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ON $A(m,p)$- EXPANSIVE AND $A(m,p)$-HYPEREXPANSIVE OPERATORS ON BANACH SPACES -II

OULD AHMED MAHMOUD SID AHMED

Mathematics Departement, College of Science, Al Jouf University, Al Jouf 2014, Saudi Arabia

Abstract. The purpose of the present paper is to give some results related to a class of linear bounded operators, known as $A(m,p)$-expansive operators acting on infinite complex Banach space X recently introduced in [29]. $A(m,p)$-expansive operators is extension of $A(m,p)$-isometric operators was defined by P.B.Duggal in [19], where he has given some of their properties. A Banach space operator $T \in \mathscr{B}(X)$ is $A(m,\ p)$- expansive (resp., $A(m,p)$-hyperexpansive) for some $A \in \mathscr{B}(X)$, integer $m \geq 1$ and $p \in (0,\ \infty)$ if, for any $x \in X$

$$\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\|AT^kx\|^p \leq 0 \ \left(\text{resp., } \sum_{0\leq k\leq n}(-1)^k\binom{n}{k}\|AT^kx\|^p \leq 0,\ \text{ for all } n:\ 1\leq n\leq m\right).$$

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1. Introduction and notations

In this paper $(X,\ \|\ \|)$ denotes a infinite-dimensional Banach space on $\mathbb{K} = \mathbb{C}$ (the complex plane). $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathscr{B}(X)$ be the set of bounded linear operators from X into itself. The authors, Sid Ahmed and Saddi introduced the concept

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of (A,m)-isometric operators. They gave several generalizations of well known facts on m-isometric operators according to semi-Hilbertian space structures.We refer the reader to [31] for more information about (A,m)-isometric operators. Recently, Duggal has introduced the notion of an $A(m,p)$-isometry of a Banach space, following a definition of Bayart in the Banach space setting.

An operator acting on a Hilbert space $\mathscr{H}$ is called m-isometric for some integer $m \geq 1$ if

$$T^{*m}T^m - \binom{m}{1} T^{*m-1}T^{m-1} + ... + (-1)^{m-1}\binom{m}{m-1} T^*T + (-1)^m I = 0, \tag{1.1}$$

where $\binom{m}{k}$ be the binomial coefficient. A simple manipulation proves that (1.1) is equivalent to

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0, \text{ for all } x \in \mathscr{H}. \tag{1.2}$$

Evidently, an isometric operator (i.e., a 1-isometric operator) is m-isometric for all integers $m \geq 1$. Indeed the class of m-isometric operators is a generalization of the class of isometric operators and a detailed study of this class and in particular 2-isometric operators on a Hilbert space has been the object of some intensive study, especially by J.Agler and Stankus in [1], [2] and [3], also by Richter [34], Shimorin [36], Patel [32] and Duggal in [17] and [18]. m-Isometries are not only a natural extension of an isometry, but they are also important in the study of Dirichlet operators and some other classes of operators.

A generalization of m-isometries to operators on general Banach spaces has been presented by several authors in the last years. Botelho [14] and Sid Ahmed [30] discuss operators defined via (1.2) on (complex) Banach spaces. Bayart introduces in [9] the notion of (m,p)-isometries on general (real or complex) Banach spaces. An operator $T \in \mathscr{B}(X)$ on a Banach space X is called an (m,p)-isometry if there exists an integer $m \geq 1$ and a $p \in [1,\infty)$, with

$$\forall\, x \in X,\ \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^{m-k} x\|^p = 0. \tag{1.3}$$

It is easy to see that, if $X = \mathscr{H}$ is a Hilbert space and $p = 2$, this definition coincides with the original definition (1.1) of m-isometries. In [26] the authors took off the restriction $p \geq 1$ and defined (m,p)-isometries for all $p > 0$. They studied when an (m,p)-isometry is an (μ,q)-isometry for some pair (μ,q). In particular, for any positive real number p they gave an example of an operator T that is a $(2,p)$-isometry, but is not a $(2,q)$-isometry for any q different from

p. In [10] and 12] it is proven that the powers on an m-isometry are m-isometries and some products of m-isometries are again m-isometries. For any $T \in \mathscr{B}(\mathscr{H})$ we set

$$\theta_m(T) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*j} T^j. \tag{1.4}$$

The Concept of completely hyperexpansive operators on Hilbert space has attracted much attention of various authors. In [4], J. Agler characterized subnormality with the positivity of $\theta_m(T)$ in (1.4) and also extended his inequalities to the concept of m-isometry.

Definition 1.1. $\big([20]\big)$ *An operator $T \in \mathscr{B}(\mathscr{H})$ is*

(i) m-isometry $(m \geq 1)$ if $\theta_m(T) = 0$.

(ii) m-expansive $(m \geq 1)$ if $\theta_m(T) \leq 0$.

(iii) m-hyperexpensive $(m \geq 1)$, if $\theta_k(T) \leq 0$ for $k = 1, 2, ..., m$.

(iv) Completely hyperexpansive if $\theta_m(T) \leq 0$ for all m.

We refer the reader to [6], [7], [8], [20] ,[27] and [37] for recent articles concerning this subject.

In [9] the author defined $\beta_k^{(p)}(T,\, .) : X \longrightarrow \mathbb{R} : \quad x \longmapsto \beta_k^{(p)}(T,\, x)$ by

$$\beta_k^{(p)}(T,\, x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \|T^j x\|^p, \ \forall \ x \in X. \tag{1.5}$$

For $k, n \in \mathbb{N}$ denote the (descending Pochhammer) symbol by $n^{(k)}$, i.e.

$$n^{(k)} = \begin{cases} 0, \text{ if } n = 0, \\ \\ 0 \text{ if } n > 0 \text{ and } k > n, \\ \\ \binom{n}{k} k! \text{ if } n > 0 \text{ and } k \leq n. \end{cases}$$

Then for $n > 0,\ k > 0$ and $k \leq n$ we have

$$n^{(k)} = n(n-1)...(n-k+1).$$

Then [9, Proposition 2.1],

$$\|T^n x\|^p = \sum_{k=0}^{m-1} n^{(k)} \beta_k^{(p)}(T,\, x) \tag{1.6}$$

for all integers $n \geq 0$ and $x \in X$. In particular,

$$\beta_{m-1}^{(p)}(T,\, x) = \lim_{n \longrightarrow \infty} \frac{\|T^n x\|^p}{\binom{n}{m-1}(m-1)!} \geq 0$$

with equality if and only if T is $(m-1,\ p)$-isometric.

Definition 1.2. $\big([19]\big)$ *Let T and $A \in \mathscr{B}(X)$, m is a positive integer and $p > 0$ a real number. We say that T is an $A(m,p)$-isometry if, for every $x \in X$*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \|AT^{m-k}x\|^p = 0. \tag{1.7}$$

For any $p > 0$; $A(1,\ p)$-isometry coincide with A-isometry, that is $\|ATx\| = \|Ax\|$ for all $x \in X$. Every A-isometry is an $A(m,\ p)$-isometry for all $m \geq 1$ and $p > 0$.
It is clear that the definition of m-isometry given by Agler [1] is equivalent to $I(m,2)$-isometry. It is well known that if T is an $A(m,\ p)$-isometry, then T is $A(n,p)$-isometry for all $n \geq m$.

Let T and $A \in \mathscr{B}(X)$ such that T is an $A(m,\ p)$-isometry. In [19] the author defined $\beta_k^{(p)}(A,\ T,\ x)$ by :

$$\beta_k^{(p)}(A,\ T,\ x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \|AT^j x\|^p,\ \forall\ x \in X. \tag{1.8}$$

We have from (1.8) that

$$\|AT^n x\|^p = \sum_{k=0}^{m-1} n^{(k)} \beta_k^{(p)}(A,\ T,\ x) \tag{1.9}$$

for all $n \geq 0$ and $x \in X$. Furthermore,

$$\beta_{m-1}^{(p)}(A,\ T,\ x) = \lim_{n \longrightarrow \infty} \frac{\|AT^n x\|^p}{\binom{n}{m-1}(m-1)!} \geq 0$$

with equality if and only if T is $A(m-1,\ p)$-isometric . (See $[19]$).

The concept of (A,m)-expansive operators on Hilbert space is introduced in $[28]$.

Throughout this paper, fix a bounded operator $A \in \mathscr{B}(X)$, and we denote

$$\Theta_m^{(p)}(A,T,x) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} \|AT^j x\|^p$$

for an operator $T \in \mathscr{B}(X)$ and a nonnegative integer m.

As an extension of the classes of expansive and hyperexpansive operators on Hilbert space, the following definition describes the classes of operators we will study in this paper.

Definition 1.3. *([29]) Let $T \in \mathscr{B}(X)$, $m \in \mathbb{N}$ and $p > 0$. We say that*

(i) T is $A(m,p)$-expansive if $\Theta_m^{(p)}(A,\ T,\ x) \le 0$ for some positive integer m and $\forall\, x \in X$

(ii) T is $A(m,p)$-hyperexpansive if $\Theta_k^{(p)}(A,\ T,\ x) \le 0\ \ \forall\, k = 1,2,...,m$ and $x \in X$.

(iii) T is completely A- hyperexpansive if $\Theta_m^{(p)}(A,\ T,\ x) \le 0$ for all $m \in \mathbb{N}$.

(iv) T is $A(m,p)$-contractive if $\Theta_m^{(p)}(A,T,x) \ge 0$ for some positive integer m and $\forall\, x \in X$.

(v) T is $A(m,p)$-hypercontractive if $\Theta_k^{(p)}(A,\ T,\ x) \ge 0\ \ \forall\, k = 1,2,...,m$ and $x \in X$.

(vi) T is completely A-contractive if T is $A(k,p)$-contractive for all positive integer k.

It is clear that this definition coincides with Definition 1.1 if $X = \mathscr{H}$ is a Hilbert space , $A = I$ and $p = 2$.

Remark 1.1. *We make the following remarks*

(1) $A(m,p)$-isometries are special cases of the class of $A(m,p)$-expansive operators.

(2)

$$\Theta_m^{(p)}(A,\ T,\ x) = \sum_{\substack{0 \le k \le m \\ (k\ \text{even})}} \binom{m}{k} \|AT^k x\|^p - \sum_{\substack{0 \le k \le m \\ (k\ \text{odd})}} \binom{m}{k} \|AT^k x\|^p, \qquad \forall\, x \in X.$$

In this article we are interested in some of the properties of the $A(m,p)$-expansive operators class. The contents of the paper are the following. In Section 1 we set up notation and terminology. Furthermore, we collect some facts about A(m; p)-isometries. We prove in section 2 that $A(2,p)$-hyperexpansive operators which are $A(m,p)$-expansive must be $A(m-1,p)$-expansive for $m \geq 2$. Recall that if T is m-isometric (resp.k-expansive or (A,m)-expansive) operator, then so are all its power T^n; for $n \geq 1$ (see [10], [20], , [28] and [32]). It turns out that the same assertion remains true for completely A-hyperexpensive operators (Theorem 2.3). Moreocver, we prove that the intersection of the class of completely A-hyperexpansive operators and the class of $A(m,p)$-isometries for $m \geq 2$ is the class of $A(2,p)$-isometries (Proposition 2.4). The section 3 of this paper is an attempt to develop some properties of the class of $A(m,p)$-isometries parallel to those of m-isometries.

2. Main results

In this section we collect some further results about our classes and we begin with the following Lemma inspired from [30].

Lemma 2.1. *([29]) Let $T \in \mathscr{B}(X)$ be an $A(2,\ p)$-expansive, then the following properties hold*

(1) $\|ATx\|^p \geq \dfrac{n-1}{n}\|Ax\|^p,\ \ n \geq 1,\ x \in X.$

(2) $\|ATx\| \geq \|Ax\|,\ \ x \in X.$

(3) If A is left invertible, then T is one-to-one.

(4) $\|AT^n(x)\|^p + (n-1)\|Ax\|^p \leq n\|ATx\|^p,\ x \in X,\ n \in \mathbb{N}_0.$

(5) $\|AT^{2n}x\|^p \leq n\|AT^{n+1}x\|^p - n(n-1)\|ATx\|^p + (n-1)^2\|Ax\|^p,\ n \geq 1,\ x \in X.$

(6) If T is an invertible , then T is $A(1,p)$-isometric.

Proposition 2.1. *Let $T,S \in \mathscr{B}(X)$ such that $TS = ST$ and $\mathscr{R}(S) \subset \mathscr{N}(A)$, then the following are true*

(i) T is $A(m,p)$-expansive if and only if, $T+S$ is $A(m,p)$-expansive.

(ii) T is $A(m,p)$-expansive if and only if, λT is $A(m,p)$-expansive for all λ: $|\lambda| = 1$,

(iii) If T is $A(2,p)$-expansive, then

(1) λT is $A(2,p)$-expansive for $|\lambda| < 1$. if λT^2 is A-expansive.

(2) λT is $A(2,p)$-expansive for $|\lambda| > 1$, if λT^2 is A-contractive.

Proof. (i) Note that, for all $p>0$ and all $x\in X$,

$$\begin{aligned}
\Theta_m^{(p)}(A,T+S,x) &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left\|A(T+S)^jx\right\|^p\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left\|A\sum_{i=0}^{j}\binom{j}{i}T^iS^{j-i}x\right\|^p\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left\|AT^jx\right\|^p \quad (\text{since } \mathscr{R}(S)\subset\mathscr{N}(A))\\
&= \Theta_m^{(p)}(A,T,x).
\end{aligned}$$

Hence, $\Theta_m^{(p)}(A,T+S,x)\le 0$ if and only if, $\Theta_m^{(p)}(A,T,x)\le 0$.

(ii) Let $x\in X$ and $\lambda\in\mathbb{C}$, we have

$$\Theta_m^{(p)}(A,T,x)=\Theta_m^{(p)}(A,\lambda T,x),\ |\lambda|=1.$$

(iii) If T is $A(2,p)$-expansive, then

$$-2|\lambda|^p\|ATx\|^p\le|\lambda|^p\big[-\|AT^2x\|^p-\|Ax\|^p\big]\ \text{ for every }\ \lambda\in\mathbb{C}.$$

So we have for every $\lambda\in\mathbb{C}$

$$|\lambda|^{2p}\|AT^2x\|^p-2|\lambda|^p\|ATx\|^p+\|Ax\|^p\le\big(|\lambda|^p-1\big)\big(|\lambda|^p\|AT^2x\|^p-\|Ax\|^p\big)$$

This completes the proof of the Proposition.

Proposition 2.2. *Let T be a $A(2,p)$-expansive operator. Then the following statements hold.*

(1) T is $A(2,p)$-hyperexpansive.

(2) $\|ATx\|^{2p}\ge\|Ax\|^p\|AT^2x\|^p$ for all $x\in X$.

(3) For each n and a non-zero $x\in X$ such that $x\notin\mathscr{N}(A)$,the sequence

$$\left(\frac{\|AT^{n+1}x\|^p}{\|AT^nx\|^p}\right)_{n\ge 0} \tag{2.1}$$

monotonically decreases to 1,

Proof. (1) Follows from part (2) of Lemma 2.1.

(2) Since T is $A(2,p)$-hyperexpansive, we have

$$\begin{aligned}\|ATx\|^{2p} &\geq \left(\frac{\|Ax\|^p+\|AT^2x\|^p}{2}\right)^2\\ &\geq \left(\|Ax\|^{\frac{p}{2}}\|AT^2x\|^{\frac{p}{2}}\right)^2\\ &\geq \|Ax\|^p\|AT^2x\|^p.\end{aligned}$$

(3) Observe that the $A(2,p)$-expansivity of T implies that

$$\|AT^{n+1}x\|^p-2\|AT^nx\|^p+\|AT^{n-1}x\|^p\leq 0, \tag{2.2}$$

and it follows that

$$\begin{aligned}\|AT^{n-1}x\|^{\frac{p}{2}}\|AT^{n+1}x\|^{\frac{p}{2}} &\leq \frac{\|AT^{n+1}x\|^p+\|AT^{n-1}x\|^p}{2}\\ &\leq \|AT^nx\|^p\end{aligned}$$

Therefore,

$$\frac{\|AT^{n+1}x\|^p}{\|AT^nx\|^p}\leq\frac{\|AT^nx\|^p}{\|AT^{n-1}x\|^p},$$

so the sequence (2.1) is monotonically decreasing. To calculate its limit,in view of part (2) of Lemma 2.1 , we observe that $\|AT^{n-1}x\|\neq 0$ for $x\notin\mathcal{N}(A)$. Divided (2.2) by $\|AT^{n-1}x\|^p$ to get

$$1-2\frac{\|AT^nx\|^p}{\|AT^{n-1}x\|^p}+\frac{\|AT^{n+1}x\|^p}{\|AT^nx\|^p}\frac{\|AT^nx\|^p}{\|AT^{n-1}x\|^p}\leq 0.$$

Hence, we have

$$\left(1-\frac{\|AT^nx\|^p}{\|AT^{n-1}x\|^p}\right)^2\leq 0$$

and let n tend to infinity.

Proposition 2.3. *For any integer $m\geq 1$,real number $p>0$ and $x\in X$,*

$$\Theta_m^{(p)}(A,T,x)=\Theta_{m-1}^{(p)}(A,T,x)-\Theta_{m-1}^{(p)}(A,T,Tx). \tag{2.3}$$

Proof. By the standard formula $\binom{m}{j}=\binom{m-1}{j}+\binom{m-1}{j-1}$ for binomial coefficients we have the equalities

$$\begin{aligned}
\Theta_m^{(p)}(A,\,T,\,x) &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|AT^jx\|^p \\
&= \|Ax\|^p+\sum_{j=1}^{m-1}(-1)^j\binom{m}{j}\|AT^jx\|^p+(-1)^m\|AT^mx\|^p \\
&= \|Ax\|^p+\sum_{j=1}^{m-1}(-1)^j\left(\binom{m-1}{j}+\binom{m-1}{j-1}\right)\|AT^jx\|^p+(-1)^m\|AT^mx\|^p \\
&= \Theta_{m-1}^{(p)}(A,T,x)-\Theta_{m-1}^{(p)}(A,T,Tx).
\end{aligned}$$

Equation (2.3) immediately implies the next statements

Corollary 2.1. *The following are then true.*

(i) If T is an $A(m,\,p)$-isometry such that T is an $A(m-1,\,p)$-isometry on $\mathscr{R}(T)$, then T is an $A(m-1,\,p)$-isometry on X.

(ii) If T is $A(m,\,p)$-expansive and $A(m-1,\,p)$-expansive on $\mathscr{R}(T)$, then T is $A(m-1,\,p)$-expansive.

Remark 2.1. *Proposition 2.2 (1) shows that the notion of $A(2,p)$-expansive and $A(2,p)$ - hyperexpansive coincide. However this result does not true for the class $A(3,p)$-expansive operators as shown the following example.*

Example 2.1. *Let $T=\alpha I$, where I is the identity operator and $\alpha\in\mathbb{C}$. It is easy to see that*

$$\|Ax\|^p-3\|ATx\|^p+3\|AT^2x\|^p-\|AT^3x\|^p=\big(1-|\alpha|^p\big)^3\|Ax\|^p\le 0 \text{ for all } \alpha:|\alpha|\ge 1$$

and

$$\|Ax\|^p-2\|ATx\|^p+\|AT^2x\|^p=\big(1-|\alpha|^p\big)^2\|Ax\|^p\ge 0.$$

Thus, T is a $A(3,p)$-expansive but not a $A(2,p)$-expansive for any $p>0$.

The following theorem gives a sufficient condition for which the $A(m,p)$-expansivity implies the $A(m-1,p)$-expansivity for $m\ge 2$.

Theorem 2.1. *Let T be a $A(2,p)$-hyperexpansive and assume that T is $A(m,p)$-expansive for some $m\ge 2$. Then T is $A(m-1,p)$-expansive.*

Proof. The conditions $\|Ax\|^p - \|ATx\|^p \leq 0$ and $\|Ax\|^p - 2\|ATx\|^p + \|AT^2x\|^p \leq 0$ guarantee that the sequence $\left(\|AT^{n+1}x\|^p - \|AT^nx\|^p\right)_{n\geq 0}$ is monotonically non-increasing and bounded, so that is converges. Thus there exists a positive constant C such that

$$\|AT^{n+1}x\|^p - \|AT^nx\|^p \longrightarrow C \text{ as } n \longrightarrow \infty.$$

Suppose that $\Theta_m^{(p)}(A,T,x) \leq 0$ with $m \geq 3$. Since

$$\Theta_m^{(p)}(A,T,x) = \Theta_{m-1}^{(p)}(A,T,x) - \Theta_{m-1}^{(p)}(A,T,Tx),$$

we have

$$\Theta_{m-1}^{(p)}(A,T,x) \leq \Theta_{m-1}^{(p)}(A,T,Tx).$$

An induction argument shows that

$$\Theta_{m-1}^{(p)}(A,T,x) \leq \Theta_{m-1}^{(p)}(A,T,T^nx),\ n \geq 1.$$

Thus, it suffices to show that

$$\Theta_{m-1}^{(p)}(A,T,T^nx) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Note that

$$\Theta_{m-1}^{(p)}(A,T,x) = \Theta_{m-2}^{(p)}(A,T,x) - \Theta_{m-2}^{(p)}(A,T,Tx),$$

so that

$$\Theta_{m-1}^{(p)}(A,T,T^nx) = \sum_{j=0}^{m-2}(-1)^j\binom{m-2}{j}\left[\|AT^{n+j}x\|^p - \|AT^{n+1+j}x\|^p\right].$$

Letting $n \longrightarrow \infty$ in the preceding equality leads to

$$\Theta_{m-1}^{(p)}(A,T,T^nx) \longrightarrow \sum_{j=0}^{m-2}(-1)^j\binom{m-2}{j}C = 0.$$

This completes the proof.

Note that every power of k-expansive (resp. (A,m)-expansive) operators on Hilbert space are k-expansive (resp. (A,m)-expansive) operators. (See [20], Theorem 2.3 and [28], Proposition 3.9).

For the class of $A(m,p)$-expansive operators it was proved in [29] that positive integral power of $A(2,p)$-expansive operators on Banach space is again a $A(2,p)$-expansive operators.

Theorem 2.2. *([29]) Let $T \in \mathscr{B}(X)$ be an $A(2,p)$-expansive. Then for any positive integer n , T^n is $A(2,p)$-expansive.*

In the following theorem we investigate the powers of completely A-hyperexpansive operator on Banach space as well as completely A-hyperexpansive operators.

According to [[6],Proposition 2 and Remark 2] for every completely A-hyperexpansive operator, the condition that $n \longmapsto \|AT^n x\|^p$ be completely alternating on $\mathbb{N}$ forces, for every $x \in X$, the representation

$$\|AT^n x\|^p = \|Ax\|^p + n\mu_x(\{1\}) + \int_{[0,1)} (1-t^n)\frac{d\mu_x(t)}{1-t}, \tag{2.4}$$

where μ_x is a positive regular Borel measure on $[0;\ 1]$ (for more details see [6]).

Theorem 2.3. *Any positive integral power of a completely A-hyperexpansive operator is completely A-hyperexpansive.*

Proof. Let T be a completely A-hyperexpansive operator and let $k \geq 1$. In view of (2.4) we have that

$$\begin{aligned} \|A(T^k)^n x\|^p = \|AT^{nk}x\|^p &= \|Ax\|^p + nk\mu_x(\{1\}) + \int_{[0,1)} (1-t^{nk})\frac{d\mu_x(t)}{1-t} \\ &= \|Ax\|^p + n(k\mu_x(\{1\})) + \int_{[0,1)} (1-s^n)\frac{d\mu'_x(s)}{1-s^{\frac{1}{k}}}. \end{aligned}$$

Whence $n \longmapsto \|AT^{nk}x\|^p$ is completely alternating and so that T^k is completely A-hyperexpansive.

The next proposition describes the intersection of the class of completely A-hyperexpansive operators with the class of $A(m,p)$-isometries.

It is proved in [37, Proposition 3.4] that if $T \in \mathscr{B}(\mathscr{H})$ (Hilbert space operators) is completely hyperexpansive and m-isometry, then T is an 2-isometry. It turns out that this assertion remains true for completely A-hyperexpansive operators on Banach space.

Proposition 2.4. *If $T \in \mathscr{B}(X)$ is completely A-hyperexpansive and $A(m,p)$-isometric then T must be $A(2,p)$-isometric.*

Proof. First, if T is A-isometric, then T is $A(2,p)$-isometric. Assume that T is $A(m,p)$-isometric with $m \geq 2$. Then we have that $\Theta_m^{(p)}(A,T,x) = 0$ and from (2.4) it follows that

$$\begin{aligned}0 &= \sum_{k=0}^{m}(-1)^k\binom{m}{k}k\mu_x(\{1\}) + \sum_{k=0}^{m}(-1)^k\binom{m}{k}\int_{[0,\,1)}(1-t^k)\frac{d\mu_x(t)}{1-t}\\ &= -\int_{[0,\,1)}(1-t)^{m-1}d\mu_x(t)\end{aligned}$$

Now $\int_{[0,\,1)}(1-t)^{m-1}d\mu_x(t) = 0$ gives that

$$\|AT^kx\|^p = \|Ax\|^p + k\mu_x(\{1\}) \quad \text{for all } k$$

and therefore

$$\Theta_2^{(p)}(A,T,x) = 0.$$

3. $A(m,p)$-Isometries

In this section,we collect some results about $A(m,p)$-isometries as a special case of $A(m,p)$-expansive operators.Our inspiration came from [1], [9], [10], [13], [16], [19], [21], [30], and [33].

3.1. General properties.

Proposition 3.1. *([29]) Let $T \in \mathscr{B}(X)$ be an invertible $A(m,p)$-isometry, then T^{-1} is also an $A(m,p)$-isometry.*

Recall that an operator $T : X \longrightarrow X$ is called power bounded provided there exists a positive number M such that $\|T^n\| \leq M$ for every positive integer n.

Theorem 3.1. *Let $A \in \mathscr{B}(X)$ and let $T \in \mathscr{B}(X)$. If T is a power bounded $A(m,p)$-isometric, then T is A-isometric.*

Proof. Since T is $A(m,\ p)$-isometric, we have $\beta_n^{(p)}(A,T,x) = 0$ for all $n \geq m$. Using equality (1.9) we obtained

$$\|AT^nx\|^p = n^{(m-1)}\beta_{m-1}^{(p)}(A,T,x) + \sum_{0\leq k\leq m-2} n^{(k)}\beta_k^{(p)}(A,T,x)$$

or equivalently

$$\frac{\|AT^nx\|^p}{(m-1)!\binom{n}{m-1}} = \beta_{m-1}^{(p)}(A,T,x) + \frac{1}{(m-1)!\binom{n}{m-1}}\Big(\sum_{0\le k\le m-2} n^{(k)}\beta_k^{(p)}(A,T,x)\Big).$$

The assumption that T is power bounded implies that $\beta_{m-1}^{(p)}(A,T,x)=0$ by setting $n\longrightarrow\infty$.

Therefore by (1.9), we have

$$\frac{\|AT^nx\|^p}{(m-2)!\binom{n}{m-2}} = \beta_{m-1}^{(p)}(A,T,x) + \frac{1}{(m-2)!\binom{n}{m-2}}\Big(\sum_{0\le k\le m-3} n^{(k)}\beta_k^{(p)}(A,T,x)\Big).$$

Form the assumption, we see that $\beta_{m-2}^{(p)}(A,T,x)=0$.

Using similar arguments and (1.9) we can obtain that

$$\beta_{m-3}^{(p)}(A,T,x)=...=\beta_1^{(p)}(A,T,x)=\beta_0^{(p)}(A,T,x)=0.$$

This is a contradiction, so our theorem is therefore established.

The following proposition is a straightforward generalization of Proposition 2.2 in [9] and Proposition 4.2 in [13].

Proposition 3.2. *If T is an $A(m,p)$-isometry,then the following properties hold*

(1) For all $x\in X$, $\beta_{m-1}^{(p)}(A,T,x)\ge 0$ and if $r=\sup\limits_k\{\beta_k^{(p)}(A,T,x)\neq 0\}$, $\beta_r^{p}(A,T,x)\ge 0$.

(2) For $x\in X$. Define the map $N_p := \big(\beta_{m-1}^{(p)}(A,T,.)\big)^{\frac{1}{p}} : X\longrightarrow\mathbb{R}$. Then N_p is a semi-norm satisfying

$$\big(\beta_{m-1}^{(p)}(A,T,x)\big)^{\frac{1}{p}} \le \|A\|(1+\|T\|^p)^{\frac{m-1}{p}}\|x\|.$$

(3) $T\Big(\mathscr{N}\big(\beta_{m-1}^{(p)}(A,T,.)\big)^{\frac{1}{p}}\Big)\subset\mathscr{N}\Big(\big(\beta_{m-1}^{(p)}(A,T,.)\big)^{\frac{1}{p}}\Big)$. Moreover if T is invertible then

$$T\Big(\mathscr{N}\big(\beta_{m-1}^{(p)}(A,T,.)\big)^{\frac{1}{p}}\Big)=\mathscr{N}\Big(\big(\beta_{m-1}^{(p)}(A,T,.)\big)^{\frac{1}{p}}\Big)$$

Proof. The hypothesis T is an $A(m,p)$-isometry implies that

$$\|AT^nx\|^p=\sum_{k=0}^{m-1} n^{(k)}\beta_k^{(p)}(A,\ T,\ x),\ \text{for all}\ \ x\in X,$$

and so $0 \leq \beta_{m-1}^{(p)}(A,T;x) = \lim_{n\longrightarrow\infty} \frac{\|AT^n x\|^p}{n^{(m-1)}}$. Moreover if $q \geq r$ we have

$$0 \leq \beta_q^{(p)}(A,T,x) = \lim_{n\longrightarrow\infty} \frac{\|AT^n x\|^p}{n^{(q-1)}}.$$

(2) We show that N_p is a semi-norm. By (1) it is clear that $N_p \geq 0$.

The homogeneity property follows from

$$N_p(\lambda x) = \beta_{m-1}^{(p)}(A,T,\lambda x)^{\frac{1}{p}} = \lim_{n\longrightarrow\infty} \frac{\|AT^n \lambda x\|}{\sqrt[p]{n^{(m-1)}}} = |\lambda| \lim_{n\longrightarrow\infty} \frac{\|AT^n x\|}{\sqrt[p]{n^{(m-1)}}}.$$

Next to prove the triangle inequality, we have that for all $x, y \in X$

$$\begin{aligned} N_p(x+y) &= \beta_{m-1}^{(p)}(A,T,x+y)^{\frac{1}{p}} = \lim_{n\longrightarrow\infty} \frac{\|AT^n(x+y)\|}{\sqrt[p]{n^{(m-1)}}} \\ &\leq \lim_{n\longrightarrow\infty} \frac{\|AT^n x\|}{\sqrt[p]{n^{(m-1)}}} + \lim_{n\longrightarrow\infty} \frac{\|AT^n y\|}{\sqrt[p]{n^{(m-1)}}} \quad (\text{since } A \text{ and } T \text{ are bounded }) \\ &\leq N_p(x) + N_p(y). \end{aligned}$$

From (1.8) it follows that

$$\begin{aligned} \beta_{m-1}^{(p)}(A,T,x) &\leq \sum_{k=0}^{m-1} \binom{m-1}{k} \|AT^k x\|^p \\ &\leq \|A\|^p \sum_{k=0}^{m-1} \binom{m-1}{k} \|T\|^{kp} \|x\|^p \\ &\leq \|A\|^p \big(1+\|T\|^p\big)^{m-1} \|x\|^p. \end{aligned}$$

i.e.,

$$\beta_{m-1}^{(p)}(A,T,x)^{\frac{1}{p}} \leq C\|x\|.$$

(3) Let $x \in \mathcal{N}(N_p)$

$$\begin{aligned} N_p(Tx) = \beta_{m-1}^{(p)}(A,T,Tx)^{\frac{1}{p}} &= \lim_{n\longrightarrow\infty} \frac{\|AT^{n+1}x\|}{\sqrt[p]{n^{(m-1)}}} \\ &= \lim_{n\longrightarrow\infty} \frac{\|AT^{n+1}x\|}{\sqrt[p]{(n+1)^{(m-1)}}} \frac{\sqrt[p]{(n+1)^{(m-1)}}}{\sqrt[p]{n^{(m-1)}}} \\ &= \lim_{n\longrightarrow\infty} \frac{\|AT^n x\|}{\sqrt[p]{n^{(m-1)}}} \\ &= \beta_{m-1}^{(p)}(A,T,x)^{\frac{1}{p}}. \end{aligned}$$

Hence,

$$\beta_{m-1}^{(p)}(A,T,Tx)^{\frac{1}{p}} = \beta_{m-1}^{(p)}(A,T,x)^{\frac{1}{p}}. \tag{3.1}$$

This equation now immediately implies that $T(\mathcal{N}(N_p)) \subset \mathcal{N}(N_p)$.

On the other hand, if we assume that T is invertible,we have for $x \in \mathcal{N}(N_p)$

$$\begin{aligned} N_p(T^{-1}x) = \beta_{m-1}^{(p)}(A,T,T^{-1}x)^{\frac{1}{p}} &= \lim_{n\longrightarrow\infty} \frac{\|AT^{n-1}x\|}{\sqrt[p]{n^{(m-1)}}} \\ &= \lim_{n\longrightarrow\infty} \frac{\|AT^{n-1}x\|}{\sqrt[p]{(n-1)^{(m-1)}}} \frac{\sqrt[p]{(n-1)^{(m-1)}}}{\sqrt[p]{n^{(m-1)}}} \\ &= 0. \end{aligned}$$

Thus,$\mathcal{N}(N_p) \subset T(\mathcal{N}(N_p))$.

The following result is a direct consequence of Proposition 2.3 and Proposition 3.2.

Corollary 3.1. *For $T \in \mathscr{B}(X)$ be an $A(m,p)$-isometry, then*

$$T : (X,\ \beta_{m-1}^{(p)}(A,T,.)^{\frac{1}{p}}) \longrightarrow (X,\ \beta_{m-1}^{(p)}(A,T,.)^{\frac{1}{p}})$$

is an isometry.

In [1, Proposition 1.23], J. Agler and M. Stankus have proved that, for an even integer m, every invertible m-isometry is also an $(m-1)$-isometry. This result was proved in [16, Proposition A].The following theorem shows that this property is also satisfied by the class of $A(m,p)$-isometries.

Theorem 3.2. *Let $T \in \mathscr{B}(X)$ be an invertible $A(m,p)$-isometry and m is even. then T is an $A(m-1,p)$-isometry.*

Proof. Since T and T^{-1} are an $A(m,p)$-isometry, we have by Proposition 3.2 (1) that

$$\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|AT^kx\|^p \geq 0,\ \ \forall\ x \in X$$

and

$$\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|AT^{-k}x\|^p \geq 0,\ \ \forall\ x \in X.$$

Then one has

$$\begin{aligned}
&\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|AT^{-k}x\|^p \geq 0, \ \forall \ x\in X\\
\Longrightarrow\ &\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{m-1-k}\|AT^{m-1-k}x\|^p \geq 0\\
\Longrightarrow\ &\sum_{k=0}^{m-1}(-1)^{k}\binom{m-1}{k}\|AT^{k}x\|^p \geq 0, \ \forall \ x\in X\\
\Longrightarrow\ &-\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|AT^{k}x\|^p \geq 0 \ \text{ (since } m \text{ is even integer)}\\
\Longrightarrow\ &\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|AT^{k}x\|^p \leq 0, \ \forall \ x\in X.
\end{aligned}$$

Hence we have

$$\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|AT^{k}x\|^p = 0, \ \forall \ x\in X.$$

So the proof is complete.

In the next theorem, we show that if T is an $A(m,p)$-isometry, then $\|AT^n\|^p$ have the same behavior as n^{m-1}. (Similar to [11], Proposition 2.3).

Theorem 3.3. *Let $T\in\mathscr{B}(X)$ be an $A(m,p)$-isometry, then the following properties hold*

(1) $\dfrac{\|AT^nx\|^p}{n^{m-1}}$ *converge uniformly to* $\beta_{m-1}^{(p)}(A,T,x)$ *on the unit ball of* X.

(2) $\dfrac{\|AT^n\|^p}{n^{m-1}}$ *converge to* $\sup\limits_x \beta_{m-1}^{(p)}(A,T,x)$.

Proof. By (1.9) it follows that

$$\frac{\|AT^nx\|^p}{n^{m-1}}-\beta_{m-1}^{(p)}(A,T,x)=\left(\frac{n^{(m-1)}}{n^{m-1}}-1\right)\beta_{m-1}^{(p)}(A,T,x)+\sum_{k=0}^{m-2}\frac{n^{(k)}}{n^{m-1}}\beta_k^{(p)}(A,T,x)\longrightarrow 0 \ as\, n\longrightarrow\infty.$$

On the other hand we have

$$\begin{aligned}
\left|\frac{\|AT^nx\|^p}{n^{m-1}}-\beta^{(p)}_{m-1}(A,T,x)\right| &\leq \left(\frac{n^{(m-1)}}{n^{m-1}}-1\right)\left|\beta^{(p)}_{m-1}(A,T,x)\right|+\sum_{k=0}^{m-2}\frac{n^{(k)}}{n^{m-1}}\left|\beta^{(p)}_{k}(A,T,x)\right| \\
&\leq \left(\frac{n^{(m-1)}}{n^{m-1}}-1\right)\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\|AT^kx\|^p \\
&\quad +\sum_{k=0}^{m-2}\frac{n^{(k)}}{n^{m-1}}\sum_{j=0}^{k}\frac{1}{j!(k-j)!}\|AT^jx\|^p \\
&\leq \left(\frac{n^{(m-1)}}{n^{m-1}}-1\right)\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}M \\
&\quad +\sum_{k=0}^{m-2}\frac{n^{(k)}}{n^{m-1}}\sum_{j=0}^{k}\frac{1}{j!(k-j)!}M \longrightarrow 0 \text{ as } n\longrightarrow\infty,
\end{aligned}$$

where $M=\sup\limits_{0\leq k\leq m-1}\|AT^k\|^p$. Hence the result.

(2) Since

$$\frac{\|AT^n\|^p}{n^{m-1}}=\sup_{\|x\|\leq 1}\frac{\|AT^nx\|^p}{n^{m-1}}$$

we deduce from (1) that

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\frac{\|AT^n\|^p}{n^{m-1}}=\lim_{n\longrightarrow\infty}\sup_{\|x\|\leq 1}\frac{\|AT^nx\|^p}{n^{m-1}} &= \sup_{\|x\|\leq 1}\lim_{n\longrightarrow\infty}\frac{\|AT^nx\|^p}{n^{m-1}} \\
&= \sup_{\|x\|\leq 1}\beta^{(p)}_{m-1}(A,T,x).
\end{aligned}$$

Theorem 3.4. *Let $T\in\mathscr{B}(X)$ be an $A(m,p)$-isometry.Then, for all $x\in X$ the sequence $(\|AT^nx\|^p)_n$ is eventually increasing; that is, there is a positive integer n_0 such that*

$$\|AT^{n+1}x\|^p-\|AT^nx\|^p\geq 0$$

for all $n\geq n_0$.

Proof. We prove the statement by induction on m. For $m=1$, it is clear. Suppose it is true for $m-1$ and let us prove it for m. Assume that $m>1$. Now, for every positive integer n, using (1.9), we observe that

$$\|AT^nx\|^p=\sum_{k=0}^{m-2}n^{(k)}\beta^{(p)}_{k}(A,T,x)+n^{(m-1)}\beta^{(p)}_{m-1}(A,T,x).$$

Since $\beta_{m-1}^{(p)}(A,T,x) \geq 0$, it follows that if $\beta_{m-1}^{(p)}(A,T,x) = 0$, T is an $A(m-1,p)$-isometry and the result is true. If $\beta_{m-1}^{(p)}(A,T,x) > 0$ we have that

$$\lim_{n \longrightarrow +\infty} \|AT^n x\|^p = \infty.$$

hence, there exists a positive integer n_0 so that the sequence $(\|AT^n x\|^p)_{n \geq n_0}$ is strictly increasing.

Definition 3.1. *([15]) An operator T acting on X is called recurrent if for every open set $U \subset X$ there exists some $k \in \mathbb{N}$ such that*

$$U \cap T^{-k}(U) \neq \emptyset.$$

A vector $x \in X$ is called recurrent for T if there exists a strictly increasing sequence of positive integers $(k_n)_{n \geq 0} \subset \mathbb{N}$ such that

$$T^{k_n} x \longrightarrow x$$

as $n \longrightarrow \infty$. We will denote by $Rec(T)$ the set of recurrent vectors for T.

Proposition 3.3. *([15]) Let $T : X \longrightarrow X$ be a bounded linear operator acting on a Banach space X. The following are equivalent*

(i) *The operator T is recurrent.*

(ii) $\overline{Rec(T)} = X$.

The following proposition is a generalization of the result in [15].

Proposition 3.4. *If the operator $T \in \mathscr{B}(X)$ is an $A(m,p)$-isometry and recurrent then T is A-expansive operator.*

Proof Let $x \in X$ be a recurrent vector for T,then there exists a strictly increasing sequence of positive integers $(k_n)_{n \geq 0} \subset \mathbb{N}$ such that $T^{k_n} x \longrightarrow x$ as $n \longrightarrow \infty$ and hence $T^{k_n+1} x \longrightarrow Tx$. By Theorem 3.4 we deduce that $\|Ax\| \leq \|ATx\|$. Since $Rec(T)$ is dense in X we have that $\|Ax\| \leq \|ATx\|$ $,\forall\ x \in X$.

3.2. **Supercyclicity and N-supercyclicity of $A(m,p)$-isometric Operators.**

We first fix some notation.Consider any subset C of X and let $T \in \mathscr{B}(X)$. The symbol $\mathscr{O}(T,C)$ denotes the orbit of C under T i.e. $\mathscr{O}(T,C) = \{T^n x \ : x \in C \ , n = 0,1,2,...\}$. If $C = \{x\}$ is a singleton we write the orbit $\mathscr{O}(T,C) = \mathscr{O}(T,x)$.

Definition 3.2. *A vector $x \in X$ is said hypercyclic for T if its orbit*

$$\mathscr{O}(x,T) := \{T^n x\}_0^\infty = \{ \ x, Tx, T^2x, T^3x, ... \ \}$$

is dense in X. The set of all hypercyclic vectors for T is denoted by $\mathscr{HC}(T)$,i.e., $\mathscr{HC}(T) = \{x \in X : \ \overline{\mathscr{O}(x,T)} = X \ \}$. The operator T is said to be hypercyclic if $\mathscr{HC}(T) \neq \emptyset$.

One may remove linearity in this definition, then under the same.

Definition 3.3. *A vector $x \in X$ is said supercyclic for T if its projective orbit*

$$\mathscr{O}_{pr} := \{\lambda T^n x\}_0^\infty = \{\lambda x, \lambda Tx, \lambda T^2 x, \ \lambda T^3 x, ...,, \lambda \in \mathbb{C}\}$$

is dense in X. The set of all hypercyclic vectors for T is denoted by $\mathscr{SC}(T)$. The operator T is called supercyclic if $\mathscr{SC}(T) \neq \emptyset$.

A nice source of examples and properties of hypercyclic and supercyclic operators is the survey article [24]. Observe that in case the operator T is hypercyclic the underlying Banach space X should be separable. Then it is well known and easy to show that an operator $T : X \longrightarrow X$ is hypercyclic if and only if for every pair of non-empty open sets U, V of X there exists a positive integer n such that $T^n(U) \cap V \neq \emptyset$.

During the past years much research has been done about hypercyclic operators. Hilden and Wallen in [25] proved that isometries on Hilbert spaces with dimension more than one are not supercyclic. Ansari and Bourdon in [5] proved this fact on Banach spaces. Moreover, recently it is shown in [21] that m-isometric operators on Hilbert spaces,which forms a larger class than isometries,are neither supercyclic nor weakly hypercyclic. In [11], it is proven that an m-isometry acting on a Hilbert space $\mathscr{H}$ and whose covariance operator is injective cannot be N-supercyclic.Recently,Bayart [9] extended this result by showing that, for any N and $m \geq 1$:

Any m-isometries on Banach spaces cannot be N-supercyclic.Yarmahmoodi,Hedayatian and Yousefi [38] proved that if A is an isometry and Q is a nilpotent operator that commutes with A, then the operator $A+Q$ is not supercyclic.

In this section we shows that certain class of $A(m,p)$-isometries are not supercyclic.

Definition 3.4. *An operator T is said to be N-supercyclic, $N \geq 1$, if there is a subspace of dimension N in X with dense orbit.*

P.B.Duggal proved that if T is an $A(m,p)-$ isometry with A is left invertible, then T can not be supercyclic (see [19], Corollary 2.6).

Theorem 3.5. *Let $T \in \mathscr{B}(X)$ be an power bounded A-isometry, then T cannot be supercyclic.*

Proof. Suppose that T is a supercyclic A-isometry and suppose that x_0 is supercyclic vector for T.Thus,for any $x \in X$ there is a sequence $(n_k)_k$ of positive integers and a sequence $(a_k)_k$ of scalars such that $a_k T^{n_k} x_0 \longrightarrow x$ as $k \longrightarrow \infty$.
Furthermore,

$$a_k A T^{n_k} x_0 \longrightarrow Ax \ \text{ or } \ |a_k| \|AT^{n_k} x_0\| \longrightarrow \|Ax\|.$$

The assumption that T is an A-isometry implies that

$$|a_k| \|Ax_0\| \longrightarrow \|Ax\| \ \text{ as } \ k \longrightarrow \infty.$$

Note that $x_0 \notin \mathscr{N}(A)$, otherwise $A \equiv 0$. If $\mathscr{N}(A) \neq \{0\}$. Let $x \in \mathscr{N}(A)$ it follows that $|a_k| \longrightarrow 0$ as $k \longrightarrow \infty$ and hence ,$\|AT^{n_k} x_0\| \longrightarrow \infty$ as $k \longrightarrow \infty$ but this is impossible. If $\mathscr{N}(A) = \{0\}$, let $x \in X, x \neq 0$ and it follows that $\lim\limits_{k \longrightarrow \infty} |a_k|$ exists and nonzero.So, $(\|AT^{n_k} x_0\|)_k$ converges and $AT^{n_k} x_0 \not\longrightarrow 0$ (or $T^{n_k} x_0 \not\longrightarrow 0$), which is impossible(see [23], Theorem 2.2). Hence, the proof is complete.

Proposition 3.5. *([11, Lemma 3.1]) Let $T_i : X_i \longrightarrow X_i$ be a (linear and continuous) operator on the Banach space $X_i (i = 1,2)$ and let $S : X_1 \longrightarrow X_2$ be an operator with dense range, such that*

$T_2S = ST_1$, that is, such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ X_2 & \xrightarrow[T_2]{} & X_2 \end{array}$$

If T_1 is N-supercyclic, then T_2 is N-supercyclic.

Theorem 3.6. *On a complex infinite-dimensional Banach space X, an $A(m,p)$- isometry $T \in \mathscr{B}(X)$ with A invertible is not N-supercyclic, given any $N \in \mathbb{N}$.*

Proof. Since A is invertible it follows that T is an $A(m,p)$-isometry if and only if, $S = ATA^{-1}$ is an (m,p)-isometry. By [9, Theorem 3.3]) it is know that an (m,p)-isometry is not N-supercyclic.This implies that S can not be N-supercyclic and the desired result follows from Proposition 3.5 .

The proof of the following theorem is inspired from [9, Theorem 3.3] and [11, Theorem 3.4]

Theorem 3.7. *Let $T \in \mathscr{B}(X)$ be an $A(m,p)$-isometry and assume that $\mathscr{N}(N_p) = \{0\}$. Then T cannot be N-supercyclic.*

Proof. Since $\mathscr{N}(N_p) = \{0\}$ by Proposition 3.2 N_p defines a new norm on X satisfies $N_p(x) \leq C\|x\|$ and $N_p(Tx)) = N_p(x)$ for all $x \in X$. Moreover Corollary 3.1 gives that T is an isometry form (X, N_p) to itself and that $N_p(x) \leq C\|x\|$.

Let $\widetilde{X}$ denote the completion of X with respect to this new norm. Then, T extends to an isometry $\widetilde{T}$ from $(\widetilde{X}, N_p)$ to itself. The density of X in $\widetilde{X}$ and the estimation $N_p(x) \leq C\|x\|$ show that every supercyclic vector x in $(X, \| \ \|)$ is supercyclic for $\widetilde{T}$. [9, Theorem 3.4] implies that $\widetilde{T}$ is not N-supercyclic and the desired result follows.

Theorem 3.8. *Let $A \in \mathscr{B}(X)$ such that $0 \notin \sigma_{ap}(A)$ (the approximate point spectrum) and m is a positive even integer. Let $T \in \mathscr{B}(X)$ be an $A(m,p)$-isometry. If T is not an $A(m-1,p)$-isometry then T is not N-supercyclic.*

Proof. We argue by contradiction. Suppose that T is N-supercyclic $A(m,p)$-isometry. As was observed in [19], if $0 \notin \sigma_{ap}(A)$ then T is bounded below. Combining the observation and the N-supercyclicity gives the invertibility of T. Applying Theorem 3.2 we have that T is an $A(m-1,p)$-isometry. But this leads to a contradiction of the assumption that T is not an $A(m-1,p)$-isometry.

Corollary 3.2. *Let $T, A \in \mathscr{B}(X)$ and assume that $0 \notin \sigma_{ap}(A)$. The following properties hold*

(i) If T is A-isometric, then T is never N- supercyclic.

(ii) If T is $A(2,p)$-isometric, then T is never N- supercyclic.

Proof. (i) Since $0 \notin \sigma_{ap}(A)$ we have that $\mathscr{N}(\beta_0^{(p)}(A,T,.)^{\frac{1}{p}}) = \{0\}$ and the result follows from Theorem 3.7.

(ii) If T is A-isometric it is clear by (i) that (ii) holds. If T is not A-isometric. The required result is now immediate from Theorem 3.8.

3.3. **Weak hypercyclicity of $A(m,p)$-isometries.**

In this section we will use the deep theorem of K. Ball (see [22] and as well as his references) to prove that some of $A(m,p)$-isometric operator cannot be wealky hypercyclic. In a separable, infinite dimensional Banach space X, the weak topology is strictly weaker than the norm topology. A vector $x \in X$ is weakly hypercyclic for T if its orbit $\{x, Tx, T^2x, ...,\}$ is weakly dense in X. An operator T is called weakly hypercyclic if it has a weakly hypercyclic vector. Despite this fact, a weakly hypercyclic operator shares many of the same properties as a hypercyclic operator. For example, it clearly follows from the definitions that every hypercyclic vector for a bounded linear operator $T : X \longrightarrow X$ is automatically a cyclic vector for T. These operators have been studied in [35] and many other articles.

In [21] it is proven that m-isometries are never weakly supercyclic. Similar results was found in [33] for $A-m$-isometries. Our next goal is to get a similar result for the class of $A(m,p)$-isometric operators.

Definition 3.5. *([22]) Let n be a positive integer, X a locally convex space and $S \subseteq X$. Then we have the following definitions*

(1) The set S is n-weakly open if for every $x_0 \in S$ there is an $\varepsilon > 0$ and a set $\mathfrak{F} \subset X^*$ with $|\mathfrak{F}| \leq n$ such that $N(x_0, \mathfrak{F}, \varepsilon) \subseteq S$.

(2) The set S is n-weakly closed if the complement of S is weakly open.

Where, for $\mathfrak{F} = \{f_1, f_2 .... f_n\} \subset X^*$ and $\varepsilon > 0$, let

$$N(x_0, \mathfrak{F}, \varepsilon) = N(x_0, f_1, ..., f_n, \varepsilon) = \{x \in X \ : \ |f(x) - f(x_0)| < \varepsilon \ \forall \ f \in \mathfrak{F}\ \}$$

and $|\mathfrak{F}|$ the cardinality of $\mathfrak{F}$.

Theorem 3.9. *([22],Ball's Theorem) Let $S = \{(x_n)_{n=0}^{\infty}\}$ be a sequence of nonzero vectors in Banach space X.*

(1) If $\sum_{n=0}^{\infty} \frac{1}{\|x_n\|} < \infty$, then S is 1-weakly closed in X.

(2) If X is a Hilbert space and $\sum_{n=0}^{\infty} \frac{1}{\|x_n\|^2} < \infty$, then the following hold

(a) If X is a complex Hilbert space, then S is 1-weakly closed in X.

(b) If X is a real Hilbert space, then S is 2-weakly closed in X.

Theorem 3.10. *Let $A \in \mathscr{L}(X)$ and let $T \in \mathscr{L}(X)$ be an $A(m,p)$-isometric. If A is left invertible, then T can not be weakly hypercyclic.*

Proof. First, assume that T is A-isometric. Then for $x \in X$, the $\mathscr{O}(T,x)$ lies in the ball $B(0, C\|Ax\|)$, $C > 0$. and so T cannot be weakly hypercyclic.

If T is $A(2,p)$-isometric, then T is bounded below (see Lemma 2.1). If we assume that T is weakly hypercyclic, then T will be a dense range operator. Hence, T is invertible and part (6) of Lemma 2.1 implies that T is A- isometric this leads to a contradiction.

Let $m > 2$ and assume, on the contrary, that T is a weakly hypercyclic $A(m,p)$-isometry with a weakly hypercyclic vector x_0 and set

$$N_p(x) = \left(\beta_{m-1}^{(p)}(A, T, x)\right)^{\frac{1}{p}} = \lim_{n \longrightarrow \infty} \frac{\|AT^n x\|}{n^{\frac{m-1}{p}}}.$$

By Proposition 3.2 we have that N_p is a semi-norm on X satisfies

$$T(\mathscr{N}(N_p)) \subset \mathscr{N}(N_p) \ \text{ and } \ N_p(x) = N_p(Tx).$$

We have from (1.9) that

$$\|AT^nx_0\|^p = \sum_{k=0}^{m-1} n^{(k)}\ \beta_k^{(p)}(A,T,x_0)\ ,\ n=0,1,2,....$$

If $\beta_{m-1}^{(p)}(A,T,x_0) \neq 0$, then the positivity of $\beta_{m-1}^{(p)}(A,T,x_0)$ shows that $\beta_{m-1}^{(p)}(A,T,x_0) > 0$ and hence

$$\|AT^nx_0\|^{-p} \approx^{\infty} \frac{1}{\beta_{m-1}^{(p)}(A,T,x_0)} \frac{1}{n^{(m-1)}}.$$

This, implies the convergence of the series $\sum_{n=1}^{\infty} \|AT^nx_0\|^{-p}$ and hence thus, in view of the Theorem 3.9, we get a contradiction. Hence, $\beta_{m-1}^{(p)}(A,T,x_0) = 0$. Since for every n, T^nx_0 is also a weakly hypercyclic vector for T, we see that

$$\beta_{m-1}^{(p)}(A,T,T^nx_0) = 0, \text{ for } n = 0,1,2,....$$

This along with the fact that $\mathscr{N}(N_p)$ is weakly closed, implies that $\beta_{m-1}^{(p)}(A,T,x) = 0$. Hence, T is an $A(m-1,p)$-isometry. Repeating the argument (as above) it follows that T is a $A(2,p)$-isometry, which is impossible. The proof is complete.

Conflict of Interests

The author declares that there is no conflict of interests.

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