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ORDER FOUR CONTINUOUS HYBRID BLOCK METHOD FOR THE SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We consirdered method of collocation of the differential system and interpolation of the approximate solution to generate a continuous hybrid linear multistep method, which is solved for the independent solution to give a continuous block method. The resultant method is evaluated at selected grid points to generate a discrete block method. The basic properties of the method was investigated and found to be zero stable, consistent and convergent. The method was tested on some numerical examples and found to compete favourably with the existing methods.

Keywords: differential system; interpolation; collocation; zero stable consistent; convergent.

2010 AMS Subject Classification: 65L05, 65L06, 65D30.

1. Introduction

Ordinary differential equation have long been an important area of study because of their wide application in physics and engineering and more recently in biology chemistry, ecology and economics. In sciences and engineering, usually mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such mathematical equations are called differential equations. In these paper, our focus is on the solution to general second order initial value problems of the form:

$$y'' = f\left(x, y(x), y'(x)\right) \qquad y(x_0) = \eta_0, y'(x_0) = \eta_1 \tag{1}$$

In most application, the conventional method of solving (1) is by reduction to system of first order ordinary differential equation and appropriate numerical method could be employed to solve the resultant systems. Adesanya *et al.* [1], Awoyemi [2], reported that in spite of the success of this approach, the setback of the method is in writing computer program which is often complicated especially when subroutine are incorporated to supply the starting values required for the method. The consequence is in longer computer time and human effort. Furthermore, Adesanya *et al.*[3] reported that this method does not utilize additional information associated with specific ordinary differential equation, such as the oscillatory nature of the solution. In addition, Bun and Varsa'Yel [4] reported that more serious disadvantage of the method of reduction is the fact that the given system of equations to be solved cannot be solved explicitly with respect to the derivatives of the highest order. For these reasons, this method is inefficient and not suitable for general purpose. Scholars later developed method to solve (1) directly without reducing it to systems of first order ordinary differential equations and concluded that direct method is better than method of reduction, among these authors are Twizel and Khaliq [5], Awoyemi and Kayode [6], Vigo-Aguiar and Ramos [7], Adesanya *et al.*[8]. In order to cater for the above mentioned setback, researchers came up with block methods which simultaneously generate approximation at different grid points within the interval of integration. Block method is less expensive in terms of the number of function evaluations compared to the linear multistep method or Runge-Kutta method.

In this paper, we proposed continuous hybrid block method for the solution of general second order initial value problem of ordinary differential equations which when evaluated at selected grid points gives discrete block. The Continuous block method possesses the same properties as the continuous hybid linear multistep method which is discussed by Awoyemi *et al.*[9]

2. Methodology

Consider power series approximate solution in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{2}$$

The second derivative of (2) gives

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} \tag{3}$$

Substituting (3) into (1) gives

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} = f(x,y,y') \tag{4}$$

$x \in [a,b]$. $a_j's$ are the parameters to be determined. The step sise (h) is given as $h = x_{n+i} - x_{n}, n = 0\,(1)\,N$

Interpolating (2) at $x_{n+j}, j = x_{n+1}, x_{n+\frac{5}{3}}$ and collocating (3) at $x_{n+j}, j = 0(\frac{2}{3})2$ gives a system of non linear equation

$$AX = U \tag{5}$$

where

$$A = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+\frac{5}{3}} & x_{n+\frac{5}{3}}^2 & x_{n+\frac{5}{3}}^3 & x_{n+\frac{5}{3}}^4 & x_{n+\frac{5}{3}}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x_{n+\frac{2}{3}}^2 & 20x_{n+\frac{2}{3}}^3 \\ 0 & 0 & 2 & 6x_{n+\frac{4}{3}} & 12x_{n+\frac{4}{3}}^2 & 20x_{n+\frac{4}{3}}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \end{bmatrix}$$

$$X = \begin{bmatrix} a_o & a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_{n+1} & y_{n+\frac{5}{3}} & f_n & f_{n+\frac{2}{3}} & f_{n+\frac{4}{3}} & f_{n+2} \end{bmatrix}^T$$

Solving (5) for the unknown constant $a_j's$ and subtituting back into (2) gives a continuous hybrid linear multistep method in the form

$$y(x) = \alpha_0 y_0 + \alpha_{\frac{5}{3}} y_{\frac{5}{3}} + h^2 \left[\beta_0(x) f_n + + \beta_{\frac{2}{3}}(x) f_{n+\frac{2}{3}} + \beta_{\frac{4}{3}}(x) f_{n+\frac{4}{3}} + \beta_2(x) f_{n+2} \right] \tag{6}$$

where

$$f_{n+j} = f(x_n + jh) = f(x_n + jh, y(x_n + jh), y'(x_n + jh))$$

$$\alpha_0 = 1 - t$$
$$\alpha_{\frac{5}{3}} = \tfrac{3}{2} t$$

$$\beta_0 = -\tfrac{1}{960}\left(27t^5 - 180t^4 + 440t^3 - 480t^2 + 193t\right)$$

$$\beta_{\frac{2}{3}} = \tfrac{1}{320}\left(27t^5 - 150t^4 + 240t^3 - 117t\right)$$

$$\beta_{\frac{4}{3}} = -\tfrac{1}{320}(27t^5 - 120t^4 + 120t^3 + 27t)$$

$$\beta_2 = \tfrac{1}{960}(27t^5 - 90t^4 + 80t^3 - 17t)$$
$$t = \tfrac{x - x_n}{h}$$

Solving (6) for the independet solution gives a continuous block method in the form

$$y_{n+k} = \sum_{j=0}^{\mu-1} \frac{(kh)^m}{m!} y_n^{(m)} + h^2 \left[\begin{array}{c} \sigma_0(x) f_n + \sigma_{\frac{1}{3}}(x) f_{n+\frac{1}{3}} + \sigma_{\frac{2}{3}}(x) f_{n+\frac{2}{3}} + \sigma_{\frac{4}{3}}(x) f_{n+\frac{4}{5}} \\ + \sigma_{\frac{5}{3}}(x) f_{n+\frac{5}{5}} + \sigma_2(x) f_{n+2} \end{array} \right], \tag{7}$$

$k = \frac{1}{3}\left(\frac{1}{3}\right)2$

where

$$\beta_0 = -\tfrac{1}{960}\left(27t^5 - 180t^4 + 440t^3 - 480t^2\right)$$

$$\beta_{\frac{2}{3}} = \tfrac{1}{320}\left(27t^5 - 150t^4 + 240t^3\right)$$

$$\beta_{\frac{4}{3}} = -\tfrac{1}{320}(27t^5 - 120t^4 + 120t^3)$$

$$\beta_2 = \tfrac{1}{960}(27t^5 - 90t^4 + 80t^3)$$

Evaluating (7) at $t = \frac{1}{3}\left(\frac{1}{3}\right)2$ gives a discrete block method in the form

$$A^0 Y_m^{(i)} = \sum_{i}^{2} h^{(i)} e_i y_n^{(i)} + h^2 \left[D F_{m-1} + E F_m\right] \tag{8}$$

where

$$Y_m = \left[y_{n+\frac{1}{3}},\, y_{n+\frac{2}{3}},\, y_{n+1},\, y_{n+\frac{4}{3}},\, y_{n+\frac{5}{3}},\, y_{n+2}\right]^T$$

$$y_m = \left[y_{n-1},\, y_{n-2},\, y_{n-3},\, y_{n-4},\, y_{n-5},\, y_n\right]^T$$
$$F_m = \left[f_{n+\frac{1}{3}},\, f_{n+\frac{2}{3}},\, f_{n+1},\, f_{n+\frac{4}{3}},\, f_{n+\frac{5}{3}},\, f_{n+2}\right]^T$$
$$f_{m-1} = \left[f_{n-1},\, f_{n-2},\, f_{n-3},\, f_{n-4},\, f_{n-5},\, f_n\right]^T$$
$$A^0 = 6 \times 6 \text{ identity matrix}$$

when $i = 0$

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1057}{25920} \\ 0 & 0 & 0 & 0 & 0 & \frac{97}{810} \\ 0 & 0 & 0 & 0 & 0 & \frac{193}{960} \\ 0 & 0 & 0 & 0 & 0 & \frac{112}{405} \\ 0 & 0 & 0 & 0 & 0 & \frac{1825}{5184} \\ 0 & 0 & 0 & 0 & 0 & \frac{13}{30} \end{bmatrix}, E = \begin{bmatrix} 0 & \frac{193}{8640} & 0 & \frac{-83}{8640} & 0 & \frac{53}{25920} \\ 0 & \frac{19}{135} & 0 & \frac{-13}{270} & 0 & \frac{4}{405} \\ 0 & \frac{117}{320} & 0 & \frac{-27}{320} & 0 & \frac{17}{960} \\ 0 & \frac{88}{135} & 0 & \frac{-8}{135} & 0 & \frac{8}{405} \\ 0 & \frac{1625}{1728} & 0 & \frac{125}{1728} & 0 & \frac{125}{5184} \\ 0 & \frac{6}{5} & 0 & \frac{3}{10} & 0 & \frac{1}{15} \end{bmatrix}$$

When $i = 1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{119}{576} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{64} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{9} \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{64} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & \frac{107}{576} & 0 & \frac{-43}{576} & 0 & \frac{1}{64} \\ 0 & \frac{19}{36} & 0 & \frac{-5}{36} & 0 & \frac{1}{36} \\ 0 & \frac{51}{64} & 0 & \frac{-3}{64} & 0 & \frac{1}{64} \\ 0 & \frac{8}{9} & 0 & \frac{2}{9} & 0 & 0 \\ 0 & \frac{475}{576} & 0 & \frac{325}{576} & 0 & \frac{25}{576} \\ 0 & \frac{3}{4} & 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}$$

3. Analysis of the block

let the linear operator $L\{y(x):h\}$ associated with the bloch methed (9) be defind as

$$L\{y(x):h\} = A^0 Y_m^{(i)} - \sum_{i}^{2} h^{(i)} e_i y_n^{(i)} - h^2 \left[DF_{m-1} + EF_m\right] \tag{9}$$

Expanding (9)in Taylor series gives

$$L\{y(x):h\} = c_0 y(x) + c_1 h y(x) + \ldots c_p h^p y(x) + c_{p+1} h^{p+1} y(x) + \ldots \tag{10}$$

Definition 1. *Order: The linear operator and associated block* (9) *are said to be of order* p *if* $c_0 = c_1 \ldots = c_{p+1} = 0$ *and* $c_{p+2} \neq 0, c_{p+2}$ *is called the error constant*

The order of our block (9) is 4 with error constant $\left[\frac{-49}{174960}\,\frac{-14}{10935}\,\frac{-1}{432}\,\frac{-32}{10935}\,\frac{-121}{34992}\,\frac{-2}{405}\right]^T$

Definition 2. *Stability: Block method is said to be zero stable if as* $h \longrightarrow 0$, *the roots* $r_j, j = 1(1)k$ *of the first characteristic polynomial* $\rho(r) = 0$ *that is* $\rho(r) = \det\left[\sum A^0 R^{k-1}\right] = 0$*satisfy by* $|R| \leq 1$ *for those root with* $|R| = 1$ *must be simple*

For our method

$$\rho(r) = \left| R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$R = 0,0,0,0,0,1$$

Hence our method is zero stable

Definition 3. *Consistency: A block method is said to be consistent, if it has order greater than one.*

Hence our method is consistent

Definition 4. *Convergence:A block method is said to be convergent if and only if it is consistent and zero stable.*

From the above, it is shown clearly that our method is convergent.

4. Numerical Example

Example one: We consider the linear initial value problem (IVP)

$y'' - x(y') = 0, y(0) = 1, y'(0) = \frac{1}{2}, h = 0.01$

Exact solution : $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

This problem was solved by Adesanya *et al.* [3] where a method of order six implimented in block predictor- block correctorwas proposed. We compared our result with their result as shown in Table 1

Example Two: We consider the non linear initial value problem

$y'' = \left(\frac{y'}{2y}\right) - 2y, y\left(\frac{\pi}{6}\right) = \frac{1}{4}, y'\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right), h = 0.01$

Exact solution: $y(x) = (\sin x)^2$

This problem was solved by Anake *et al.*[10] where theyproposed a method of order six implimented in block method. We compared our method with their method as shown in Table II

Table 1 showing the result of problem 1

x	Exact result	Computed result	Error	ERA
0.1	1.050041729278491	1.05004172927851	2.3314(-14)	1.3300(-13)
0.2	1.100335347731076	1.10033534773126	1.8918(-13)	9.9198(-13)
0.3	1.151144043593646	1.15114043593712	6.5836(-13)	3.4396(-12)
0.4	1.202732554054082	1.20273255405572	1.6406(-12)	8.6803(-12)
0.5	1.255412811882995	1.25541281188643	3.4350(-12)	1.8373(-11)
0.6	1.309519604203112	1.30951960420960	6.4921(-12)	3.5087(-11)
0.7	1.365443754271396	1.36544375428292	1.1529(-11)	6.3313(-11)
0.8	1.423648930193602	1.42364893021330	1.9728(-11)	1.4013(-10)
0.9	1.484700278594052	1.48470027862716	3.3111(-11)	1.8818(-10)
1.0	1.549306144334055	1.54930614438934	5.5286(-11)	3.2211(-10)

Note: ERA$\rightarrow$ Error in Adesanya *et al.*[3]

Table II showing the result of problem II

x	Exact result	Computed result	Error	EAN
1.1	0.794465126268689	0.794465126278411	9.7221(-12)	6.6348(-08)
1.2	0.868876102282285	0.868876102286973	4.6878(-12)	6.1995(-08)
1.3	0.928581152153741	0.928581152147770	5.9711(-12)	3.1135(-08)
1.4	0.971200023955486	0.971200023931598	2.3264(-11)	3.0572(-08)
1.5	0.995033637758565	0.995033637710881	4.7683(-11)	1.2473(-07)
1.6	0.999131822593813	0.999131822514726	7.9086(-11)	2.4989(-07)
1.7	0.983331196763794	0.983331196647171	1.1662(-10)	4.0149(-07)
1.8	0.948261681358595	0.948261681199868	1.5872(-10)	5.7196(-07)
1.9	0.893213872891642	0.895321387086013	2.0315(-10)	7.5116(-07)
2.0	0.826620877017291	0.826620876770221	2.4707(-10)	9.2698(-07)

Note: EAN$\rightarrow$ Error in Anake *et al.*[10]

5. Conclusion

We have proposed an order four continuous hybrid block method for the solution of $y'' = f(x,y,y')$. The method was found to be consistent and zero stable hence convergent. The numerical examples shows that the method compete favourably with the existing method.

Conflict of Interests

The authors declare that there is no conflict of interests.

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