}{\sqrt{y}}\int_{-\infty}^{0}\int_1^{\infty}\Big|\gamma^{\alpha}_{a,b,0}(\phi(t,x))\frac{e^{-st}x^{\frac{1}{2}}t}{e^{bt}x^{\alpha+1}}\Big|dxdt \\
&= \frac{M}{\sqrt{y}}\gamma^{\alpha}_{a,b,0}(\phi(t,x))\int_0^{\infty}\int_0^{1}\Big|x^{-\alpha+\frac{1}{2}}te^{-}(a+s)^{t}\Big|\,dx\,dt \\
&+\frac{M}{\sqrt{y}}\gamma^{\alpha}_{a,b,0}(\phi(t,x))\int_{-\infty}^{0}\int_1^{\infty}\Big|e^{-(b+s)t}tx^{-\alpha-\frac{a}{2}}\Big|\,dx\,dt.
\end{aligned} \tag{3.3}$$

Now consider the first integral in (3.3):

$$\begin{aligned}
&\int_0^{\infty}\int_0^{1}\Big|x^{-\alpha+\frac{1}{2}}e^{-(a+s)t}.t\Big|\,dx\,dt \\
&= \Big[\int_0^{\infty}\Big|e^{-(a+s)^{t}}.t\Big|dt\Big]\times\Big[\int_0^{1}\Big|x^{-\alpha+\frac{1}{2}}\Big|dx\Big] \\
&= \Big[\Big|\frac{1}{-(a+\Re(s)).e^(a+\Re(s))^{t}}-\frac{e^{-}(a+\Re(s))t}{(a+\Re(s))^2}\Big|\Big]_0^{\infty}\times\frac{1}{(\frac{3}{2}-\alpha)} \\
&= \frac{1}{(\frac{3}{2}-\alpha)}\Big[\frac{2}{(a+\Re(s))^2}\Big] \quad \Big[\because a+\Re(s)>0\Big] \\
&= \frac{2}{(\frac{3}{2}-\alpha))(a+\Re(s))^2} \\
&< \infty.
\end{aligned} \tag{3.4}$$

Again, for the second integral in (3.3), we have

$$\begin{aligned}
&\int_{-\infty}^{0}\int_{1}^{\infty}\left|x^{-\alpha-\frac{-}{3}2}e^{(-b+\Re(s))t}.t\right|dx\,dt \\
&= \left[\int_{-\infty}^{0}\left|e^{-(b+Re(s))t}.t\right|dt\right]\times\left[\int_{1}^{\infty}\left|x^{-\alpha-\frac{3}{2}}\right|dx\right] \\
&= \left[\frac{-t}{(b+\Re(s)).e^(b+\Re(s))t}-\frac{1}{(b+\Re(s))^2}e^{(b+Re(s))t}\right]_{-\infty}^{0} \\
&\times\left[\frac{x^{-\alpha-2+\frac{3}{2}}-\alpha-2+\frac{3}{2}}{-\alpha-2+\frac{3}{2}}\right]_{1}^{\infty} \\
&= \left|\left[\frac{-1}{(b+Re(s))}.e^{(b+\Re(s))t}-\frac{1}{(b+\Re(s))^2}e^{(b+\Re(s))t}\right]_{-\infty}^{0}\right| \\
&\times\left|\left[-\frac{x^{-\alpha-\frac{1}{2}}}{(\alpha+\frac{1}{2})}\right]_{1}^{\infty}\right| \\
&= \frac{2}{(b+\Re(s))^2}.\frac{1}{(\alpha+\frac{1}{2})}\quad[\because b+\Re(s)>0] \\
&< \infty.
\end{aligned}\tag{3.5}$$

Now from inequalities (3.4) and (3.5) we get $|F(s,y)|<\infty$ for fixed $s\in\mathbb{C}$ and $y>0$. This shows that the Laplace-Hardy transformation exists for $a+\Re(s)>0$ and $b+\Re(s)<0$. This completes the proof.

Now for $f(t,x)\in\mathscr{L}\mathscr{H}^{t}_{\alpha}(\Omega)$ we define its distributional Laplace-Hardy transformation by the relation

$$\begin{aligned}
F(s,y) &= \mathscr{L}\mathscr{H}\{f(t,x)\} \\
&= \langle m'(t,x)f(t,x),\, e^{-st}C_\nu(xy)\rangle,
\end{aligned}\tag{3.6}$$

where, $s\in\mathbb{C}$ and $y\in\mathbb{R}_+$, $|\nu|\le\alpha\le\frac{1}{2}$, $a<\Re(s)<b$ and $a+\Re(s)<0$; $b+\Re(s)>0$.

By the Theorem 2.1, $e^{-st}C_\nu(xy)\in\mathscr{L}\mathscr{G}_\alpha(\Omega)$ and $m'(t,x)f(t,x)\in\mathscr{L}\mathscr{G}^{1}_{\alpha}(\Omega)$, and therefore the relation (3.6) is meaningful.

All other properties of the distributional Laplace-Hardy transformation such as analytical,representation and boundedness theorems etc. can be proved by closely observing the proofs giving in Ahirrao and More [1] with appropriate modifications. We omit the details.

Conflict of Interests

The authors declare that there is no conflict of interests.

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