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MODELING AND ANALYSIS OF A PREY-WHOLLY DEPENDENT PREDATOR SYSTEM WITH RESERVED AREA

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Abstract: In this paper, a mathematical model for a prey-wholly dependent predator system with reserved area has been proposed and analyzed. It is assumed that the habitat consisting of unreserved area, where the predator attack its sole food the prey, and reserved area, where the prey lived safely. The predator consumes the prey according to the Beddington-DeAngelis type of functional response. The existence, uniqueness and boundedness of the solution of the system are discussed. The dynamical behavior of the system has been investigated locally as well as globally with the help of Lyapunov function. The persistence conditions are established. Local bifurcation near the equilibrium points have been investigated. Finally numerical simulation has been used to confirm our obtained analytical results and specify the control set of parameters.

Keywords: prey-predator; stability; local bifurcation.

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1. Introduction

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiology, Physiology, Ecology, Immunology, Genetics, Physics are some of those disciplines. This mathematical modeling has raised to the highest level in recent years and spread to all branches of life and drew the attention of every one. The application of mathematical models to problems in ecology has resulted in a branch of ecology known as mathematical ecology, while the population dynamics deals with the dynamical behavior of the model ecological systems.

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In the beginning of twentieth century number of attempts has been made to predict the evolution and existence of species mathematically. In deed, the first major attempt in this direction was due to the well known classical Lotka-Volterra model. It proposed independently by the American ecologist Alfred J. Lotka in 1925 as a description of a prey-predator system consisting of a plant population and of a herbivorous animal which relies on this plant as its only food source and by the Italian mathematician Vito Volterra in 1926 as a model to describe the interaction between sharks and fishes in the Adriatic sea. Since then many complicated model for two or more interacting species has been proposed according to the Lotka-Volterra model by taking in to account the effect of competition, time delay, switching, over-exploitation, over-predation, environmental pollution, disease etc., see for example [1-9] and the references their in. All these models are formulated in terms of a system of nonlinear ordinary differential equations and discussed the coexistence and extinction of interacting biological species.

Now, in order to protect the species from driven to extinction, appropriate measures such as restriction on harvesting, creating reserved zones, etc. should be taken that will decrease the interaction of these species with external forces. The role of reserve zones in prey-predator dynamics has received considerable attention in literatures [1]. In particular, Krivan [4] proposed and analyzed the effects of optimal anti predator behavior of prey in predator-prey system. He showed that optimal anti predator behavior of prey leads to persistence and reduction of oscillation in population densities. Chattopadhyay et al. [5] studied a prey-predator model with some cover on prey species. They observed that global stability of the system around positive equilibrium does not necessarily imply the permanence of the system. Dubey et al. [6] proposed and analyzed a mathematical model to study the dynamics of a fishery resource system in an aquatic environment consisting of two areas, namely a free fishing area and a reserve area where fishing is strictly prohibited. It was suggested that even if fishery is exploited continuously in the unreserved area, fish populations can be maintained at an appropriate equilibrium level in the habitat. Later on, Kar [7] proposed a harvests predator-prey model incorporating a prey refuge. He showed that, it is possible to break the cyclic behavior of the system. In the above investigations, the dynamics of predator living in unreserved area together with prey has not been studied explicitly. Dubey [8] proposed and analyzed the dynamics of a prey-predator model with a reserved area; it is assumed that the habitat is divided in to two disjoint areas (unreserved area and reserved area). The predators are not allowed to enter in to the reserved area; however it consumes the prey in unreserved area according to linear type (Lotka-volterra) of functional response. He concluded that the existence of reserved area has a stabilizing effect on prey-predator model. Naji and Kasim [9] proposed and analyzed the effects of switching on the dynamics of two prey-one predator model. They observed that adding the defensive switching behavior to the model has a stabilizing effect on the dynamical behavior of the model. Recently, Mukherjee in [10] proposed and analyzed a Holling type-II prey-predator system with a reserved area and he concluded that under certain conditions reserved zone has destabilizing effect on prey-predator dynamics. However Mukherjee in [11] studied a generalized prey-predator system with a reserved area. He assumed that the migration rate of prey population from free area to reserved area is predator density dependent.

In this paper, the idea of models of Dubey [8] and Mukherjee [11] are adopted together and a new mathematical model is proposed and studied. It is assumed that the more general predator functional response of Beddington-DeAngelis type, which was proposed independently by DeAngelis et al.(1975) and Beddington (1975) [12-13], are used for predation process and the migration rate of prey population from unreserved area to reserved area is depends on both a constant rate and predator density. The local and global stability of the proposed model are investigated. The persistence conditions are established. The local bifurcation analysis is carried out. Finally numerical simulation is used to investigate the global dynamics and confirm our obtained results.

2. Mathematical model

Consider a prey-predator system in which the predator dependent on a sole prey in its feeding living in habitat consisting of two zones namely reserved area and unreserved area. In order to formulate the mathematical model that describes the above real system the following hypotheses are adopted:

1. The prey in a reserved area is capable of reproducing in logistic fashion with carrying capacity $K > 0$ and intrinsic growth rate $r_1 > 0$. While the prey in unreserved area is capable of reproducing in logistic fashion with carrying capacity $L > 0$ and intrinsic growth rate $r_2 > 0$.
2. The transition of prey from unreserved area to reserved area is proportional with a natural moving rate $\alpha > 0$ as well as predator density, while the transition in opposite direction

is proportional with a natural moving rate $\beta > 0$ only. However, the transition of predator species from unreserved area is not allowed.

3. The predator species consumes the prey species in an unreserved area according to Beddington-DeAngelis type of functional response with maximum attack rate $a > 0$, half-saturation constant $b > 0$ and a scale of the impact of the predator interference that given by $c > 0$. Finally, in the absence of prey species the predator will decay exponentially with a death rate given by $d > 0$.

Now, let $x(t)$ be the density of prey species in unreserved area, $y(t)$ be the density of prey species in reserved area and $z(t)$ be the density of predator species at time $t \geq 0$, then according to the above hypothesis the dynamics of the above system can be describe by the following set of differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= r_1 x\left(1 - \frac{x}{K}\right) - (\alpha + z)x + \beta y - \frac{ax}{b + x + cz} z = F_1(x, y, z) \\
\frac{dy}{dt} &= r_2 y\left(1 - \frac{y}{L}\right) + (\alpha + z)x - \beta y = F_2(x, y, z) \\
\frac{dz}{dt} &= \frac{eaxz}{b + x + cz} - dz = F_3(x, y, z)
\end{aligned} \tag{1}$$

with $x(t) \geq 0$, $y(t) \geq 0$ and $z(t) \geq 0$. Clearly the interaction functions in the right hand side of system (1) given by the vector $F = (F_1, F_2, F_3)^t$ are continuously differential function on R_+^3, Hence they are Lipschitizian. Therefore the solution of system (1) exists and is unique. Further, all the solutions of system (1) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (1) which initiate in R_+^3 are uniformly bounded.

Proof: Let $(x(t), y(t), z(t))$ be any solution initiate in R_+^3 and consider the function

$$w(t) = x(t) + y(t) + \frac{1}{e} z(t)$$

By differentiate $w(t)$ with respect to time and then simplifying the resulting terms we get that

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{1}{e}\frac{dz}{dt}$$

$$\frac{dw}{dt} = (r_1 + 1)x - \frac{r_1}{K} x^2 + (r_2 + 1)y - \frac{r_2}{L} y^2 - x - y - \frac{d}{e} z$$

$$\frac{dw}{dt} \le (r_1+1)x\left[1-\frac{x}{K(r_1+1)/r_1}\right]+(r_2+1)y\left[1-\frac{y}{L(r_2+1)/r_2}\right]-\mu_1\left(x+y+\frac{z}{e}\right)$$

where $\mu_1 = \min.\{1, d\}$. Now, since the logistic terms are bounded, then straight forward computation shows that

$$\frac{dw}{dt}+\mu_1 w \le \frac{K(r_1+1)^2}{4r_1}+\frac{L(r_2+1)^2}{4r_2}=\mu_2$$

Consequently by using the comparison theorem, We obtain that $w(t) \le \frac{\mu_2}{\mu_1}$ for sufficiently large t. Hence all the species are uniformly bounded for any initial value in R_+^3. ■

3. Existence of equilibrium points and stability analysis

There are at most three non-negative equilibrium points of system (1), the existence conditions and stability analyses of them are described below:

The vanishing equilibrium point $E_0 = (0,0,0)$ always exists.

The predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$, where

$$\bar{y} = \frac{\bar{x}}{\beta}\left[\frac{r_1\bar{x}}{K}+\alpha-r_1\right] \tag{2}$$

while $\bar{x}$ is a positive root of the third degree polynomial

$$\frac{A_1^2 r_2}{L}x^3+\frac{2A_1A_2r_2}{L}x^2-\left[(r_2-\beta)A_1-\frac{A_2^2 r_2}{L}\right]x-\left[(r_2-\beta)A_2+\alpha\right]=0 \tag{3}$$

where $A_1 = \frac{r_1}{K\beta} > 0$ and $A_2 = \frac{\alpha - r_1}{\beta}$, that exists uniquely in the positive quadrant of $xy-$plane if and only if the Eq.(3) has a unique positive root $\bar{x}$ that satisfy the following condition

$$\frac{r_1\bar{x}}{K}+\alpha > r_1 \tag{4}$$

Further, according to the Discard rule, Eq.(3) has a unique positive root if and only if one sets of conditions hold,

$$(r_2-\beta)A_2+\alpha > 0 \text{ with } \alpha > r_1 \tag{5a}$$

Or

$$(r_2-\beta)A_1 > \frac{A_2^2 r_2}{L} \tag{5b}$$

The positive equilibrium point, $E_2 = (x^*, y^*, z^*)$ exists uniquely in the interior of R_+^3 ($Int.R_+^3$) provided that there is a positive solution to the following set of algebraic equations.

$$\begin{aligned} & r_1 x\left(1-\frac{x}{K}\right)-(\alpha+z)x+\beta y-\frac{axz}{b+x+cz}=0 \\ & r_2 y\left(1-\frac{y}{L}\right)+(\alpha+z)x-\beta y=0 \\ & \frac{eax}{b+x+cz}-d=0 \end{aligned} \tag{6}$$

Straight forward computation shows that

$$y^* = B_3 x^{*^2} + B_4 x^{*^2} - B_5 \text{ and } z^* = B_1 x^* - B_2 \tag{7}$$

here $B_1 = \frac{ea-d}{cd}$, $B_2 = \frac{b}{c} > 0$. $B_3 = \frac{1}{\beta}(\frac{r_1}{K} + B_1) > 0$, $B_4 = \frac{1}{\beta}(\frac{dB_1}{e} - r_1 + \alpha - B_2)$ and $B_5 = \frac{dB_2}{\beta e} > 0$,

however x^* is a positive root of the following fourth order polynomial equation

$$\begin{aligned} -\frac{r_2}{L}B_3^2 x^4 - \frac{r_2}{L}B_3 B_4 x^3 + \Bigg[\frac{r_2}{L}(L+2B_5)B_3 \\ + B_1 - \frac{r_2}{L}B_4^2 - \beta B_3\Bigg]x^2 + \Bigg[\frac{r_2}{L}(L+2B_5)B_4 \\ + \alpha - B_2 - \beta B_4\Bigg]x + B_5\left[\beta - \frac{r_2}{L}(L+B_5)\right] = 0 \end{aligned} \tag{8}$$

Consequently, it is easy to verify that $E_2 = (x^*, y^*, z^*)$ exists uniquely in $Int.R_+^3$ provided that the following set of conditions hold

$$ea > d \tag{9a}$$

$$\frac{d}{e}B_1 + \alpha > r_1 + B_2 \tag{9b}$$

$$\beta > \frac{r_2}{L}(L+B_5) \tag{9c}$$

$$\left.\begin{array}{l}\frac{r_2}{L}(L+2B_5)B_3+B_1<\frac{r_2}{L}B_4^2+\beta B_3 \\ OR \\ \frac{r_2}{L}(L+2B_5)B_4+\alpha>B_2+\beta B_4\end{array}\right\} \tag{9d}$$

$$x^*>\frac{B_2}{B_1} \tag{9e}$$

$$B_3{x^*}^2+B_4x^*>B_5 \tag{9f}$$

Now, in order to investigate the local stabilities of the above equilibrium points, we need to consider the Jacobian matrix $DF=J(x,y,z)$ of system (1) that can be written as

$$J(x,y,z)=(C_{ij})_{3\times 3} \tag{10}$$

where $C_{11}=r_1-\frac{2r_1}{K}x-(\alpha+z)-\frac{az(b+cz)}{(b+x+cz)^2}$, $C_{12}=\beta$, $C_{13}=-x-\frac{a(b+x)x}{(b+x+cz)^2}$

$C_{21}=\alpha+z$, $C_{22}=r_2-\frac{2r_2}{L}y-\beta$, $C_{23}=x$

$C_{31}=\frac{ae(b+cz)z}{(b+x+cz)^2}$, $C_{32}=0$, $C_{33}=\frac{eax}{b+x+cz}-d-\frac{acexz}{(b+x+cz)^2}$

Clearly, straight forward computation shows that the Jacobian matrix near the vanishing equilibrium point $E_0=(0,0,0)$ is

$$J(E_0)=\begin{pmatrix} r_1-\alpha & \beta & 0 \\ \alpha & r_2-\beta & 0 \\ 0 & 0 & -d\end{pmatrix} \tag{11}$$

Thus the characteristic equation can be written as:

$$\left[\lambda^2-\left((r_1-\alpha)+(r_2-\beta)\right)\lambda+(r_1-\alpha)(r_2-\beta)-\alpha\beta\right][-d-\lambda]=0 \tag{12}$$

Hence the eigenvalues of $J(E_0)$ are

$$\lambda_{0x},\lambda_{0y}=\frac{(r_1-\alpha)+(r_2-\beta)}{2} \pm\frac{1}{2}\sqrt{[(r_1-\alpha)+(r_2-\beta)]^2-4[(r_1-\alpha)(r_2-\beta)-\alpha\beta]} \tag{13a}$$

$$\lambda_{0z}=-d \tag{13b}$$

here $\lambda_{0x}, \lambda_{0y}$ and λ_{0z} represent the eigenvalues of $J(E_0)$ in the *x*-direction ,*y*-direction and *z*-direction respectively. Clearly all the above eigenvalues will be negative provided that the following conditions hold

$$r_1 < \alpha \tag{14a}$$

$$r_2 < \beta \tag{14b}$$

$$r_1 r_2 > r_1\beta + r_2\alpha \tag{14c}$$

Since condition (14c) can't satisfied simultaneously with conditions (14a) and (14b), hence $J(E_0)$ has one positive eigenvalues and then E_0 is a saddle point.

The Jacobian matrix of the system (1) near the predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$ can be written as

$$J(E_1) = \begin{pmatrix} r_1 - \frac{2r_1}{K}\bar{x} - \alpha & \beta & -\bar{x}\left(1 + \frac{a(b+\bar{x})}{(b+\bar{x})^2}\right) \\ \alpha & r_2 - \frac{2r_2}{L}\bar{y} - \beta & \bar{x} \\ 0 & 0 & \frac{ea\bar{x}}{b+\bar{x}} - d \end{pmatrix} = (b_{ij}) \tag{15}$$

Therefore the characteristic equation and the eigenvalue of $J(E_1)$ can be written respectively as

$$\begin{aligned} &\left[\lambda^2 - \left[\left(r_1 - \frac{2r_1}{K}\bar{x} - \alpha\right) + \left(r_2 - \frac{2r_2}{L}\bar{y} - \beta\right)\right]\lambda \right. \\ &\left. \quad + \left(r_1 - \frac{2r_1}{K}\bar{x} - \alpha\right)\left(r_2 - \frac{2r_2}{L}\bar{y} - \beta\right) - \alpha\beta\right]\left[\left(\frac{ea\bar{x}}{b+\bar{x}} - d\right) - \lambda\right] = 0 \end{aligned} \tag{16}$$

$$\lambda_{1x}, \lambda_{1y} = \frac{b_{11} + b_{22}}{2} \pm \frac{1}{2}\sqrt{[b_{11} + b_{22}]^2 - 4[b_{11}b_{22} - b_{12}b_{21}]} \tag{17a}$$

$$\lambda_{1z} = b_{33} = \frac{ea\bar{x}}{b+\bar{x}} - d \tag{17b}$$

Now, since $r_1 - \frac{2r_1}{K}\bar{x} - \alpha < 0$ due to existence condition (4), thus all these eigenvalues are negative or have negative real parts and hence E_1 is locally asymptotically stable in R_+^3 provided that

$$r_2 < \frac{2r_1}{L}\bar{y} + \beta \tag{18a}$$

$$\frac{ea\bar{x}}{b+\bar{x}} < d \tag{18b}$$

$$\left(r_1 - \frac{2r_1}{K}\bar{x}\right)\left(r_2 - \frac{2r_2}{L}\bar{y}\right) > \left(r_1 - \frac{2r_1}{K}\bar{x}\right)\beta + \left(r_2 - \frac{2r_2}{L}\bar{y}\right)\alpha \tag{18c}$$

Finally, the Jacobain matrix of the system(1) near the positive equilibrium point E_2 can be written as

$$J(E_2) = (a_{ij})_{3\times 3} \tag{19}$$

where $a_{11} = r_1 - \frac{2r_1}{K}x^* - (\alpha + z^*) - \frac{az^*(b+cz^*)}{(b+x^*+cz^*)^2}$, $a_{12} = \beta > 0$,

$a_{13} = -x^*\left[1 + \frac{a(b+x^*)}{(b+x^*+cz^*)^2}\right] < 0$, $a_{21} = \alpha + z^* > 0$, $a_{22} = r_2 - \frac{2r_2}{L}y^* - \beta$,

$a_{23} = x^* > 0$, $a_{31} = \frac{ae(b+cz^*)z^*}{(b+x^*+cz^*)^2} > 0$, $a_{32} = 0$, $a_{33} = \frac{-acex^*z^*}{(b+x^*+cz^*)^2} < 0$

Therefore the characteristic equation of E_2 can be written as follow

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{20}$$

where

$$A_1 = -(a_{11} + a_{22} + a_{33})$$

$$A_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33}$$

$$A_3 = -a_{33}[a_{11}a_{22} - a_{12}a_{21}] - a_{31}[a_{12}a_{23} - a_{13}a_{22}]$$

while

$$\begin{aligned}\Delta = A_1A_2 - A_3 = &-(a_{11} + a_{22})[a_{11}a_{22} - a_{12}a_{21}] - (a_{11} + a_{33})[a_{11}a_{33} - a_{13}a_{31}] \\ &- a_{22}a_{33}(a_{22} + a_{33}) - 2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31}\end{aligned}$$

From the Routh-Hurwitz criterion [14], all the roots (eigenvalues of $J(E_2)$) of Eq. (20) have negative real parts and hence $E_2 = (x^*, y^*, z^*)$ is locally asymptotically stable if and only if A_1, A_3 and Δ are positive. Therefore in the following theorem we present the sufficient conditions of local stability of E_2.

Theorem (2): Suppose that the positive equilibrium point E_2 of system (1) exists in $Int.R_+^3$. Then E_2 is locally asymptotically stable if

$$K < 2x^* \tag{21a}$$

$$L < 2y^* \tag{21b}$$

Proof: Straight forward computation gives that conditions (21a)-(21b) guarantee that a_{11} and a_{22} are negative, hence by substituting the elements of $J(E_2)$ and then doing simple calculation, we get that A_1, A_3 and Δ are positive. Hence according to Routh-Hurwitz criterion E_2 is locally asymptotically stable in $Int.R_+^3$. ■

Now, we will study the global stability of the equilibrium points of system (1) with the help of Lyapunov method. The results of this study can be summarized in the following theorems.

Theorem (3): Suppose that the predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$ is locally asymptotically stable in the R_+^3, then it is a globally asymptotically stable provided that

$$\frac{\beta}{\alpha}\bar{y} < \bar{x} < \frac{d}{e} \tag{22a}$$

$$\frac{a}{b}\left(\frac{\bar{x}}{\bar{y}}\right)^2 < \frac{\beta x}{\alpha y} \tag{22b}$$

Proof. Consider the following function $V_1 = c_1\left[x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right] + c_2\left[y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right] + c_3 z$, where $c_i; i = 1,2,3$ are positive constants to be determined. Clearly $V_1 : R_+^3 \to R$, is a continuously differentiable positive definite real valued function with $V_1(\bar{x}, \bar{y}, 0) = 0$ and $V_1(x, y, z) > 0$ otherwise. Further, since

$$\frac{dV_1}{dt} = c_1\left(\frac{x - \bar{x}}{x}\right)\frac{dx}{dt} + c_2\left(\frac{y - \bar{y}}{y}\right)\frac{dy}{dt} + c_3\frac{dz}{dt}$$

Then by substituting the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ from system (1) and then simplifying the resulting terms we obtains that

$$\frac{dV_1}{dt} = -c_1 \frac{r_1}{K}(x-\bar{x})^2 - c_2 \frac{r_2}{L}(y-\bar{y})^2 + \frac{x\bar{y}-\bar{x}y}{x\bar{x}y\bar{y}}\big[(c_2\alpha\bar{x} - c_1\beta\bar{y})xy$$
$$+(c_1\beta y - c_2\alpha x)\bar{x}\bar{y}\big] - (c_1 - c_2)xz - (c_3 d - c_1\bar{x})z$$
$$-(c_1 - c_3 e)\frac{axz}{M} - \left(c_2 \frac{x\bar{y}}{y} - c_1 \frac{a\bar{x}}{M}\right)z$$

here $M = b + x + cz$. So, by choosing the positive constants as $c_1 = 1, c_2 = \frac{\beta\bar{y}}{\alpha\bar{x}}$ and $c_3 = \frac{1}{e}$ we get that

$$\frac{dV_1}{dt} = -\frac{r_1}{K}(x-\bar{x})^2 - \frac{\beta\bar{y}}{\alpha\bar{x}}\frac{r_2}{L}(y-\bar{y})^2 - \frac{\beta}{x\bar{x}y}(x\bar{y}-\bar{x}y)^2 - \left(1 - \frac{\beta\bar{y}}{\alpha\bar{x}}\right)xz$$
$$-\left(\frac{d}{e} - \bar{x}\right)z - \left(\frac{\beta x\bar{y}^2}{\alpha\bar{x}y} - \frac{a\bar{x}}{M}\right)z$$

Clearly, $\frac{dV_1}{dt} < 0$ provided that the given conditions hold. Hence V_1 is a Lyapunov function and hence $E_1 = (\bar{x}, \bar{y}, 0)$ is a globally asymptotically stable. ■

According to the above theorem its easy to concludes that, the basin of attraction of the predator free equilibrium point is

$$\mathrm{B}(E_1) = \left\{(x,y,z) \in R_+^3 : \frac{a}{b}\left(\frac{\bar{x}}{\bar{y}}\right)^2 < \frac{\beta x}{\alpha y}, z \ge 0\right\}$$

Finally, in the following theorem the conditions of globally asymptotically stable for a positive equilibrium point are established.

Theorem (4). Suppose that the positive equilibrium point $E_2 = (x^*, y^*, z^*)$ is locally asymptotically stable in the R_+^3, then it is a globally asymptotically stable provided that

$$\frac{az^*}{bM^*} < \frac{r_1}{K} \tag{23a}$$

$$\gamma_{12}{}^2 < \gamma_{11}\gamma_{22} \tag{23b}$$

$$\gamma_{13}{}^2 < \gamma_{11}\gamma_{33} \tag{23c}$$

$$\gamma_{23}{}^2 < \gamma_{22}\gamma_{33} \tag{23d}$$

here $M^* = b + x^* + cz^*$, $\gamma_{11} = \frac{1}{2}\left(\frac{r_1}{K} - \frac{az^*}{MM^*}\right)$, $\gamma_{12} = \frac{\beta y^*}{\alpha x^*}$, $\gamma_{22} = \frac{1}{2}\frac{\beta y^*}{\alpha x^*}\left(\frac{r_2}{L} + \frac{x^* z^*}{yy^*}\right)$,

$\gamma_{13} = \frac{ae(b+cz^*)}{MM^*} - \left(1 + \frac{a(b+x^*)}{MM^*}\right)$, $\gamma_{23} = \frac{\beta y^*}{\alpha y}$ and $\gamma_{33} = \frac{aecx^*}{2MM^*}$.

Proof. Consider the following function

$$V_2 = d_1\left[x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right] + d_2\left[y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right] + d_3\left[z - z^* - z^* \ln\left(\frac{z}{z^*}\right)\right],$$

where $d_i; i = 1,2,3$ are positive constants to be determined. Clearly $V_2 : R_+^3 \to R$, is a continuously differentiable positive definite real valued function with $V_2(x^*, y^*, z^*) = 0$ and $V_2(x, y, z) > 0$ otherwise. Further, since

$$\frac{dV_2}{dt} = d_1\left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + d_2\left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + d_3\left(\frac{z - z^*}{z}\right)\frac{dz}{dt}$$

Then by substituting the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ from system (1) and then simplifying the resulting terms we obtains that

$$\begin{aligned}\frac{dV_2}{dt} = &-d_1\left[\frac{r_1}{K} - \frac{az^*}{MM^*}\right](x - x^*)^2 - d_2\left[\frac{r_2}{L} + \frac{x^* z^*}{yy^*}\right](y - y^*)^2 - d_3\frac{aecx^*}{MM^*}(z - z^*)^2 \\ &+ \left[d_3\frac{ae(b + cz^*)}{MM^*} - d_1\left(1 + \frac{a(b + x^*)}{MM^*}\right)\right](x - x^*)(z - z^*) \\ &+ d_2\frac{z}{y}(x - x^*)(y - y^*) + d_2\frac{x^*}{y}(y - y^*)(z - z^*) \\ &+ \frac{xy^* - x^* y}{xx^* yy^*}\left[xy(d_2\alpha x^* - d_1\beta y^*) - x^* y^*(d_2\alpha x - d_1\beta y)\right]\end{aligned}$$

By choosing the positive constants as $d_1 = 1, d_2 = \frac{\beta y^*}{\alpha x^*}$, $d_3 = 1$ and using the given conditions we get after some algebraic manipulation that:

$$\begin{aligned}\frac{dV_2}{dt} \le &-\left[\sqrt{\gamma_{11}}(x - x^*) - \sqrt{\gamma_{22}}(y - y^*)\right]^2 - \left[\sqrt{\gamma_{11}}(x - x^*) - \sqrt{\gamma_{33}}(z - z^*)\right]^2 \\ &-\left[\sqrt{\gamma_{22}}(y - y^*) - \sqrt{\gamma_{33}}(z - z^*)\right]^2 - \frac{\beta}{xx^* y}\left(xy^* - x^* y\right)^2\end{aligned}$$

Clearly, $\frac{dV_2}{dt} < 0$ under the given conditions then V_2 is a Lyapunov function and hence $E_2 = (x^*, y^*, z^*)$ is a globally asymptotically stable. ■

4. Persistence of system (1)

In this section, the persistence of system (1) is studied. It is well known that the system (1) is said to be persistence if and only if each species persists. Mathematically this is meaning that the solution of system (1) do not have omega limit set in the boundaries of R_+^3. Now before we go further to establish the persistence conditions of system (1), we need to show weather there is a periodic dynamics in the $xy-$plane or not.

Consider the system (1) in the interior of $xy-$plane, which can be written as:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\left(1-\frac{x}{K}\right) - \alpha x + \beta y = g_1(x,y) \\ \frac{dy}{dt} &= r_2 y\left(1-\frac{y}{L}\right) + \alpha x - \beta y = g_2(x,y) \end{aligned} \qquad (24)$$

Clearly Eq. (24) represents a subsystem of system (1) that has the predator free equilibrium point E_1 of system (1) as a unique positive equilibrium point. Define $H(x,y) = \frac{1}{xy}$ that is obviously $H(x,y) > 0$ and C^1 function in the $Int.R_+^2$ of the $xy-$ plane. Now, since

$$\Delta(x,y) = \frac{\partial(g_1 H)}{\partial x} + \frac{\partial(g_2 H)}{\partial y} = -\frac{r_1}{Ky} - \frac{\beta}{x^2} - \frac{r_2}{Lx} - \frac{\alpha}{y^2} \neq 0$$

Then it is clear that $\Delta(x,y)$ does not change sign and it is not identically zero in the $Int.R_+^2$ of the $xy-$ plane. So by using Dulic-Bendixon's criterion there is no closed curve in the $Int.R_+^2$ of the $xy-$ plane. Moreover according to the Poincare-Bendixon theorem, E_1 will be a globally asymptotically stable in the $Int.R_+^2$ of the $xy-$ plane whenever it exists and locally stable.

Consequently, in the following theorem, the necessary and sufficient conditions for the uniform persistence of the system (1) are derived.

Theorem (5): The system (1) is uniformly persistence if the following conditions hold

$$r_1\left(1-\frac{\bar{x}}{K}\right)+\frac{\beta\bar{y}}{\bar{x}}>\alpha \tag{25a}$$

$$r_2\left(1-\frac{\bar{y}}{L}\right)+\frac{\alpha\bar{x}}{\bar{y}}>\beta \tag{25b}$$

$$\frac{ea\bar{x}}{b+\bar{x}}>d \tag{25c}$$

Proof: Consider the average Lyapunov function of the form $\sigma(x,y,z)=x^{p_1}y^{p_2}z^{p_3}$, where each $p_i; i=1,2,3$ is assumed positive constant. Obviously $\sigma(x,y,z)$ is a C^1 positive function defined in $Int.R_+^3$, and $\sigma(x,y,z)\to 0$ if $x\to 0$ or $y\to 0$ or $z\to 0$. Consequently we obtain

$$\Omega(x,y,z)=\frac{\sigma'(x,y,z)}{\sigma(x,y,z)}=p_1\left(r_1\left(1-\frac{x}{K}\right)-(\alpha+z)+\frac{\beta y}{x}-\frac{az}{b+x+cz}\right)$$
$$+p_2\left(r_2\left(1-\frac{y}{L}\right)+\frac{(\alpha+z)x}{y}-\beta\right)+p_3\left(\frac{eax}{b+x+cz}-d\right)$$

Now, since there are no periodic attractors in the boundary planes then, for any initial point in the $Int.R_+^3$, the only possible omega limit set in the boundary planes of the system (1) is the equilibrium points E_1. Thus according to the Gard technique [15] the proof is follows and the system is uniformly persists if we can proof that $\Omega()>0$ at each of these points. Since

$$\Omega(E_1)=p_1\left(r_1\left(1-\frac{\bar{x}}{K}\right)-\alpha+\frac{\beta\bar{y}}{\bar{x}}\right)$$
$$+p_2\left(r_2\left(1-\frac{\bar{y}}{L}\right)+\frac{\alpha\bar{x}}{\bar{y}}-\beta\right)+p_3\left(\frac{ea\bar{x}}{b+\bar{x}}-d\right)$$

Obviously, $\Omega(E_1)>0$ for any positive constants $p_i; i=2,3$ provided that the given conditions hold. Then strictly positive solution of system (1) do not have omega limit set in the boundary planes. Hence, system (1) is uniformly persistence. ■

5. The local bifurcation analysis

In this section, an application of the Sotomayor's theorem [16] is used to investigate the occurrence of the local bifurcation near the possible stable equilibrium points of system (1). Since the existence of a non-hyperbolic equilibrium point is a necessary but not sufficient condition for bifurcation to occurs, a parameter that makes the Jacobian matrix has a zero real part eigenvalue will be adopted as a candidate bifurcation parameter as shown in the following theorems.

Consider now the Jacobian matrix of system (1) at (x, y, z) that given by Eq. (10). Then, with straight forward computation, it is easy to verify that

$$D^2F(x,y,z)(V,V) = (\bar{\bar{d}}_{ij})_{3\times1} \tag{26}$$

here $$\bar{\bar{d}}_{11} = \left(\frac{-2r_1}{K} + \frac{2a(b+cz)z}{M^3}\right)v_1^2 - 2v_1v_3 - \frac{2ac(b+x)x}{M^3}v_3^2$$

$$\bar{\bar{d}}_{21} = 2v_1v_3 - \frac{2r_2}{L}v_2^2$$

$$\bar{\bar{d}}_{31} = -\frac{2ae(b+cz)z}{M^3}v_1^2 + [(b+2cz)x + b(b+cz)]\frac{2ae}{M^3}v_1v_3 - \frac{acex}{M^2}\left(1 + \frac{b+x-cz}{M}\right)v_3^2$$

here $V = (v_1, v_2, v_3)^t$ is any vector in R^3. Moreover

$$D^3F(x,y,z)(V,V,V) = (\bar{\bar{\bar{d}}}_{ij})_{3\times1} \tag{27}$$

here

$$\bar{\bar{\bar{d}}}_{11} = -\frac{6a(b+cz)z}{M^4}v_1^3 - \frac{2ac}{M^4}[(b+2x)M - 3(b+x)x]v_1v_3^2 + \frac{2a}{M^4}[(b+2cz) - 3c(b+cz)z]v_1^2v_3 + \frac{6ac(b+x)x}{M^4}v_3^3$$

$$\bar{\bar{\bar{d}}}_{21} = 0$$

$$\bar{\bar{\bar{d}}}_{31} = \frac{6ae(b+cz)z}{M^4}v_1^3 - \frac{6ae}{M^4}[(b+2cz)x + (b+cz)(b-cz)]v_1^2v_3 + \frac{ace}{M^4}[6(b+x)x - 4(b+2cz)x - 4b(b+cz)]v_1v_3^2 + \frac{acex}{M^4}[2M + cM - 3c(b+x-cz)]v_3^3$$

Theorem (6). The system (1) at the predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$ with the parameter $\tilde{d} = \frac{ae\bar{x}}{b+\bar{x}}$ has

1. No saddle-node bifurcation.

2. Transcritical bifurcation provided that

$$b\frac{\beta\bar{x}+\tilde{A}_2\tilde{A}_3}{\tilde{A}_1\tilde{A}_3-\alpha\beta}\neq c\bar{x} \tag{28a}$$

3. Pitchfork bifurcation provided that

$$b\frac{\beta\bar{x}+\tilde{A}_2\tilde{A}_3}{\tilde{A}_1\tilde{A}_3-\alpha\beta}=c\bar{x} \text{ and } c\neq\tfrac{1}{3} \tag{28b}$$

Proof. According to the Jacobian matrix at the predator free equilibrium point $J(E_1)$ that given by Eq. (15) and their characteristic equation given in Eq. (16), its easy to verify that $J(E_1)$ has zero eigenvalue $\tilde{\lambda}=0$ at $\tilde{d}=\frac{ae\bar{x}}{b+\bar{x}}$ and hence E_1 will be a non-hyperbolic point. Let $\tilde{V}=(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\tilde{\lambda}=0$ of the Jacobian matrix $\tilde{J}=J(E_1,\tilde{d})$, then

$$\left[\tilde{J}-\tilde{\lambda}I\right]\tilde{V}=0\Rightarrow\ \tilde{V}=\left(\frac{\beta\bar{x}+\tilde{A}_2\tilde{A}_3}{\tilde{A}_1\tilde{A}_3-\alpha\beta}\tilde{v}_3,-\frac{\tilde{A}_1\bar{x}+\alpha\tilde{A}_2}{\tilde{A}_1\tilde{A}_3-\alpha\beta}\tilde{v}_3,\tilde{v}_3\right)^t$$

where $\tilde{v}_3$ represents any nonzero real number and $\tilde{A}_1=r_1-\frac{2r_1}{K}\bar{x}-\alpha$, $\tilde{A}_2=-\bar{x}(1+\frac{a}{b+\bar{x}})$ and $\tilde{A}_3=r_2-\frac{2r_2}{L}\bar{y}-\beta$.

Let $\tilde{\Psi}=(\tilde{\psi}_1,\tilde{\psi}_2,\tilde{\psi}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\tilde{\lambda}=0$ of the transpose of Jacobian matrix $\tilde{J}^t=J^t(E_1,\tilde{d})$, then

$$\left[\tilde{J}^t-\tilde{\lambda}I\right]\tilde{\Psi}=0\Rightarrow\ \tilde{\Psi}=\left(0,0,\tilde{\psi}_3\right)^t$$

where $\tilde{\psi}_3$ represents any nonzero real number.

Now let $X=(x,y,z)$ then since

$$F_d(X,d)=\begin{pmatrix}0\\0\\-z\end{pmatrix}\Rightarrow F_d(E_1,d)=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

here $F_d(X,d)$ represents the derivative of $F=(F_1,F_2,F_3)^t$ with respect to d . Then we get that

$$\tilde{\Psi}^tF_d(E_1,0)=0$$

Thus according to the Sotomayor's theorem for local bifurcation, the saddle-node bifurcation can't occur while the first condition of transcritical and pitchfork bifurcation is satisfied. Further, since

$$DF_d(X,d)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-1\end{pmatrix}\Rightarrow DF_d(E_d,d)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

here $DF_d(X,d)$ represents the derivative of $F_d(X,d)$ with respect to $X=(x,y,z)$, consequently we get

$$\tilde{\Psi}^t\left[DF_d(E_1,\tilde{d})\tilde{V}\right]=-\tilde{\psi}_3\tilde{v}_3\neq 0$$

Moreover, by substituting E_1, $\tilde{d}$ and $\tilde{V}$ in Eq. (26), it is observed that

$$\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]=\frac{2abe}{(b+\bar{x})^2}\tilde{v}_1\tilde{v}_3\tilde{\psi}_3-\frac{2ace\bar{x}}{(b+\bar{x})^2}\tilde{v}_3{}^2\tilde{\psi}_3$$
$$=\frac{2ae}{(b+\bar{x})^2}\left[b\frac{\beta\bar{x}+\tilde{A}_2\tilde{A}_3}{\tilde{A}_1\tilde{A}_3-\alpha\beta}-c\bar{x}\right]\tilde{v}_3{}^2\tilde{\psi}_3$$

Cleary, if condition (28a) holds then $\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]\neq 0$ and hence transcritical bifurcation occurs. However if $b\frac{\beta\bar{x}+\tilde{A}_2\tilde{A}_3}{\tilde{A}_1\tilde{A}_3-\alpha\beta}=c\bar{x}$ then $\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]=0$, and hence the transcritical bifurcation can't occur. Further by substituting E_1, $\tilde{d}$ and $\tilde{V}$ in Eq. (27), it is observed that

$$\tilde{\Psi}^t[D^3F(E_1,\tilde{d})(\tilde{V},\tilde{V},\tilde{V})]=\frac{2ace\bar{x}}{(b+\bar{x})^3}(1-3c)\tilde{v}_3{}^3\tilde{\psi}_3$$

Cleary, if condition (28b) holds then $\tilde{\Psi}^t[D^3F(E_1,\tilde{d})(\tilde{V},\tilde{V},\tilde{V})]\neq 0$. Hence pitchfork bifurcation occurs. ■

Theorem (7). Assume that condition (21a) holds while condition (21b) is revised and let the following conditions hold

$$r_2KM^*[b+c\alpha+2cz^*]>(\beta-r_2)[r_1cM^*(2x^*-K)+aK(b+cz^*)] \quad (29a)$$

$$\frac{r_2x^*}{\breve{L}}\left(2y^*-\breve{L}\right)\neq\frac{ax^*(b+x^*)}{M^{*2}}\left(r_2-\frac{2r_2}{\breve{L}}y^*-\beta\right) \quad (29b)$$

where

$$\breve{L} = \frac{2r_2[r_1 c y^* M^*(2x^* - K) + KM^* y^*(b + c\alpha + 2cz^*) + aKy^*(b + cz^*)]}{r_2 KM^*[b + c\alpha + 2cz^*] - (\beta - r_2)[r_1 cM^*(2x^* - K) + aK(b + cz^*)]}$$

$$M^* = b + x^* + cz^*.$$

Then system (1) at the positive equilibrium point $E_2 = (x^*, y^*, z^*)$, with the parameter value given by $\breve{L}$, has

1. saddle-node bifurcation.
2. No transcritical bifurcation.
3. No pitchfork bifurcation.

Proof. According to the Jacobian matrix at the positive equilibrium point $J(E_2)$ that given by Eq. (19) and their characteristic equation given in Eq. (20), its observed that

$$A_3 = \frac{aex^* z^*}{{M^*}^2}\left(\frac{2r_2}{L}\left[\frac{r_1 c}{K} y^*\left(2x^* - K\right) + y^*\left[b + c\alpha + 2cz^*\right] + \frac{ay^*(b + cz^*)}{M^*}\right]\right.$$
$$\left. - r_2\left[b + c\alpha + 2cz^*\right] + (\beta - r_2)\left[\frac{r_1 c}{K}\left(2x^* - K\right) + \frac{a(b + cz^*)}{M^*}\right]\right)$$

Thus its easy to verify that $J(E_2)$ has zero eigenvalue $\breve{\lambda} = 0$ at the parameter value $\breve{L}$, which is positive under condition (29a). Hence E_2 is a non-hyperbolic equilibrium point.

Let $\breve{V} = (\breve{v}_1, \breve{v}_2, \breve{v}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\breve{\lambda} = 0$ of the Jacobian matrix $\breve{J} = J(E_2, \breve{L}) = (\breve{a}_{ij})_{3\times 3}$, where $\breve{a}_{ij} = a_{ij;\ \forall i,j=1,2,3}$ with $a_{22} = r_2 - \frac{2r_2}{\breve{L}} y^* - \beta$. Then

$$\left[\breve{J} - \breve{\lambda} I\right]\breve{V} = 0 \Rightarrow \breve{V} = \left(-\frac{\breve{a}_{33}}{\breve{a}_{31}}\breve{v}_3, \frac{\breve{a}_{21}\breve{a}_{33} - \breve{a}_{23}\breve{a}_{31}}{\breve{a}_{22}\breve{a}_{31}}\breve{v}_3, \breve{v}_3\right)^t$$

here $\breve{v}_3$ is any nonzero real number and $\breve{a}_{21}\breve{a}_{33} - \breve{a}_{23}\breve{a}_{31} < 0$.

Let $\breve{\Psi} = (\breve{\psi}_1, \breve{\psi}_2, \breve{\psi}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\breve{\lambda} = 0$ of the transpose of Jacobian matrix $\breve{J}^t = J^t(E_2, \breve{L})$, then

$$\left[\breve{J}^t - \breve{\lambda} I\right]\breve{\Psi} = 0 \Rightarrow \breve{\Psi} = \left(-\frac{\breve{a}_{22}}{\breve{a}_{12}}\breve{\psi}_2, \breve{\psi}_2, \frac{\breve{a}_{13}\breve{a}_{22} - \breve{a}_{12}\breve{a}_{23}}{\breve{a}_{12}\breve{a}_{33}}\breve{\psi}_2\right)^t$$

here $\breve{\psi}_2$ represents any nonzero real number and $\breve{a}_{13}\breve{a}_{22} \neq \breve{a}_{12}\breve{a}_{23}$ under condition (29b).

Now let $X = (x, y, z)$ then since

$$F_L(X, L) = \begin{pmatrix} 0 \\ \frac{r_2 y^2}{L^2} \\ 0 \end{pmatrix} \Rightarrow F_L(E_2, \breve{L}) = \begin{pmatrix} 0 \\ \frac{r_2 {y^*}^2}{\breve{L}^2} \\ 0 \end{pmatrix}$$

here $F_L(X, L)$ represents the derivative of $F = (F_1, F_2, F_3)^t$ with respect to L. Then we get that

$$\breve{\Psi}^t F_L(E_2, \breve{L}) = \frac{r_2 {y^*}^2}{\breve{L}} \breve{\psi}_2 \neq 0$$

Thus according to the Sotomayor's theorem for local bifurcation, the transcritical and pitchfork bifurcation can't occur while the first condition of saddle-node bifurcation is satisfied. Further, straight forward computation gives that

$$D^2 F(E_2, \breve{L})(\breve{V}, \breve{V}) = (\breve{\breve{d}})_{3\times 1}$$

where

$$\breve{\breve{d}}_{11} = -\frac{2cx^*}{b + cz^*} \breve{v}_3^2 \left[\frac{r_1 c x^*}{K(b + cz^*)} + 1 \right] + \frac{2acx^*}{{M^*}^3} \breve{v}_3^2 \left[\frac{cx^* z^*}{b + cz^*} - (b + x^*) \right]$$

$$\breve{\breve{d}}_{21} = \frac{2cx^*}{b + cz^*} \breve{v}_3^2 - \frac{2r_2}{\breve{L}} \left(\frac{-x^*(\alpha c + b + 2cz^*)}{\left(\frac{r_2}{\breve{L}} (\breve{L} - 2y^*) - \beta \right)(b + cz^*)} \right)^2 \breve{v}_3^2$$

$$\breve{\breve{d}}_{31} = \frac{2aecx^*}{{M^*}^3} (b + x^*) \breve{v}_3^2 - \frac{2acex^*(b + x^*)}{{M^*}^3} \breve{v}_3^2 = 0$$

Hence we obtain that

$$\breve{\Psi}^t D^2 F(E_2,\breve{L})(\breve{V},\breve{V}) = \breve{\psi}_1 \breve{\breve{d}}_{11} + \breve{\psi}_2 \breve{\breve{d}}_{21}$$

$$= \left[\frac{r_2}{\breve{L}\beta}(\breve{L}-2y^*)-1\right]\left[\frac{2cx^*[r_1 cx^* M^{*2} + abK(b+cz^*)]}{K(b+cz^*)^2 M^{*2}}\right]\breve{\psi}_2 \breve{v}_3{}^2$$

$$+\frac{2r_2 x^*}{\breve{L}}\left[\frac{c(\breve{L}-2y^*)}{\beta(b+cz^*)} - x^*\left(\frac{\alpha c + b + 2cz^*}{(\frac{r_2}{\breve{L}}(\breve{L}-2y^*)-\beta)(b+cz^*)}\right)^2\right]\breve{\psi}_2 \breve{v}_3{}^2$$

Straight forward computation shows that $\breve{\Psi}^t D^2 F(E_2,\breve{L})(\breve{V},\breve{V}) \neq 0$. and hence system (1) has saddle-node bifurcation at E_2 with the bifurcation point given by $\breve{L}$. ■

6. Numerical Simulation

In this section the global dynamics of system (1) is studied numerically. The objectives of this study are confirming our analytical results and understand the effects of varying the system's parameters on the dynamics of system (1). Consequently, system (1) is solved numerically, for different sets of parameters and different sets of initial conditions.

It is observed that for the following biologically feasible set of hypothetical parameters values, different set of parameters values can be adopted too, system (1) is solved for different sets of initial values and then the trajectories of system (1) as a function of time are drawn in Fig. (1a)-(1c).

$$\begin{aligned} &r_1 = 1.5,\ K = 200, \alpha = 0.5,\ \beta = 0.9,\ a = 0.5, b = 10, \\ &c = 0.1,\ r_2 = 0.75,\ L = 100,\ e = 0.75,\ d = 0.1 \end{aligned} \qquad (30)$$

Obviously, Fig. (1a)-(1c) shows clearly the convergent of system (1) to the globally asymptotically stable positive equilibrium point $E_2 = (4.42, 104.59, 21.6)$, which confirm our analytical results.

Now, in order to discuss the effect of varying the maximum attack rate on the dynamical behavior of system (1), the system is solved numerically for different values of the maximum attack rate keeping other parameters fixed as given in Eq. (30), and then the trajectories of system (1) as a function of time are drawn in Fig. (2a)-(2b) for the typical values of maximum attack rate $a = 0.2, 0.1$ respectively.

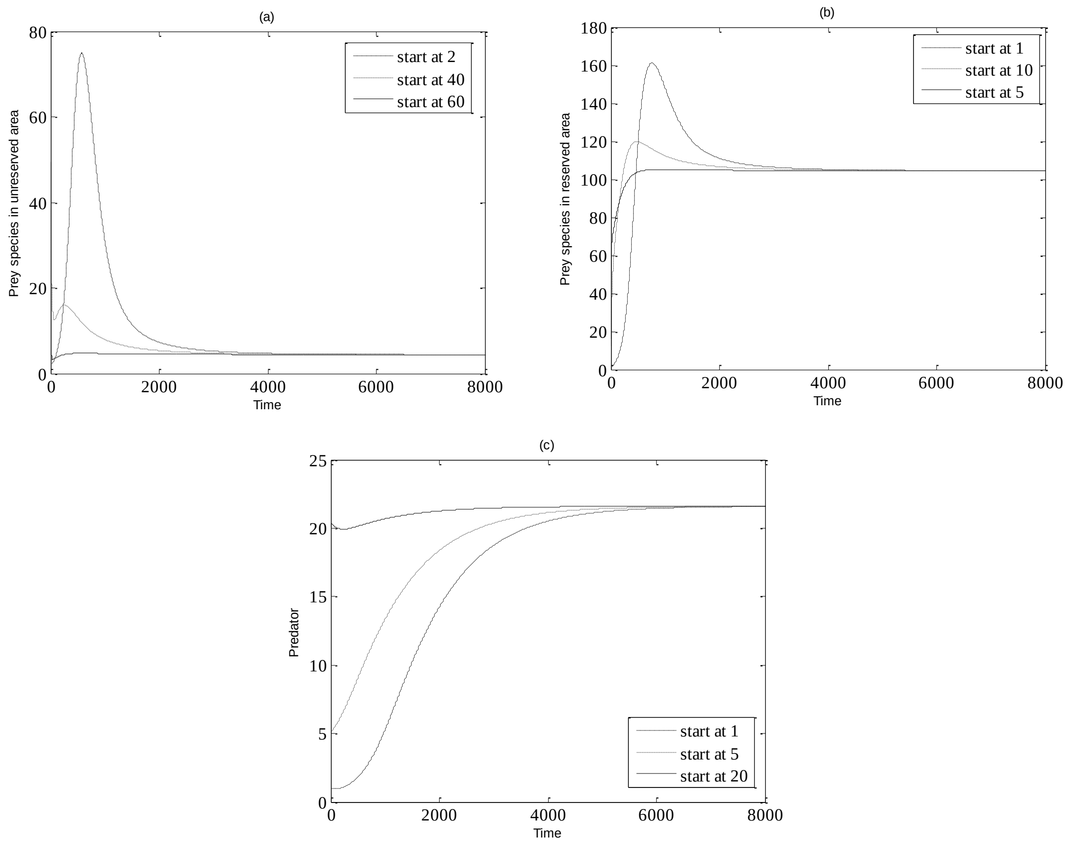

Fig. (1): *Time series of the solutions of system (1) that approaches asymptotically to* $E_2 = (4.42, 104.59, 21.6)$.

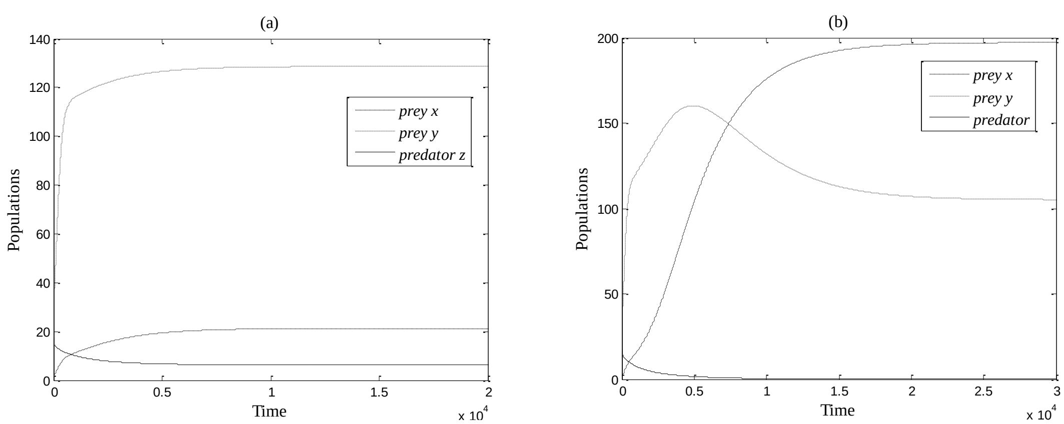

Fig. (2): *Time series of the solutions of system (1).(a) The solution approaches to* $E_2 = (21.25, 128.65, 6.25)$ *for* $a = 0.2$. *(b) The solution approaches to* $E_1 = (197.27, 105.11, 0)$ *for* $a = 0.1$.

Obviously the solution of system (1) approaches asymptotically to the positive equilibrium point for data given in Eq. (30) with $a = 0.2$, while its approaches asymptotically to the predator free equilibrium point for the data given in Eq. (30) with $a = 0.1$.

Further the effect of varying the conversion rate of the amount of food from prey to predator on the dynamical behavior of system (1) is studied. The system (1) is solved numerically for different values of the conversion rate keeping other parameters fixed as given in Eq. (30), and then the trajectories of system (1) as a function of time are drawn in Fig. (3a)-(3b) for the typical values of conversion rate $e = 0.25, 0.1$ respectively.

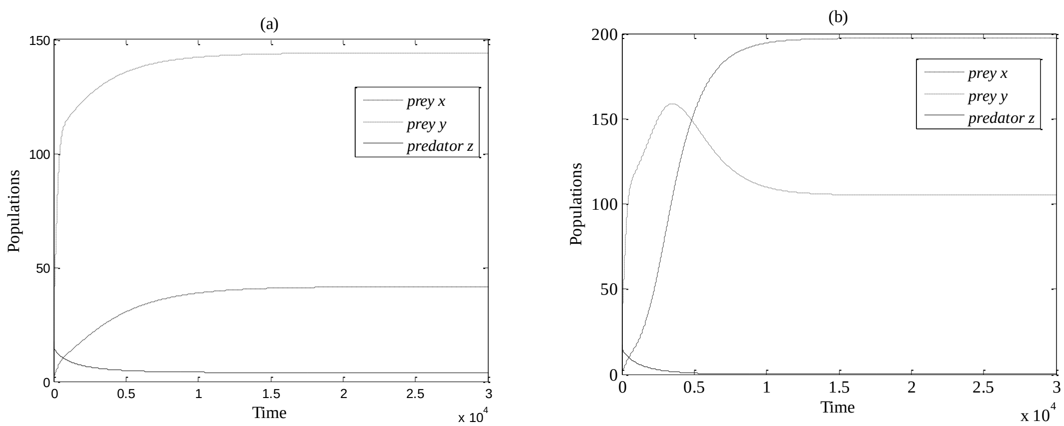

Fig. (3): *Time series of the solutions of system (1).(a) The solution approaches to* $E_2 = (41.51, 144.22, 3.77)$ *for* $e = 0.25$. *(b) The solution approaches to* $E_1 = (197.27, 105.11, 0)$ *for* $e = 0.1$.

According to the above figure the solution of system (1) approaches asymptotically to the positive equilibrium point for data given in Eq. (30) with $e = 0.25$, while its approaches asymptotically to the predator free equilibrium point for the data given in Eq. (30) with $e = 0.15$.

Now the effect of varying the predator's death rate on the dynamical behavior of system (1) is also studied. The system (1) is solved numerically for different values of the death rate keeping other parameters fixed as given in Eq. (30), and then the trajectories of system (1) as a function of time are drawn in Fig. (4a)-(4b) for the typical values of death rate $d = 0.2, 0.4$ respectively.

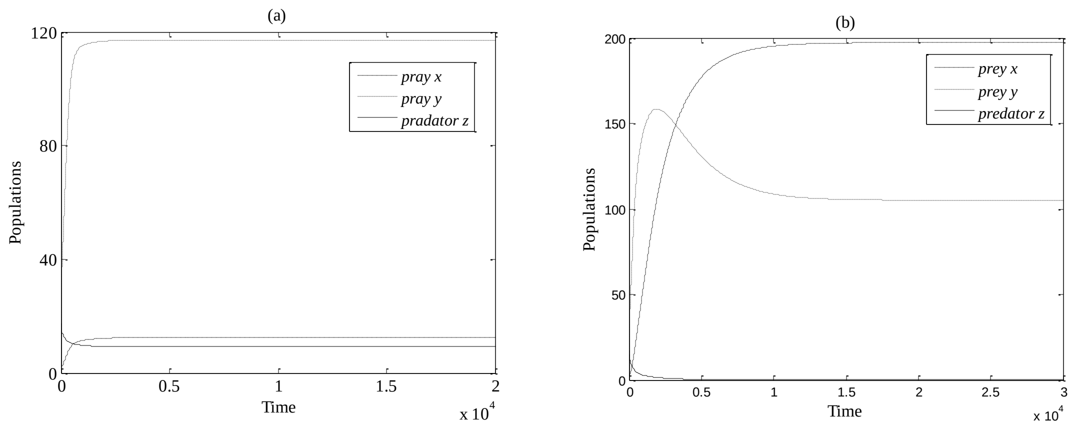

Fig. (4): *Time series of the solutions of system (1).(a) The solution approaches to* $E_2 = (12.47, 117.18, 9.16)$ *for* $d = 0.2$. *(b) The solution approaches to* $E_1 = (197.27, 105.11, 0)$ *for* $d = 0.4$.

Finally, varying other parameters one at the time keeping the rest of parameters as given in Eq. (30) is also investigated numerically, but the solution of the system (1) still approaches to the positive equilibrium point.

7. Discussion and Conclusions:

In this paper, a mathematical model has been proposed and analyzed to study the real world system consisting of a prey interacting with predator that depends on the prey as a sole food. It is assumed that the habitat consisting of unreserved area, where the interacting occurs, and reserved area, where the prey lived safely. The predator is consumed the prey in an unreserved area according to the Beddington-DeAngelis type of functional response. The dynamical behavior of the proposed model represented by system (1) has been investigated locally as well as globally. The persistence conditions are established. Local bifurcation near the equilibrium points have been investigated. It is observed that the system has at most three nonnegative equilibrium points, the vanishing equilibrium point that is always exists and unstable saddle point while the predator free equilibrium point and the coexistence (positive) equilibrium point are exist and asymptotically stable provided that specific conditions are satisfied. Finally in order to confirm our obtained analytical results and specify the control parameters on the global dynamics of the system (1), the system is solved numerically for biologically feasible set of hypothetical parameters values that given in Eq. (30) and the obtained results can be summarized in the following:

1. For the data given in Eq. (30) the system approaches asymptotically to the positive equilibrium point starting from different initial sets of points, which indicates to the existence of a globally stable positive equilibrium point and the system is persist.
2. Increasing the value of maximum attack rate in the range $a \geq 0.15$ (conversion rate in the range $e \geq 0.22$) causes increasing in the density of the predator z but the system (1) still approaches asymptotically to the positive equilibrium point. However decreasing the value of maximum attack rate in the range $a < 0.15$ (conversion rate in the range $e < 0.22$) causes extinction in the predator species and the system (1) approaches asymptotically to the predator free equilibrium point in the $xy-$plane.
3. Decreasing the value of predator's death rate in the range $d \leq 0.35$ causes increasing in the density of the predator z but the system (1) still approaches asymptotically to the positive equilibrium point. However increasing the value of predator's death rate in the range $d > 0.35$ causes extinction in the predator species and the system (1) approaches asymptotically to the predator free equilibrium point in the $xy-$plane.
4. According to the above second and third point its clear that the parameters a, e and d work as a control parameters on the persistence and bifurcation of the system (1).
5. Finally varying the other parameters one at a time keeping the rest of parameters as given in Eq. (30) don't change the dynamics of the system (1) and the system still persist and has a stable positive equilibrium point. This results may be changed depending on the selected set of data.
6. The system doesn't have periodic dynamics.

Conflict of Interests

The author declares that there is no conflict of interests.

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