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GENERALIZATION OF POPOVICIU TYPE INEQUALITIES FOR SYMMETRIC MEANS GENERATED BY CONVEX FUNCTIONS

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Abstract. In this paper an inequality of Popoviciu, which was improved by Vasić and Stanković [13], is generalized by using Green function. An extension of Popoviciu type inequality is introduced. The mean value theorems, n-exponential convexity and exponential convexity are presented for the differences of these inequalities and related Cauchy means are also generated.

Keywords: Cauchy means; convex function; exponentially convex function.

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1. Introduction

The inequality of Popoviciu as given by Vasić and Stanković in [13] can be written in the following form (see page 173 [10]):

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Theorem 1.1. *Let $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_n) \in [\alpha,\beta]^n$, $\mathbf{p} = (p_1,...,p_n)$ be a positive n-tuple such that $\sum_{i=1}^n p_i = 1$. Also let $f : [\alpha,\beta] \to \mathbb{R}$ be a convex function. Then*

$$f_{k,n}(\mathbf{x},\mathbf{p}) \leq \frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}), \tag{1}$$

where

$$f_{k,n}(\mathbf{x},\mathbf{p}) := \frac{1}{C_{k-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum_{j=1}^{k} p_{i_j} \right) f \left(\frac{\sum\limits_{j=1}^{k} p_{i_j} x_{i_j}}{\sum\limits_{j=1}^{k} p_{i_j}} \right).$$

By inequality (1), we write

$$\Upsilon_1(\mathbf{x},\mathbf{p};f) := \frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p}) \geq 0.$$

The mean value theorems and exponential convexity of the functional $\Upsilon_1(\mathbf{x},\mathbf{p};f)$ are given in [5] for a positive n-tuple $\mathbf{p}$. We used some special classes of convex functions to construct the exponential convexity in [5]. But in this paper, we generalize the results related to $\Upsilon_1(\mathbf{x},\mathbf{p};f)$. First, we give the generalization of Theorem 1.1, then the mean value theorems with new methods from [9]. A new extension of Popoviciu type inequality is introduced. We also employ the new exponential convexity method from [8] for functionals that appear in the sequel. In this way our results are more general than the corresponding results given in [5] as well as in [2].

2. Generalization of Popoviciu's Inequality

Consider the Green function $G : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ defined as

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{2}$$

The function G is convex and continuous w.r.t s and due to symmetry also w.r.t t.

For any function $h : [\alpha,\beta] \to \mathbb{R}$, $h \in C^2([\alpha,\beta])$, we have

$$h(x) = \frac{\beta - x}{\beta - \alpha} h(\alpha) + \frac{x - \alpha}{\beta - \alpha} h(\beta) + \int_{\alpha}^{\beta} G(x,s) h''(s) ds, \tag{3}$$

where the function G is defined in (2) (see [14]).

In Theorem 1.1 we have that p_i $(i=1,\dots,n)$ are positive real numbers. Now we give the generalization of that result for real values of p_i $(i=1,\dots,n)$ with $\sum_{i=1}^n p_i = 1$ using the Green function as defined in (2).

Theorem 2.1. *Let $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\dots,x_n) \in [\alpha,\beta]^n$, $\mathbf{p} = (p_1,\dots,p_n)$ be a real n-tuple such that $\sum_{j=1}^k p_{i_j} \neq 0$ for any $1 \leq i_1 < \dots < i_k \leq n$ and $\sum_{i=1}^n p_i = 1$. Also let $\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < \dots < i_k \leq n$. Then the following statements are equivalent:*

(i) For every continuous convex function $f : [\alpha,\beta] \to \mathbb{R}$

$$f_{k,n}(\mathbf{x},\mathbf{p}) \leq \frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}), \tag{4}$$

where

$$f_{k,n}(\mathbf{x},\mathbf{p}) := \frac{1}{C_{k-1}^{n-1}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k p_{i_j} \right) f \left(\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}} \right).$$

(ii) For all $s \in [\alpha,\beta]$

$$G_{k,n}(\mathbf{x},s;\mathbf{p}) \leq \frac{n-k}{n-1} G_{1,n}(\mathbf{x},s;\mathbf{p}) + \frac{k-1}{n-1} G_{n,n}(\mathbf{x},s;\mathbf{p}), \tag{5}$$

where

$$G_{k,n}(\mathbf{x},s;\mathbf{p}) := \frac{1}{C_{k-1}^{n-1}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k p_{i_j} \right) G \left(\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}}, s \right); \quad 1 \leq k \leq n,$$

for the function $G : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ defined in (2).

Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both (4) and (5).

Proof. (i)$\Rightarrow$(ii): Let (i) be valid. Then as the function $G(\cdot,s)$ $(s \in [\alpha,\beta])$ is also continuous and convex, it follows that also for this function (4) holds, i.e. (5) is valid.

(ii)$\Rightarrow$(i): Let $f : [\alpha,\beta] \to \mathbb{R}$ be a convex function, $f \in C^2([\alpha,\beta])$ and (ii) holds. Then, we can represent function f in the form (3). Now by means of some simple calculations we can write

$$
\begin{aligned}
&\tfrac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \tfrac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p}) \\
&= \int_{\alpha}^{\beta} \left(\tfrac{n-k}{n-1} G_{1,n}(\mathbf{x},s;p) + \tfrac{k-1}{n-1} G_{n,n}(\mathbf{x},s;p) - G_{k,n}(\mathbf{x},s;p) \right) f''(s)\,ds.
\end{aligned} \tag{6}
$$

By the convexity of f, we have $f''(s) \geq 0$ for all $s \in [\alpha,\beta]$. Hence, if for every $s \in [\alpha,\beta]$, (5) is valid then it follows that for every convex function $f : [\alpha,\beta] \to \mathbb{R}$, with $f \in C^2([\alpha,\beta])$, (4) is valid.

Here we can eliminate the differentiability condition due to the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials (see [10], page 172).

Analogous to the above proof we can give the proof of the last part of our theorem.

Remark 2.2. *Consider $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_n) \in [\alpha,\beta]^n$, $\mathbf{p} = (p_1,...,p_n)$ be a real n-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq n$ and $\sum_{i=1}^{n} p_i = 1$. Also assume that $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < ... < i_k \leq n$.*
(a) If for all $s \in [\alpha,\beta]$ the inequality holds in (5) then from above theorem we have

$$
\Upsilon_2(f) := \Upsilon_2(\mathbf{x},\mathbf{p};f) := \frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p}) \geq 0. \tag{7}
$$

(b) If for all $s \in [\alpha,\beta]$ the reverse inequality holds in (5) then from above theorem we have

$$
\bar{\Upsilon}_2(f) := \bar{\Upsilon}_2(\mathbf{x},\mathbf{p};f) := f_{k,n}(\mathbf{x},\mathbf{p}) - \frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) - \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) \geq 0. \tag{8}
$$

Remark 2.3. *Note that in the case when* **p** *is a positive n-tuple, the inequality (4) gives (1).*

Now we give two mean value theorems.

Theorem 2.4. *Let $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_n) \in [\alpha,\beta]^n$, $\mathbf{p} = (p_1,...,p_n)$ be a real n-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq n$ and $\sum_{i=1}^{n} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < ... < i_k \leq n$ and $f : [\alpha,\beta] \to \mathbb{R}$ be a function such that*

$f \in C^2([\alpha,\beta])$. If for all $s \in [\alpha,\beta]$ the inequality holds in (5) or if for all $s \in [\alpha,\beta]$ the reverse inequality holds in (5), then there exists $\xi \in [\alpha,\beta]$ such that

$$\frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p}) = \frac{1}{2} f''(\xi) \Upsilon_2(\mathbf{x},\mathbf{p};f_0),$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$ and $f_0(x) = x^2$.

Proof. By the assumption, we have that the function $f^{''}$ is continuous and

$$\frac{n-k}{n-1} G_{1,n}(\mathbf{x},s;p) + \frac{k-1}{n-1} G_{n,n}(\mathbf{x},s;p) - G_{k,n}(\mathbf{x},s;p)$$

does not change its positivity on $[\alpha,\beta]$. Also for our function f the equality (6) is valid and now by applying the integral mean value theorem we get that there exists some $\xi \in [\alpha,\beta]$ such that

$$\begin{aligned} &\tfrac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \tfrac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p}) \\ &= f''(\xi) \int_{\alpha}^{\beta} \left(\tfrac{n-k}{n-1} G_{1,n}(\mathbf{x},s;p) + \tfrac{k-1}{n-1} G_{n,n}(\mathbf{x},s;p) - G_{k,n}(\mathbf{x},s;p) \right) ds. \end{aligned} \tag{9}$$

Next by the definition of the function G, we observe that

$$\int_{\alpha}^{\beta} G(t,s)\,ds = \frac{1}{2}(t-\alpha)(t-\beta). \tag{10}$$

We calculate the integral on right side of (9) with the help of (10) as follows:

$$\frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p})$$

$$= f''(\xi) \left(\begin{array}{l} \frac{n-k}{n-1} \sum_{i=1}^{n} p_i \int_{\alpha}^{\beta} G(x_i,s)\,ds + \frac{k-1}{n-1} \int_{\alpha}^{\beta} G(x^2,s)\,ds \\ -\frac{1}{C_{k-1}^{n-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} p_{i_j} \right) \int_{\alpha}^{\beta} G\left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}, s \right) ds \end{array} \right)$$

$$= \frac{f''(\xi)}{2} \left(\begin{array}{l} \frac{n-k}{n-1} \sum_{i=1}^{n} p_i (x_i-\alpha)(x_i-\beta) + \frac{k-1}{n-1} (\bar{x}-\alpha)(\bar{x}-\beta) \\ -\frac{1}{C_{k-1}^{n-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} p_{i_j} \right) \left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} - \alpha \right) \left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} - \beta \right) \end{array} \right)$$

$$= \frac{f''(\xi)}{2}\left(\frac{n-k}{n-1}\sum_{i=1}^{n} p_i x_i^2 + \frac{k-1}{n-1}\bar{x}^2 - \frac{1}{C_{k-1}^{n-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{j=1}^{k} p_{i_j}\right)\left(\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right)^2\right),$$

$= \frac{1}{2}f''(\xi)\Upsilon_2(\mathbf{x},\mathbf{p};f_0)$, which completes the proof.

Theorem 2.5. *Let $n,k \in \mathbb{N}$, $n \ge 3$, $2 \le k \le n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_n) \in [\alpha,\beta]^n$, $\mathbf{p} = (p_1,...,p_n)$ be a real n-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \le i_1 < ... < i_k \le n$ and $\sum_{i=1}^{n} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \le i_1 < ... < i_k \le n$ and $f,g : [\alpha,\beta] \to \mathbb{R}$ be functions such that $f,g \in C^2([\alpha,\beta])$. If for all $s \in [\alpha,\beta]$ (5) holds or if for all $s \in [\alpha,\beta]$ the reverse inequality holds in (5), then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\frac{n-k}{n-1}f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1}f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p})}{\frac{n-k}{n-1}g_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1}g_{n,n}(\mathbf{x},\mathbf{p}) - g_{k,n}(\mathbf{x},\mathbf{p})} = \frac{f''(\xi)}{g''(\xi)}. \tag{11}$$

for non zero values of denominators.

Proof. Consider the function

$$\begin{aligned} h(t) = &\left(\tfrac{n-k}{n-1}g_{1,n}(\mathbf{x},\mathbf{p}) + \tfrac{k-1}{n-1}g_{n,n}(\mathbf{x},\mathbf{p}) - g_{k,n}(\mathbf{x},\mathbf{p})\right) f(t) \\ &- \left(\tfrac{n-k}{n-1}f_{1,n}(\mathbf{x},\mathbf{p}) + \tfrac{k-1}{n-1}f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p})\right) g(t). \end{aligned}$$

which is defined on $[\alpha,\beta]$ and also $h \in C^2([\alpha,\beta])$. Therefore we can apply Theorem 2.4 on the function h and then it follows that there exists some $\xi \in [\alpha,\beta]$ such that

$$\frac{n-k}{n-1}h_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1}h_{n,n}(\mathbf{x},\mathbf{p}) - h_{k,n}(\mathbf{x},\mathbf{p}) = \frac{h''(\xi)}{2}\left[\Upsilon_2(\mathbf{x},\mathbf{p};f_0)\right], \tag{12}$$

where $f_0(x) = x^2$.

After some simple calculations we get that the LHS of this equation equals to zero. The term in the square brackets on the RHS of (12) is nonzero, because otherwise, from the Theorem 2.4 applied on the function h, we would have that the denominator on the LHS of (11) equals to zero, which contradicts the assumption of this theorem. Hence

$$h''(\xi) = 0,$$

which completes the proof.

Next we give an extension of an inequality (6.4) page 174 of [10] by Popoviciu.

Theorem 2.6. *Let $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $(0,a] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\dots,x_n) \in (0,a]^n$ such that $\sum_{i=1}^{n} x_i \in (0,a]$. Also let $f : (0,a] \to \mathbb{R}$ be a function such that $\frac{f(x)}{x}$ is convex. Then*

$$f_{k,n}(\mathbf{x}) \leq \frac{n-k}{n-1} f_{1,n}(\mathbf{x}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}), \tag{13}$$

where

$$f_{k,n}(\mathbf{x}) := \frac{1}{C_{k-1}^{n-1}} \sum_{1 \leq i_1 < \dots < i_k \leq n} f\left(\sum_{j=1}^{k} x_{i_j}\right).$$

Proof. For $k=2$ and $n=3$ the result is followed by inequality (6.4) on page 174 of [10]. Next for $k>2$ and $n>3$ the result is followed by Theorem 6.9 on page 176 of [10]. □

Hence for convex function $\frac{f(x)}{x}$, (13) gives

$$\Upsilon_3(f) := \Upsilon_3(\mathbf{x};f) := \frac{n-k}{n-1} f_{1,n}(\mathbf{x}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}) - f_{k,n}(\mathbf{x}) \geq 0. \tag{14}$$

The following lemma is given in [1]:

Lemma 2.7. *Let $h \in C^2(I)$ for an interval $I \subset \mathbb{R} \setminus \{0\}$ and consider $m, M \in \mathbb{R}$ such that*

$$m \leq \frac{x^2 h''(x) - 2x h'(x) + 2h(x)}{x^3} \leq M.$$

Also let h_1, h_2 be real valued functions defined on I as follows

$$h_1(x) = M\frac{x^3}{2} - h(x),$$

$$h_2(x) = h(x) - m\frac{x^3}{2}.$$

Then $\frac{h_1(x)}{x}$ and $\frac{h_2(x)}{x}$ are convex.

Theorem 2.8. *Let $[\alpha,\beta] \subset \mathbb{R}^+$ and $f \in C([\alpha,\beta])$ then there exists $\xi \in [\alpha,\beta]$ such that*

$$\Upsilon_3(\mathbf{x};f) = \frac{\xi^2 f''(\xi) - 2\xi f'(\xi) + 2f(\xi)}{2\xi^3} \Upsilon_3(\mathbf{x};x^3).$$

Proof. The idea of proof is same as given in Theorem 2.3 of [5]. □

Theorem 2.9. *Let $[\alpha,\beta]\subset\mathbb{R}^+$ and $f,g\in C([\alpha,\beta])$ then there exists $\xi\in[\alpha,\beta]$ such that*

$$\frac{\Upsilon_3(\mathbf{x};f)}{\Upsilon_3(\mathbf{x};g)}=\frac{\xi^2 f''(\xi)-2\xi f'(\xi)+2f(\xi)}{\xi^2 g''(\xi)-2\xi g'(\xi)+2g(\xi)},$$

for non zero values of denominators.

Proof. The idea of proof is same as given in Theorem 2.4 of [5].

3. Exponential Convexity

The notion of *n-exponentially convex* function and the following properties of *exponentially convex* function defined on an interval $I\subset\mathbb{R}$, are given in [8].

Definition 1. A function $g:I\to\mathbb{R}$ is called *n-exponentially convex* in the Jensen sense if

$$\sum_{i,j=1}^{n} a_i a_j g\left(\frac{x_i+x_j}{2}\right)\geq 0$$

holds for every $a_i\in\mathbb{R}$ and every $x_i\in I$, $i=1,2,...,n$.

A function $g:I\to\mathbb{R}$ is *n-exponentially convex* if it is *n-exponentially convex* in the Jensen sense and continuous on I.

Remark 3.1. *From the definition it is clear that* 1*-exponentially convex functions in the Jensen sense are in fact the nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m\in\mathbb{N}$, $m\leq n$.*

Definition 2. A function $g:I\to\mathbb{R}$ is exponentially convex in the Jensen sense, if it is n-exponentially convex in the Jensen sense for all $n\in\mathbb{N}$.

A function $g:I\to\mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 3.2. *It is easy to see that a positive function $g:I\to\mathbb{R}$ is* log*-convex in the Jensen sense if and only if it is* 2*-exponentially convex in the Jensen sense, that is*

$$a_1^2 g(x)+2a_1a_2 g\left(\frac{x+y}{2}\right)+a_2^2 g(y)\geq 0$$

holds for every $a_1, a_2 \in \mathbb{R}$ *and* $x, y \in I$.

Similarly, if g *is* 2*-exponentially convex, then* g *is* log*-convex. Conversely, if* g *is* log*-convex and continuous, then* g *is* 2*-exponentially convex.*

Divided differences are fertile to study functions having different degree of smoothness.

Definition 3. The second order divided difference of a function $g : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$[y_i; g] = g(y_i), \quad i = 0, 1, 2$$

$$[y_i, y_{i+1}; g] = \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1$$

$$[y_0, y_1, y_2; g] = \frac{[y_1, y_2; g] - [y_0, y_1; g]}{y_2 - y_0}. \tag{15}$$

Remark 3.3. *The value* $[y_0, y_1, y_2; g]$ *is independent of the order of the points* y_0, y_1*, and* y_2*. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows:* $\forall\ y_0,\ y_1,\ y_2 \in I$ *such that* $y_2 \neq y_0$

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; g] = [y_0, y_0, y_2; g] = \frac{g(y_2) - g(y_0) - g^{'}(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}$$

provided that g' *exists, and furthermore, taking the limits* $y_i \to y_0$*,* $i = 1, 2$ *in (15), we get*

$$[y_0, y_0, y_0; g] \ = \lim_{y_i \to y_0} [y_0, y_1, y_2; g] = \frac{g^{''}(y_0)}{2} \text{ for } i = 1, 2$$

provided that $g^{''}$ *exist on* I.

Theorem 3.4. *Assume* $J \subset \mathbb{R}$ *is an interval, and assume* $\Lambda = \{\phi_t \mid t \in J\}$ *is a family of continuous functions defined on an interval* $I \subset \mathbb{R}$*, such that the function* $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ *is n-exponentially convex in the Jensen sense on* J *for every three mutually different points* $y_0, y_1, y_2 \in I$*. Consider* $\Upsilon_2(f)$ *as given in (7). Then* $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ *is an n-exponentially convex function in the Jensen sense on* J*. If the function* $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ *is continuous, then it is n-exponentially convex on* J*.*

Proof. Let $t_k, t_l \in J$, $t_{kl} := \frac{t_k+t_l}{2}$ and $b_k, b_l \in \mathbb{R}$ for $k,l = 1,2,...,n$, and define the function ω on I by

$$\omega := \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}.$$

Then ω is continuous on I being the linear combination of continuous functions. Also by hypothesis the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is n-exponentially convex in the Jensen sense, therefore we have

$$[y_0, y_1, y_2; \omega] = \sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] \geq 0,$$

which implies that ω is a convex function on I. Therefore we have $\Upsilon_2(\omega) \geq 0$, which yields by the linearity of Υ_2, that

$$\sum_{k,l=1}^{n} b_k b_l \Upsilon_2(\phi_{t_{kl}}) \geq 0.$$

We conclude that the function $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J.

If the function $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ is continuous on J, then it is n-exponentially convex on J by definition.

As a consequence of the above theorem we can give the following corollaries.

Corollary 3.5. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\Upsilon_2(f)$ as given in (7). Then $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on J. If the function $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ is continuous, then it is exponentially convex on J.*

Corollary 3.6. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t : t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\Upsilon_2(f)$ as given in (7). Then the following two statements hold:*

(i) *If the function $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ is continuous, then it is 2-exponentially convex on J, and thus* log-*convex.*

(ii) *If the function $t \to \Upsilon_2(\phi_t)$ $(t \in J)$ is positive, then for every $s,t,u,v \in J$, such that $s \leq u$ and $t \leq v$, we have*

$$\mathfrak{u}_{s,t}(\Upsilon_2,\Lambda) \leq \mathfrak{u}_{u,v}(\Upsilon_2,\Lambda) \tag{16}$$

where

$$\mathfrak{u}_{s,t}(\Upsilon_2,\Lambda) := \begin{cases} \left(\frac{\Upsilon_2(\phi_s)}{\Upsilon_2(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Upsilon_2(\phi_s)}{\Upsilon_2(\phi_s)}\right), & s = t \end{cases} \tag{17}$$

for $\phi_s, \phi_t \in \Lambda$ and we consider that the function $t \to \Upsilon_2(\phi_t)$ is differentiable when $t = s$.

Proof.

(i) See Remark 3.2 and Theorem 3.4.

(ii) From the definition of a convex function ψ on J, we have the following inequality (see [10, page 2])

$$\frac{\psi(s) - \psi(t)}{s - t} \leq \frac{\psi(u) - \psi(v)}{u - v}, \tag{18}$$

$\forall s,t,u,v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.

By (i), $s \to \Upsilon_2(\phi_s)$, $s \in J$ is log-convex, and hence (18) shows with $\psi(s) = \log \Upsilon_2(\phi_s)$, $s \in J$ that

$$\frac{\log \Upsilon_2(\phi_s) - \log \Upsilon_2(\phi_t)}{s-t} \leq \frac{\log \Upsilon_2(\phi_u) - \log \Upsilon_2(\phi_v)}{u-v} \tag{19}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (16). For $s = t$ or $u = v$ (16) follows from (19) by taking limit.

Remark 3.7. *Note that the results from Theorem 3.4, Corollary 3.5, Corollary 3.6 are valid when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions ϕ_t such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 3.3 and suitable characterization of convexity.*

Remark 3.8. *The results similar to Theorem 3.4, Corollary 3.5 and Corollary 3.6 can also be given for* $\bar{\Upsilon}_2(f)$ *as defined in (8).*

Remark 3.9. *A refinement of the inequality of Popoviciu from* [11] *is given by Niculescu and Popovici in* [7]. *Also an integral version of Theorem 1.1 is given by Niculescu in* [6]. *The results similar to Theorem 2.1, Theorem 2.4, Theorem 2.5, Theorem 3.4, Corollary 3.5 and Corollary 3.6 can also be given for refinement results of* [7] *as well as for integral version of Popociciu's inequality.*

Theorem 3.10. *Assume* $J \subset \mathbb{R}$ *is an interval, and assume* $\Phi = \{\phi_t \mid t \in J\}$ *is a family of functions defined on an interval* $(0,a] \subset \mathbb{R}$*, such that the function* $t \to [y_0, y_1, y_2; \frac{\phi_t(y)}{y}]$ $(t \in J)$ *is n-exponentially convex in the Jensen sense on J for every three mutually different points* $y_0, y_1, y_2 \in (0,a]$. *Consider* $\Upsilon_3(f)$ *as given in (14). Then* $t \to \Upsilon_3(\phi_t)$ $(t \in J)$ *is an n-exponentially convex function in the Jensen sense on J. If the function* $t \to \Upsilon_3(\phi_t)$ $(t \in J)$ *is continuous, then it is n-exponentially convex on J.*

Proof. Proof is similar to the proof of Theorem 3.4, but we consider the convex function $\frac{f(y)}{y}$ instead of f.

As a consequence of the above theorem we can give the following corollaries.

Corollary 3.11. *Assume* $J \subset \mathbb{R}$ *is an interval, and assume* $\Phi = \{\phi_t \mid t \in J\}$ *is a family of functions defined on an interval* $(0,a] \subset \mathbb{R}$*, such that the function* $t \to [y_0, y_1, y_2; \frac{\phi_t(y)}{y}]$ $(t \in J)$ *is exponentially convex in the Jensen sense on J for every three mutually different points* $y_0, y_1, y_2 \in (0,a]$. *Consider* $\Upsilon_3(f)$ *as given in (14). Then* $t \to \Upsilon_3(\phi_t)$ $(t \in J)$ *is an exponentially convex function in the Jensen sense on J. If the function* $t \to \Upsilon_3(\phi_t)$ $(t \in J)$ *is continuous, then it is exponentially convex on J.*

Corollary 3.12. *Assume* $J \subset \mathbb{R}$ *is an interval, and assume* $\Phi = \{\phi_t : t \in J\}$ *is a family of functions defined on an interval* $(0,a] \subset \mathbb{R}$*, such that the function* $t \to [y_0, y_1, y_2; \frac{\phi_t(y)}{y}]$ $(t \in J)$ *is* 2*-exponentially convex in the Jensen sense on J for every three mutually different points* $y_0, y_1, y_2 \in (0,a]$. *Consider* $\Upsilon_3(f)$ *as given in (14). Then the following two statements hold:*

(i) *If the function $t \to \Upsilon_3(\phi_t)$ $(t \in J)$ is continuous, then it is* 2*-exponentially convex on J, and thus* log*-convex.*

(ii) *If the function $t \to \Upsilon_3(\phi_t)$ $(t \in J)$ is positive then for every $s,t,u,v \in J$, such that $s \le u$ and $t \le v$, we have*

$$\bar{\mathrm{u}}_{s,t}(\Upsilon_3,\Phi) \le \bar{\mathrm{u}}_{u,v}(\Upsilon_3,\Phi) \tag{20}$$

where

$$\bar{\mathrm{u}}_{s,t}(\Upsilon_3,\Phi) := \begin{cases} \left(\frac{\Upsilon_3(\phi_s)}{\Upsilon_3(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Upsilon_3(\phi_s)}{\Upsilon_3(\phi_s)}\right), & s = t \end{cases} \tag{21}$$

for $\phi_s, \phi_t \in \Phi$ and we consider that the function $t \to \Upsilon_3(\phi_t)$ is differentiable when $t = s$.

Proof. Proof is similar to the proof of Corollary 3.6, but we consider the convex function $\frac{f(y)}{y}$ instead of f.

Remark 3.13. *Note that the results from Theorem 3.10, Corollary 3.11, Corollary 3.12 are valid when two of the points $y_0, y_1, y_2 \in (0,a]$ coincide, say $y_1 = y_0$, for a family of differentiable functions ϕ_t such that the function $t \to [y_0, y_1, y_2; \frac{\phi_t(y)}{y}]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 3.3 and suitable characterization of convexity.*

The following result is given in [3].

Theorem 3.14. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of twice differentiable functions defined on an interval $I \subset \mathbb{R}$ such that the function $t \mapsto \phi_t''(x)$ $(t \in J)$ is exponentially convex for every fixed $x \in I$. Then the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense for any three points y_0, y_1, $y_2 \in I$.*

4. Applications to Cauchy Means

In this section we generate new Cauchy means with the help of some classes of functions from [8].

Throughout in Examples (4.1-4.4) we mention that the functional Υ_2, defined in (7) is linear on the vector space $C(I)$ for the interval $I \subset \mathbb{R}$, and $\Upsilon_2(f) \geq 0$ for every continuous convex function defined on I. We also assume that $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $\mathbf{x} = (x_1, ..., x_n) \in I^n$, $\mathbf{p} = (p_1, ..., p_n)$ be real n-tuple such that $\sum_{i=1}^n p_i = 1$.

Example 4.1. *Assume $I = \mathbb{R}$ and consider the class of continuous convex functions*

$$\Lambda_1 := \{\phi_t : \mathbb{R} \to [0, \infty) \mid t \in \mathbb{R}\},$$

where

$$\phi_t(x) := \begin{cases} \frac{1}{t^2} e^{tx}; \ t \neq 0, \\ \frac{1}{2} x^2; \ t = 0. \end{cases}$$

Then $t \mapsto \phi_t''(x)$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in \mathbb{R}$ (see [4]*), thus by Theorem 3.14, the function $t \mapsto [y_0, y_1, y_2; \phi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in \mathbb{R}$.*

By applying Corollary 3.5 with $\Lambda = \Lambda_1$, we get the exponential convexity of $t \mapsto \Upsilon_2(\phi_t)$ $(t \in \mathbb{R})$ in the Jensen sense. This mapping is also differentiable, therefore exponentially convex, and the expression in (17) has the form

$$\mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_1) = \begin{cases} \left(\frac{\Upsilon_2(\phi_s)}{\Upsilon_2(\phi_t)} \right)^{\frac{1}{s-t}}, s \neq t, \\ \exp\left(\frac{\Upsilon_2(id\, \phi_s)}{\Upsilon_2(\phi_s)} - \frac{2}{s} \right), s = t \neq 0, \\ \exp\left(\frac{\Upsilon_2(id\, \phi_0)}{3\Upsilon_2(\phi_0)} \right), s = t = 0, \end{cases}$$

where "id" means the identity function on $\mathbb{R}$.

From (16) we have the monotonicity of the functions $\mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_1)$ in both parameters.

Suppose $\Upsilon_2(\phi_t) > 0$ $(t \in \mathbb{R})$, $a := \min\{x_1, ..., x_n\}$, $b := \max\{x_1, ..., x_n\}$, and let

$$\mathfrak{M}_{s,t}(\Upsilon_2, \Lambda_1) := \log \mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_1); \quad s, t \in \mathbb{R}.$$

Then from Theorem 2.5 we have

$$a \leq \mathfrak{M}_{s,t}(\Upsilon_2, \Lambda_1) \leq b,$$

and thus $\mathfrak{M}_{s,t}(\Upsilon_2, \Lambda_1)$ $(s,t \in \mathbb{R})$ *are means. The monotonicity of these means is followed by (16).*

Example 4.2. *Assume* $I = (0,\infty)$ *and consider the class of continuous convex functions*

$$\Lambda_2 = \{\psi_t : (0,\infty) \to \mathbb{R} \mid t \in \mathbb{R}\},$$

where

$$\psi_t(x) := \begin{cases} \frac{x^t}{t(t-1)};\ t \neq 0,1, \\ -\log x;\ t = 0, \\ x\log x;\ t = 1. \end{cases}$$

Then $t \mapsto \psi_t''(x) = x^{t-2} = e^{(t-2)\log x}$ $(t \in \mathbb{R})$ *is exponentially convex for every fixed* $x \in (0,\infty)$.

By similar arguments as given in Example 4.1 we get the exponential convexity of $t \mapsto \Upsilon_2(\psi_t)$ $(t \in \mathbb{R})$ *in the Jensen sense. This mapping is differentiable too, therefore exponentially convex. It is easy to calculate that (17) can be written as*

$$u_{s,t}(\mathbf{x},\mathbf{p},\Upsilon_2,\Lambda_2) = \begin{cases} \left(\frac{\Upsilon_2(\psi_s)}{\Upsilon_2(\psi_t)}\right)^{\frac{1}{s-t}};\ s \neq t, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Upsilon_2(\psi_s\psi_0)}{\Upsilon_2(\psi_s)}\right);\ s = t \neq 0,1, \\ \exp\left(1 - \frac{\Upsilon_2(\psi_0^2)}{2\Upsilon_2(\psi_0)}\right);\ s = t = 0, \\ \exp\left(-1 - \frac{\Upsilon_2(\psi_0\psi_1)}{2\Upsilon_2(\psi_1)}\right);\ s = t = 1. \end{cases}$$

Suppose $\Upsilon_2(\psi_t) > 0$ $(t \in \mathbb{R})$, *and let* $a := \min\{x_1,...,x_n\}$, $b := \max\{x_1,...,x_n\}$.

By Theorem 2.5, we can check that

$$a \leq \mathfrak{u}_{s,t}(\mathbf{x},\mathbf{p},\Upsilon_2,\Lambda_2) \leq b; \quad s,t \in \mathbb{R}.$$

The means $\mathfrak{u}_{s,t}(\mathbf{x},\mathbf{p},\Upsilon_2,\Lambda_2)$ $(s,t \in \mathbb{R})$ *are continuous, symmetric and monotone in both parameters (by use of (16)).*

Let $s,t,r \in \mathbb{R}$ *such that* $r \neq 0$. *By the substitutions* $s \to \frac{s}{r}$, $t \to \frac{t}{r}$, $(x_1,\ldots,x_n) \to (x_1^r,\ldots,x_n^r)$ *in (22), we get*

$$\bar{a} \leq \mathfrak{u}_{s,r}(\mathbf{x}^r, \mathbf{p}, \Upsilon_2, \Lambda_2) \leq \bar{b},$$

where $\bar{a} := \min\{x_1^r, \ldots, x_n^r\}$ *and* $\bar{b} := \max\{x_1^r, \ldots, x_n^r\}$. *Thus new means can be defined with three parameters:*

$$\mathfrak{u}_{s,t,r}(\mathbf{x}, \mathbf{p}, \Upsilon_2, \Lambda_2) := \begin{cases} (\mathfrak{u}_{s/r,t/r}(\mathbf{x}^r, \mathbf{p}, \Upsilon_2, \Lambda_2))^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\log \mathbf{x}, \mathbf{p}, \Upsilon_2, \Lambda_1); & r = 0, \end{cases}$$

where $\log \mathbf{x} = (\log x_1, ..., \log x_n)$.

The monotonicity of these three parameter means is followed by the monotonicity and continuity of the two parameter means.

Example 4.3. *Assume* $I = (0, \infty)$ *and consider the class of continuous convex functions*

$$\Lambda_3 = \{\eta_t : (0, \infty) \to (0, \infty) \mid t \in (0, \infty)\},$$

where

$$\eta_t(x) := \begin{cases} \frac{t^{-x}}{\log^2 t}; t \neq 1, \\ \frac{x^2}{2}; \quad t = 1. \end{cases}$$

$t \mapsto \eta_t''(x)$ $(t \in (0, \infty))$ *is exponentially convex for every fixed* $x \in (0, \infty)$, *being the restriction of the Laplace transform of a nonnegative function (see* [4] *or* [12] *page 210).*

We can get the exponential convexity of $t \mapsto \Upsilon_2(\psi_t)$ $(t \in \mathbb{R}^+)$ *as in Example 4.1. For the class* Λ_3, *(17) has the form*

$$\mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_3) = \begin{cases} \left(\frac{\Upsilon_2(\eta_s)}{\Upsilon_2(\eta_t)}\right)^{\frac{1}{s-t}}; s \neq t, \\ \exp\left(-\frac{2}{s \log s} - \frac{\Upsilon_2(id\eta_s)}{s\Upsilon_2(\eta_s)}\right); \ s = t \neq 1, \\ \exp\left(-\frac{\Upsilon_2(id\eta_1)}{3\Upsilon_2(\eta_1)}\right); \ s = t = 1. \end{cases}$$

The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_3)$ $(s, t \in (0, \infty))$ *comes from (16).*

Suppose $\Upsilon_2(\eta_t) > 0$ $(t \in (0, \infty))$, *and let* $a := \min\{x_1, ..., x_n\}$, $b := \max\{x_1, ..., x_n\}$, *and define*

$$\mathfrak{M}_{s,t}(\Upsilon_2, \Lambda_3) := -L(s,t) \log \mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_3), \quad s, t \in (0, \infty),$$

where $L(s,t)$ is the well known logarithmic mean

$$L(s,t) := \begin{cases} \frac{s-t}{\log s - \log t}; & s \neq t, \\ t; & s = t. \end{cases}$$

From Theorem 2.5 we have

$$a \leq \mathfrak{M}_{s,t}(\Upsilon_2, \Lambda_3) \leq b, \quad s,t \in (0,\infty),$$

and therefore we get means.

Example 4.4. *Assume $I = (0,\infty)$ and consider the class of continuous convex functions*

$$\Lambda_4 = \{\gamma_t : (0,\infty) \to (0,\infty) \mid t \in (0,\infty)\},$$

where

$$\gamma_t(x) := \frac{e^{-x\sqrt{t}}}{t}.$$

$t \mapsto \gamma_t''(x) = e^{-x\sqrt{t}}$, $t \in (0,\infty)$ *is exponentially convex for every fixed $x \in (0,\infty)$, being the restriction of the Laplace transform of a non-negative function (see* [4] *or* [12] *page 214).*

As before $t \mapsto \Upsilon_2(\psi_t)$ $(t \in \mathbb{R}^+)$ is exponentially convex and differentiable. For the class Λ_4, (17) becomes

$$\mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_4) = \begin{cases} \left(\frac{\Upsilon_2(\gamma_s)}{\Upsilon_2(\gamma_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp\left(-\frac{1}{t} - \frac{\Upsilon_2(id\gamma_t)}{2\sqrt{t}\Upsilon_2(\gamma_t)}\right); & s = t, \end{cases}$$

where 'id' means the identity function on $(0,\infty)$. The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_4)$ $(s,t \in (0,\infty))$ is followed by (16).

Suppose $\Upsilon_2(\eta_t) > 0$ $(t \in (0,\infty))$, let $a := \min\{x_1, \ldots, x_n\}$, $b := \max\{x_1, \ldots, x_n\}$, and define

$$\mathfrak{M}_{s,t}(\Upsilon_2, \Lambda_4) := -(\sqrt{s} + \sqrt{t}) \log \mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_4), \quad s,t \in (0,\infty).$$

Then Theorem 2.5 yields that

$$a \leq \mathfrak{M}_{s,t}(\Upsilon_2, \Lambda_4) \leq b,$$

thus we have new means.

For remaining Examples (4.5-4.8) we assume that $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $I = (0,\infty)$, $\mathbf{x} = (x_1,...,x_n) \in I^n$ such that $\underline{x} = \sum_{i=1}^{n} x_i \in I$ and consider the linear functionals Υ_3 defined in (14).

Example 4.5. *Consider the class of continuous convex functions*

$$\Phi_1 := \{\tau_t : (0,\infty) \to (0,\infty) \mid t \in \mathbb{R}\},$$

where

$$\tau_t(x) := \begin{cases} \frac{xe^{tx}}{t^2};\ t \neq 0, \\ \frac{x^3}{2};\ t = 0. \end{cases}$$

Then $t \mapsto \left(\frac{\tau_t(x)}{x}\right)''$ $(t \in \mathbb{R})$ *is exponentially convex for every fixed* $x \in \mathbb{R}$ *(see* [4]*), thus by Theorem 3.14, the function* $t \mapsto [y_0,y_1,y_2;\phi_t]$, $t \in \mathbb{R}$ *is exponentially convex in the Jensen sense for every three mutually different points* $y_0,y_1,y_2 \in \mathbb{R}$.

By applying Corollary 3.11 with $\Lambda = \Phi_1$, *we get the exponential convexity of* $t \mapsto \Upsilon_3(\phi_t)$ $(t \in \mathbb{R})$ *in the Jensen sense. This mapping is also differentiable, therefore exponentially convex, and the expression in (21) has the form*

$$\bar{\mathfrak{u}}_{s,t}(\Upsilon_3,\Phi_1) =$$

$$\begin{cases} \left(\frac{t^2}{s^2} \frac{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i e^{sx_i} + \frac{k-1}{n-1}\underline{x}e^{s\underline{x}} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right) e^{s\left(\sum_{j=1}^{k} x_{i_j}\right)}}{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i e^{tx_i} + \frac{k-1}{n-1}\underline{x}e^{t\underline{x}} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right) e^{t\left(\sum_{j=1}^{k} x_{i_j}\right)}} \right)^{\frac{1}{s-t}} & ; s \neq t, s,t \neq 0, \\ \exp\left(\frac{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i^2 e^{sx_i} + \frac{k-1}{n-1}\underline{x}^2 e^{s\underline{x}} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right)^2 e^{s\left(\sum_{j=1}^{k} x_{i_j}\right)}}{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i e^{sx_i} + \frac{k-1}{n-1}\underline{x}e^{s\underline{x}} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right) e^{s\left(\sum_{j=1}^{k} x_{i_j}\right)}} - \frac{2}{s} \right) & ; s = t \neq 0, \\ \exp\left(\frac{2}{s^2} \frac{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i e^{sx_i} + \frac{k-1}{n-1}\underline{x}e^{s\underline{x}} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right) e^{s\left(\sum_{j=1}^{k} x_{i_j}\right)}}{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i^3 + \frac{k-1}{n-1}\underline{x}^3 - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right)^3} \right) & ; s \neq 0, \\ \exp\left(\frac{1}{3} \frac{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i^4 e^{sx_i} + \frac{k-1}{n-1}\underline{x}^4 e^{s\underline{x}} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right)^4 e^{s\left(\sum_{j=1}^{k} x_{i_j}\right)}}{\frac{n-k}{n-1}\sum_{i=1}^{n} x_i^3 e^{sx_i} + \frac{k-1}{n-1}\underline{x}^3 e^{s\underline{x}} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{j=1}^{k} x_{i_j}\right)^3 e^{s\left(\sum_{j=1}^{k} x_{i_j}\right)}} \right) & ; s = t = 0. \end{cases}$$

From (20) we have the monotonicity of the functions $\bar{\mathfrak{u}}_{s,t}(\Upsilon_3,\Phi_1)$ *in both parameters.*

Suppose $\Upsilon_3(\phi_t) > 0$ $(t \in \mathbb{R})$, $a := \min\{x_1,...,x_n\}$, $b := \max\{x_1,...,x_n\}$, *and let*

$$\bar{\mathfrak{M}}_{s,t}(\Upsilon_3,\Phi_1) := \log \bar{\mathfrak{u}}_{s,t}(\Upsilon_3,\Phi_1); \quad s,t \in \mathbb{R}.$$

Then from Theorem 2.9 we have

$$a \le \bar{\mathfrak{M}}_{s,t}(\Upsilon_3, \Phi_1) \le b,$$

and thus $\bar{\mathfrak{M}}_{s,t}(\Upsilon_3, \Phi_1)$ $(s,t \in \mathbb{R})$ *are means. The monotonicity of these means is followed by (20).*

Example 4.6. *Consider the class of continuous convex functions*

$$\Phi_2 = \{\mu_t : (0,\infty) \to \mathbb{R} \mid t \in \mathbb{R}\},$$

where

$$\mu_t(x) := \begin{cases} \frac{x^{t+1}}{t(t-1)};\ t \neq 0,1, \\ -x\log x;\ t = 0, \\ x^2 \log x;\ t = 1. \end{cases}$$

Then $t \mapsto \left(\frac{\mu_t(x)}{x}\right)'' = x^{t-2} = e^{(t-2)\log x}$ $(t \in \mathbb{R})$ *is exponentially convex for every fixed* $x \in (0,\infty)$.

By similar arguments as given in Example 4.5 we get the exponential convexity of $t \mapsto \Upsilon_3(\mu_t)$ $(t \in \mathbb{R})$ *in the Jensen sense. This mapping is differentiable too, therefore exponentially convex. In this case (21) gives*

$\bar{\mathfrak{u}}_{s,t}(\mathbf{x}, \Upsilon_3, \Phi_2) =$

$$\begin{cases} \left(\frac{t(t-1)}{s(s-1)} \frac{\frac{n-k}{n-1}\sum_{i=1}^n x_i^{s+1} + \frac{k-1}{n-1}\underline{x}^{s+1} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^{s+1}}{\frac{n-k}{n-1}\sum_{i=1}^n x_i^{t+1} + \frac{k-1}{n-1}\underline{x}^{t+1} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^{t+1}} \right)^{\frac{1}{s-t}} & ; s \neq t, s,t \neq 0,1, \\ \exp\left(\frac{1-2s}{s(s-1)} + \frac{\frac{n-k}{n-1}\sum_{i=1}^n x_i^{s+1}\log x_i + \frac{k-1}{n-1}\underline{x}^{s+1}\log\underline{x} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^{s+1} \log\left(\sum_{j=1}^k x_{i_j}\right)}{\frac{n-k}{n-1}\sum_{i=1}^n x_i^{s+1} + \frac{k-1}{n-1}\underline{x}^{s+1} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^{s+1}} \right) & ; s = t \neq 0,1, \\ \left(\frac{-1}{s(s-1)} \frac{\frac{n-k}{n-1}\sum_{i=1}^n x_i^{s+1} + \frac{k-1}{n-1}\underline{x}^{s+1} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^{s+1}}{\frac{n-k}{n-1}\sum_{i=1}^n x_i\log x_i + \frac{k-1}{n-1}\underline{x}\log\underline{x} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right) \log\left(\sum_{j=1}^k x_{i_j}\right)} \right)^{\frac{1}{s}} & ; t = 0, s \neq 0,1, \\ \left(\frac{1}{s(s-1)} \frac{\frac{n-k}{n-1}\sum_{i=1}^n x_i^{s+1} + \frac{k-1}{n-1}\underline{x}^{s+1} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^{s+1}}{\frac{n-k}{n-1}\sum_{i=1}^n x_i^2\log x_i + \frac{k-1}{n-1}\underline{x}\log\underline{x}^2 - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^2 \log\left(\sum_{j=1}^k x_{i_j}\right)} \right)^{\frac{1}{s-1}} & ; t = 1, s \neq 0,1, \\ \exp\left(1 + \frac{\frac{n-k}{n-1}\sum_{i=1}^n x_i\log^2 x_i + \frac{k-1}{n-1}\underline{x}\log^2\underline{x} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right) \log^2\left(\sum_{j=1}^k x_{i_j}\right)}{2\left(\frac{n-k}{n-1}\sum_{i=1}^n x_i\log x_i + \frac{k-1}{n-1}\underline{x}\log\underline{x} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right) \log\left(\sum_{j=1}^k x_{i_j}\right)\right)} \right) & ; s = t = 0, \\ \exp\left(-1 + \frac{\frac{n-k}{n-1}\sum_{i=1}^n x_i^2\log^2 x_i + \frac{k-1}{n-1}\underline{x}^2(\log\underline{x})^2 - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^2 \left(\log\left(\sum_{j=1}^k x_{i_j}\right)\right)^2}{2\left(\frac{n-k}{n-1}\sum_{i=1}^n x_i^2\log x_i + \frac{k-1}{n-1}\underline{x}^2\log\underline{x} - \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^k x_{i_j}\right)^2 \log\left(\sum_{j=1}^k x_{i_j}\right)\right)} \right) & ; s = t = 1. \end{cases}$$

Suppose $\Upsilon_3(\mu_t) > 0$ $(t \in \mathbb{R})$*, and let* $a := \min\{x_1, ..., x_n\}$*,* $b := \max\{x_1, ..., x_n\}$*.*

By Theorem 2.9, we can check that

$$a \le \bar{\mathrm{u}}_{s,t}(\mathbf{x}, \Upsilon_3, \Phi_2) \le b; \quad s,t \in \mathbb{R}. \tag{22}$$

The means $\bar{\mathrm{u}}_{s,t}(\mathbf{x}, \Upsilon_3, \Phi_2)$ $(s,t \in \mathbb{R})$ *are continuous, symmetric and monotone in both parameters (by use of (20)).*

Let $s,t,r \in \mathbb{R}$ *such that* $r \neq 0$*. By the substitutions* $s \to \frac{s}{r}$*,* $t \to \frac{t}{r}$*,* $(x_1, \ldots, x_n) \to (x_1^r, \ldots, x_n^r)$ *in (22), we get*

$$\bar{a} \le \bar{\mathrm{u}}_{s,r}(\mathbf{x}^r, \Upsilon_3, \Phi_2) \le \bar{b},$$

where $\bar{a} := \min\{x_1^r, \ldots, x_n^r\}$ *and* $\bar{b} := \max\{x_1^r, \ldots, x_n^r\}$*. Thus new means can be defined with three parameters:*

$$\bar{\mathrm{u}}_{s,t,r}(\mathbf{x}, \Upsilon_3, \Phi_2) := \begin{cases} (\bar{\mathrm{u}}_{s/r,t/r}(\mathbf{x}^r, \Upsilon_3, \Phi_2))^{\frac{1}{r}}; & r \neq 0, \\ \bar{\mathrm{u}}_{s,t}(\log \mathbf{x}, \Upsilon_3, \Phi_1); & r = 0, \end{cases}$$

where $\log \mathbf{x} = (\log x_1, ..., \log x_n)$*.*

The monotonicity of these three parameter means is followed by the monotonicity and continuity of the two parameter means.

Example 4.7. *Consider the class of continuous convex functions*

$$\Phi_3 = \{\chi_t : (0,\infty) \to (0,\infty) \mid t \in (0,\infty)\},$$

where

$$\chi_t(x) := \begin{cases} \frac{xt^{-x}}{\log^2 t}; t \neq 1, \\ \frac{x^3}{2}; \quad t = 1. \end{cases}$$

$t \mapsto \left(\frac{\chi_t(x)}{x}\right)''$ $(t \in (0,\infty))$ *is exponentially convex for every fixed* $x \in (0,\infty)$*, as discussed in Example 4.3.*

We can get the exponential convexity of $t \mapsto \Upsilon_3(\psi_t)$ $(t \in \mathbb{R}^+)$ *as in Example 4.5. For the class* Φ_3*, (21) has the form*

$$\bar{\mathfrak{u}}_{s,t}(\Upsilon_3,\Phi_3) =$$

$$\begin{cases} \left(\frac{\log^2 t}{\log^2 s} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i s^{-x_i} + \frac{k-1}{n-1}\underline{\mathcal{X}} s^{-\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right) s^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i t^{-x_i} + \frac{k-1}{n-1}\underline{\mathcal{X}} t^{-\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right) t^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)}} \right)^{\frac{1}{s-t}} & ; s \neq t,\ s,t \neq 1, \\ \exp\left(-\frac{2}{s\log s} - \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i^2 s^{-x_i} + \frac{k-1}{n-1}\underline{\mathcal{X}}^2 s^{-\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right)^2 s^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)}}{s\left(\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i s^{-x_i} + \frac{k-1}{n-1}\underline{\mathcal{X}} s^{-\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right) s^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)}\right)} \right) & ; s = t \neq 1, \\ \left(\frac{2}{\log^2 s} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i s^{-x_i} + \frac{k-1}{n-1}\underline{\mathcal{X}} s^{-\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right) s^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i^3 + \frac{k-1}{n-1}\underline{\mathcal{X}}^3 - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right)^3} \right)^{\frac{1}{s-1}} & ; t = 1,\ s \neq 1, \\ \exp\left(-\frac{1}{3} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i^4 + \frac{k-1}{n-1}\underline{\mathcal{X}}^4 - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right)^4}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} x_i^3 + \frac{k-1}{n-1}\underline{\mathcal{X}}^3 - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right)^3} \right) & ; s = t = 1. \end{cases}$$

The monotonicity of $\bar{\mathfrak{u}}_{s,t}(\Upsilon_3,\Phi_3)$ $(s,t\in(0,\infty))$ *comes from (20).*

Suppose $\Upsilon_3(\chi_t) > 0$ $(t\in(0,\infty))$*, and let* $a := \min\{x_1,\ldots,x_n\}$*,* $b := \max\{x_1,\ldots,x_n\}$*, and define*

$$\bar{\mathfrak{M}}_{s,t}(\Upsilon_3,\Phi_3) := -L(s,t)\log\bar{\mathfrak{u}}_{s,t}(\Upsilon_3,\Phi_3), \quad s,t\in(0,\infty),$$

where $L(s,t)$ *is the logarithmic mean as defined in Example 4.3.*

From Theorem 2.9 we have

$$a \le \bar{\mathfrak{M}}_{s,t}(\Upsilon_3,\Phi_3) \le b, \quad s,t\in(0,\infty),$$

and therefore we get means.

Example 4.8. *Consider the class of continuous convex functions*

$$\Phi_4 = \{\delta_t : (0,\infty)\to(0,\infty) \mid t\in(0,\infty)\},$$

where

$$\delta_t(x) := \frac{xe^{-x\sqrt{t}}}{t}.$$

$t \mapsto \left(\frac{\delta_t(x)}{x}\right)'' = e^{-x\sqrt{t}}$*,* $t\in(0,\infty)$ *is exponentially convex for every fixed* $x\in(0,\infty)$*, as discussed in Example 4.4. Also as before* $t\mapsto\Upsilon_3(\delta_t)$ $(t\in\mathbb{R}^+)$ *is exponentially convex and differentiable. Hence for the class* Φ_4*, (21) becomes*

$\overline{\mathfrak{u}}_{s,t}(\Upsilon_3, \Lambda_4) =$

$$\begin{cases} \left(\frac{t}{s} \frac{\frac{n-k}{n-1} \sum\limits_{i=1}^{n} x_i e^{-x_i\sqrt{s}} + \frac{k-1}{n-1} \underline{\mathcal{X}} e^{-\sqrt{s}\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right) e^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)\sqrt{s}}}{\frac{n-k}{n-1} \sum\limits_{i=1}^{n} x_i e^{-x_i\sqrt{t}} + \frac{k-1}{n-1} \underline{\mathcal{X}} e^{-\sqrt{t}\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right) e^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)\sqrt{t}}} \right)^{\frac{1}{s-t}} & ; s \neq t, \\ \exp\left(-\frac{1}{s} - \frac{1}{2\sqrt{s}} \frac{\frac{n-k}{n-1} \sum\limits_{i=1}^{n} x_i^2 e^{-x_i\sqrt{s}} + \frac{k-1}{n-1}{}^2 \underline{\mathcal{X}}^2 e^{-\sqrt{s}\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right)^2 e^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)\sqrt{s}}}{\frac{n-k}{n-1} \sum\limits_{i=1}^{n} x_i e^{-x_i\sqrt{s}} + \frac{k-1}{n-1} \underline{\mathcal{X}} e^{-\sqrt{s}\underline{\mathcal{X}}} - \frac{1}{C_{k-1}^{n-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} x_{i_j}\right) e^{-\left(\sum\limits_{j=1}^{k} x_{i_j}\right)\sqrt{s}}} \right) & ; s = t. \end{cases}$$

The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon_3, \Phi_4)$ $(s,t \in (0,\infty))$ *is followed by (16).*

Suppose $\Upsilon_3(\eta_t) > 0$ $(t \in (0,\infty))$*, let* $a := \min\{x_1, \ldots, x_n\}$*,* $b := \max\{x_1, \ldots, x_n\}$*, and define*

$$\bar{\mathfrak{M}}_{s,t}(\Upsilon_2, \Lambda_4) := -(\sqrt{s} + \sqrt{t}) \log \mathfrak{u}_{s,t}(\Upsilon_2, \Lambda_4), \quad s,t \in (0,\infty).$$

Then Theorem 2.9 yields that

$$a \le \bar{\mathfrak{M}}_{s,t}(\Upsilon_3, \Phi_4) \le b,$$

thus we have new means.

Conflict of Interests

The authors declare that there is no conflict of interests.

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