A NOTE ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY
RUSCHEWEYH DERIVATIVE AND A NEW GENERALISED MULTIPLIER
TRANSFORMATION

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Abstract: In this paper, we consider the operator $RI^m_{\alpha, \beta, \lambda} : A(n) \rightarrow A(n)$ defined by

$$RI^m_{\alpha, \beta, \lambda} f(z) = (1 - \lambda)R^m f(z) + \lambda I^m_{\alpha, \beta} f(z),$$

where $A(n)$ denote the class of analytic functions in the unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$, of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, R^m f(z), m \in N_0 = N \cup \{0\}$ is the Ruscheweyh operator and $I^m_{\alpha, \beta} f(z) = z + \sum_{k=n+1}^{\infty} \left(\frac{\alpha + k \beta}{\alpha + \beta}\right)^m a_k z^k, n \in N, m \in N_0 = N \cup \{0\}$, $\lambda \geq 0, \beta \geq 0$, and $\alpha$ a real number with $\alpha + \beta > 0$. The new subclass $\mathcal{R}I^\lambda_n(m, \mu, \rho, \alpha, \beta)$ of $A(n)$, involving the operator $RI^m_{\alpha, \beta, \lambda}$ is introduced. Some interesting properties of the class $\mathcal{R}I^\lambda_n(m, \mu, \rho, \alpha, \beta)$ are established by making use of the concept of differential subordination.

Keywords: Analytic function, starlike function, convex function, Ruscheweyh derivative, multiplier transformation, differential subordination.

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1. INTRODUCTION

Let $A(n)$ denote the class of functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, n \in N = \{1, 2, 3, \ldots\}$,
which are analytic in the open unit disc \( U = \{ z : z \in \mathbb{C}, |z| < 1 \} \). Clearly \( A(1) = A \) is a well-known class of normalized analytic functions in \( U \). If \( f \) and \( g \) are analytic in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \), if there exists a Schwarz function \( w(z) \), which (by definition)is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1, z \in U \), such that \( f(z) = g(w(z)), z \in U \). Further, if the function \( g \) is univalent in \( U \), then we have the following equivalence \( f \prec g \iff f(0) = g(0) \) and \( f(U) \subset g(U) \).

For \( 0 \leq \rho < 1 \), we denote \( S_n^\ast(\rho) \) and \( K_n(\rho) \) the subclasses of \( A(n) \) consisting of all analytic functions which are respectively, starlike of order \( \rho \) and convex of order \( \rho \) in \( U \). It is well known that \( K_n(\rho) \subset S_n^\ast(\rho) \subset S \), where \( S \) is the class of univalent functions in \( U \). We also denote by \( R_n(\rho) \) the subclass of functions in \( A(n) \) which satisfy \( \text{Re}(f^\prime(z)) > \rho, z \in U \).

**Definition 1.1** ([16]). For \( f \in A(n), m \in N_0 = N \cup \{ 0 \}, \beta \geq 0 \) and \( \alpha \) a real number with \( \alpha + \beta > 0 \), a new generalized multiplier transformation, denoted by \( I_{\alpha,\beta}^m \), is defined by the following infinite series:

\[
I_{\alpha,\beta}^m f(z) = z + \sum_{k=n+1}^{\infty} \left( \frac{\alpha + k\beta}{\alpha + \beta} \right)^m a_k z^k, z \in U.
\]

It follows from (1.1) that

\[
I_{\alpha,0}^m f(z) = f(z),
\]

\[
(\alpha + \beta) I_{\alpha,\beta}^{m+1} f(z) = \alpha I_{\alpha,\beta}^m f(z) + \beta z (I_{\alpha,\beta}^m f(z))',
\]

We note that

- \( I_{\alpha,1}^m f(z) = I_{\alpha}^m f(z), \alpha > -1 \) (See Cho and Srivastava [10] and Cho and Kim [11]).
- \( I_{1-\beta,\beta}^m f(z) = D_{\beta}^m f(z), \beta \geq 0 \) (See Al-Oboudi [6]).
- \( I_{l+1-\beta,\beta}^m f(z) = I_{l,\beta}^m f(z), l > -1, \beta \geq 0 \) (See Catas [9]).
Remark 1.2. a) \( I^m_\alpha f(z) \) was defined and investigated in [10] and [11] for \( \alpha \geq 0 \) and \( I^m_{l,\beta} f(z) \) was defined and studied in [9] for \( l \geq 0, \beta \geq 0 \). So our results in this paper are improvement of corresponding results proved earlier for \( I^m_\alpha f(z) \) or \( I^m_{l,\beta} f(z) \) to \( \alpha > -1 \) or \( l > -1 \), respectively.
b) i) \( D^m_\beta f(z), m \geq 0 \) was due to Acu and Owa [1], ii) \( D^m f(z) \) was introduced by Salagean [15] and was considered for \( m \geq 0 \) in [7], and iii) \( I^m_1 f(z) \) was investigated by Uralegaddi and Somanath [20].

Definition 1.3 ([14]). For \( m \in N_0, f \in A(n) \), the operator \( R^m \) is defined by \( R^m : A(n) \rightarrow A(n) \),

\[
R^0 f(z) = f(z), \\
R^1 f(z) = zf'(z), \\
\vdots \\
(m + 1)R^{m+1} f(z) = z(R^m f(z))' + mR^m f(z), z \in U.
\]

Definition 1.4. Let \( m \in N_0, \lambda \geq 0, \beta \geq 0 \) and \( \alpha \) a real number with \( \alpha + \beta > 0 \). Denote by \( RI^m_{\alpha,\beta,\lambda} \), the operator given by \( RI^m_{\alpha,\beta,\lambda} : A(n) \rightarrow A(n) \),

\[
RI^m_{\alpha,\beta,\lambda} f(z) = (1 - \lambda)R^m f(z) + \lambda I^m_{\alpha,\beta} f(z), z \in U.
\]

Remark 1.5. If \( f \in A(n) \), then \( RI^m_{\alpha,\beta,\lambda} f(z) = z + \sum_{k=n+1}^{\infty} (1-\lambda)C^{m}_{m+k-1} + \lambda \left( \frac{\alpha + k\beta}{\alpha + \beta} \right)^m a_k z^k, z \in U. \)

Remark 1.6. The operator \( I^m_{\alpha,\beta} \) is introduced and investigated in [16] and [17]. The operator \( RI^m_{\alpha,\beta,\lambda} \) is studied in [18] and [19].

For \( \lambda = 0 \), \( RI^m_{\alpha,\beta,0} f(z) = R^m f(z), z \in U \), and for \( \lambda = 1 \), \( RI^m_{\alpha,\beta,1} = I^m_{\alpha,\beta} f(z), z \in U. \)

To prove our results we need the following lemma.
Lemma 1.7 [13]. Let $\frac{1}{2} \leq \rho < 1$, $u$ be analytic in $U$ with $u(0) = 1$ and suppose that

$$
(1.4) \quad \text{Re}\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\rho - 1}{2\rho}, \quad z \in U.
$$

Then $\text{Re}(u(z)) > \rho, z \in U$.

2. MAIN RESULTS

Definition 2.1. We say that a function $f \in A(n)$ is in the class $I_n(m, \mu, \rho, \alpha, \beta), m \in N_0,$ $n \in N, \mu \geq 0, \rho \in [0,1), \alpha$ a real number with $\alpha + \beta > 0$, if

$$
(2.1) \quad \left|\left(\frac{I_{\alpha, \beta}^{m+1} f(z)}{z}\right) - 1 - \frac{z}{I_{\alpha, \beta}^{m} f(z)}\right|^\mu < 1 - \rho, z \in U.
$$

Definition 2.2. We say that a function $f \in A(n)$ is in the class $RI_n^{\lambda}(m, \mu, \rho, \alpha, \beta), m \in N_0,$ $n \in N, \mu \geq 0, \rho \in [0,1), \alpha$ a real number with $\alpha + \beta > 0$, if

$$
(2.2) \quad \left|\left(\frac{RI_{\alpha, \beta, \lambda}^{m+1} f(z)}{z}\right) - 1 - \frac{z}{RI_{\alpha, \beta, \lambda}^{m} f(z)}\right|^\mu < 1 - \rho, z \in U.
$$

For $\lambda = 1$, (2.2) reduces to (2.1).

Remark 2.3. The family $RI_n^{\lambda}(m, \mu, \rho, \alpha, \beta)$ is a new comprehensive class of analytic functions which includes various well known classes of analytic univalent functions as well as some new ones. For example, i) $RI_n^{\lambda}(m, \mu, \rho, l + 1 - \beta, \beta) = RD_n^{\lambda}(m, \mu, \rho, l, \beta), l > -1$, was studied in [2] for $l \geq 0$; ii) $RI_n^{\lambda}(m, \mu, \rho, 1 - \beta, \beta) = RD_n^{\lambda}(m, \mu, \rho, 0, \beta) = RD_n^{\lambda}(m, \mu, \rho, \beta)$ was due to Lupas [3], iii) $RI_n^{\lambda}(m, \mu, \rho, \alpha, \beta) = I_n(m, \mu, \rho, \alpha, \beta)$ (Definition 2.1), iv) $I_n(m, \mu, \rho, 1 - \beta, \beta) = D_n(m, \mu, \rho, \beta)$ was introduced in [4], v) $D_n(0,1, \rho, l) = S_n^{\lambda}(\rho), D_n(1,1, \rho, l) = K_n(\rho)$ and $D_n(0,0, \rho, 1) = R_n(\rho)$, vi) $D_n(0,1, \rho, 1) = D(m,n,\rho)$ was introduced in [5,8], vii) $D_n(0, \mu, \rho, l) = D(\mu, \rho)$ was introduced
by Frasin and Jahangiri [13] and viii) $D_1(0,2,\rho,1) = D(\rho)$ which has been investigated by Frasin and Darus [12].

In this note we provide a sufficient condition for functions to be in the class $\mathcal{R}_n^\lambda(m,\mu,\rho,\alpha,\beta)$.

**Theorem 2.4.** Let $m \in \mathbb{N}_0, n \in \mathbb{N}, \lambda \geq 0, \mu \geq 0, \frac{1}{2} \leq \rho < 1, \gamma = \frac{3\rho - 1}{2\rho}, \beta > 0, \alpha$ a real number with $\alpha + \beta > 0$ and $f \in A(n)$. If

\[
(2.3) \quad (m + 2) \frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^m f(z)} = \mu(m + 1) \frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^m f(z)} + \lambda \left( \frac{\alpha + \beta}{\beta} - m - 1 \right) \frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^m f(z)} - \\
- \lambda \mu \left( \frac{\alpha + \beta}{\beta} - m - 1 \right) \frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^m f(z)} - \lambda \left( \frac{\alpha}{\beta} - m - 1 \right) \frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^m f(z)} + \\
+ \lambda \mu \left( \frac{\alpha}{\beta} - m \right) \frac{RI_{\alpha,\beta,\lambda}^m f(z)}{RI_{\alpha,\beta,\lambda}^{m+1} f(z)} + (m + 1)(\mu - 1) z^\gamma, z \in U,
\]

then $f \in \mathcal{R}_n^\lambda(m,\mu,\rho,\alpha,\beta)$.

Proof. Define the function $u(z)$ by

\[
(2.4) \quad u(z) = \left( \frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{z} \right) \left( \frac{z}{RI_{\alpha,\beta,\lambda}^m f(z)} \right)^\mu.
\]

Then the function $u(z)$ is analytic in $U$ with $u(0) = 1$. Differentiating (2.4) logarithmically with respect to $z$ and using (1.3), we obtain
\[
\frac{zu'(z)}{u(z)} = (m + 2) \frac{RI_{\mu,\beta,\lambda}^{m+2} f(z)}{RI_{\mu,\beta,\lambda}^{m+1} f(z)} - \mu(m + 1) \frac{RI_{\mu,\beta,\lambda}^{m+1} f(z)}{RI_{\mu,\beta,\lambda}^{m} f(z)} + \lambda \left( \frac{\alpha + \beta}{\beta} - m - 2 \right) \frac{RI_{\mu,\beta,\lambda}^{m+1} f(z)}{RI_{\mu,\beta,\lambda}^{m+1} f(z)} - \\
- \lambda \mu \left( \frac{\alpha + \beta}{\beta} - m - 1 \right) \frac{RI_{\mu,\beta,\lambda}^{m+1} f(z)}{RI_{\mu,\beta,\lambda}^{m} f(z)} - \lambda \left( \frac{\alpha}{\beta} - m - 1 \right) \frac{RI_{\mu,\beta,\lambda}^{m+1} f(z)}{RI_{\mu,\beta,\lambda}^{m} f(z)} + \\
+ \lambda \mu \left( \frac{\alpha}{\beta} - m \right) \frac{RI_{\mu,\beta,\lambda}^{m} f(z)}{RI_{\mu,\beta,\lambda}^{m} f(z)} + (m + 1)(\mu - 1) - 1
\]

From (1.4) and (2.3) we get \( \text{Re} \left( 1 + \frac{zu'(z)}{u(z)} \right) > \frac{3\rho - 1}{2\rho}, z \in U \). Applying Lemma 1.4 we deduce that

\[
\text{Re} \left( \left( \frac{RI_{\mu,\beta,\lambda}^{m+1} f(z)}{z} \right) \left( \frac{z}{RI_{\mu,\beta,\lambda}^{m} f(z)} \right)^m \right) > \rho, z \in U.
\]

Therefore, \( f \in R I_{\mu}^{1}(m, \rho, \alpha, \beta) \), by Definition 2.3.

Taking \( \lambda = 1 \) in Theorem 2.4, we obtain

**Theorem 2.5.** Let \( m \in N_0, n \in N, \mu \geq 0, \frac{1}{2} \leq \rho < 1, \gamma = \frac{3\rho - 1}{2\rho}, \beta > 0, \alpha \) a real number with \( \alpha + \beta > 0 \) and \( f \in A(n) \). If

\[
\left( \frac{\alpha + \beta}{\beta} \right) \left( \frac{RI_{\mu,\beta,\lambda}^{m+2} f(z)}{RI_{\mu,\beta,\lambda}^{m+1} f(z)} - \mu \frac{RI_{\mu,\beta,\lambda}^{m+1} f(z)}{RI_{\mu,\beta,\lambda}^{m} f(z)} + (\mu - 1) \right) + 1 < 1 + \gamma z, z \in U,
\]

then \( f \in I_{\mu}^{1}(m, \rho, \alpha, \beta), z \in U \).

As consequences of the above theorem, we have the following interesting corollary:
Corollary 2.6. Let \( f \in A(n), \rho = \frac{1}{2}, \lambda = 1, \beta > 0 \) and \( \alpha \) a real number with \( \alpha + \beta > 0 \).

(a) Let \( m = 1, \mu = 1 \). If \( \Re \left( \frac{\alpha + \beta}{\beta} \left( \frac{I_{\alpha,\beta} f(z)}{I_{\alpha,\beta}^3 f(z)} - \frac{I_{\alpha,\beta}^2 f(z)}{I_{\alpha,\beta} f(z)} \right) \right) > -\frac{1}{2}, z \in U, \) then

\[
\Re \left( \frac{I_{\alpha,\beta} f(z)}{I_{\alpha,\beta} f(z)} \right) > \frac{1}{2}, z \in U. \text{ That is } f \in I_n(1,1,\frac{1}{2},\alpha,\beta).
\]

(b) Let \( m = 1, \mu = 0 \) If \( \Re \left( \frac{\alpha + \beta}{\beta} \left( \frac{I_{\alpha,\beta} f(z)}{I_{\alpha,\beta}^3 f(z)} - 1 \right) \right) > -\frac{1}{2}, z \in U, \) the \( \Re \left( \frac{I_{\alpha,\beta} f(z)}{z} \right) > \frac{1}{2}, z \in U. \)

That is \( f \in I_n(1,0,\frac{1}{2},\alpha,\beta). \)

(c) Let \( m = 0, \mu = 1 \). If \( \Re \left( \frac{\alpha + \beta}{\beta} \left( \frac{I_{\alpha,\beta} f(z)}{I_{\alpha,\beta}^3 f(z)} \right) \right) > -\frac{1}{2}, z \in U, \) then

\[
\Re \left( \frac{I_{\alpha,\beta} f(z)}{f(z)} \right) > \frac{1}{2}, z \in U. \text{ That is } f \in I_n(0,1,\frac{1}{2},\alpha,\beta).
\]

(d) Let \( m = 0, \mu = 0 \) If \( \Re \left( \frac{\alpha + \beta}{\beta} \left( \frac{I_{\alpha,\beta} f(z)}{I_{\alpha,\beta}^3 f(z)} - 1 \right) \right) > -\frac{1}{2}, z \in U, \) then

\[
\Re \left( \frac{I_{\alpha,\beta} f(z)}{z} \right) > \frac{1}{2}, z \in U. \text{ That is } f \in I_n(0,0,\frac{1}{2},\alpha,\beta).
\]

\( \alpha = 0 \) in Corollary 2.6, we have

Corollary 2.7. Let \( f \in A(n). \)

(a) If \( \Re \left( \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} \right) > -\frac{1}{2}, z \in U, \) then \( f \) is convex of order \( 1/2 \)

(i.e. \( f \in K_n(1/2) \)).

(b) If \( \Re \left( \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} \right) > -\frac{1}{2}, z \in U, \) then \( \Re (f'(z) + zf''(z)) > \frac{1}{2}, z \in U. \)
(c) If \( \Re \left( \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) > \frac{3}{2}, \) \( z \in U, \) then \( f \) is starlike of order \( 1/2 \) (i.e. \( f \in S^*(1/2) \)).

(d) If \( f \) is convex of order \( 1/2 \), then \( f \in R_n(1/2) \).

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