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FIFTH ORDER NUMERICAL METHOD FOR HEAT EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. This paper deals with numerical method for the approximate solution of one dimensional heat equation $u_t = u_{xx} + q(x,t)$ with integral boundary conditions. The integral conditions are approximated by Simpson's $\frac{1}{3}$ rule while the space derivatives are approximated by fifth-order difference approximations. The method of lines, semi discretization approach is used to transform the model partial differential equation into a system of first-order linear ordinary differential equations whose solution satisfies a recurrence relation involving matrix exponential function. The method developed is L-acceptable, fifth-order accurate in space and time and do not required the use of complex arithmetic. A parallel algorithm is also developed and implemented on several problems from literature and found highly accurate when compared with the exact ones and alternative techniques.

Keywords: heat equation; nonlocal boundary condition; fifth-order numerical methods; method of lines; parallel algorithm.

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1. Introduction

Several boundary-value problems arising in plasma physics [12], heat conduction [1, 6], dynamics of ground waters [9, 13], thermoelasticity [8], control theory [14] and life sciences [2] can be converted into nonlocal boundary problems (problems with integral conditions). In this paper we have considered a non-homogeneous heat equation with nonlocal boundary conditions. Much attention has been paid in the literature for the development, analysis and implementation of accurate methods for the numerical solution of this typical problem.

Consider the heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+q(x,t),\ 0<x<1,\ 0<t\leq T \tag{1}$$

subject to initial condition

$$u(x,0)=f(x),\ 0<x<1 \tag{2}$$

and nonlocal boundary conditions

$$u(0,t)=\int_0^1 \phi(x,t)u(x,t)dx+g_1(t),\ 0<t\leq T, \tag{3}$$

$$u(1,t)=\int_0^1 \psi(x,t)u(x,t)dx+g_2(t),\ 0<t\leq T, \tag{4}$$

where f,g_1,g_2,ϕ,ψ and q are known functions and are assumed to be sufficiently smooth to produce a smooth solution of u. T is a given positive constant.

In this paper the method of lines, semi discretization approach, will be used to transform the model partial differential equation (PDE) into a system of first order, linear, ordinary differential equations (ODEs) the solution of which satisfies a recurrence relation involving matrix exponential terms. A fifth-order rational approximation will be used to approximate exponential functions which will lead to an algorithm which may be parallelized through the partial fraction splitting technique.

2. Discretization and treatment of nonlocal boundary condition

Choosing a positive integer $N \geq 9$ and dividing the interval $[0,1]$ into $N+1$ subintervals each of width h, so that $(N+1)h = 1$, and the time variable t into time steps each of length l gives a rectangular mesh of points with coordinates $(x_m, t_n) = (mh, nl)$ where $(m = 0, 1, 2, ..., N, N+1$ and $n = 0, 1, 2, ...)$ covering the region $R = [0 < x < X] \times [t > 0]$ and its boundary ∂R consisting of the lines $x = 0$, $x = 1$ and $t = 0$.

Assuming that $u(x,t)$ is at least seven times continuously differentiable with respect to variable x and that these derivatives are uniformly bounded, the space derivative $\frac{\partial^2 u}{\partial x^2}$ in (1) may be approximated to the fifth-order at some general point (x,t) of the mesh by using approximations given by [11].

Applying (1) with approximation for $\frac{\partial^2 u}{\partial x^2}$ at all interior mesh points of the grid at time level $t = t_n$ produces a system of N linear equations in $N+2$ unknowns $U_0, U_1, ..., U_{N+1}$. The integrals in (3) and (4) are approximated by using Simpson's $\frac{1}{3}$ rule as used by [10]. Here

$$\begin{aligned} u(0,t) &= \frac{h}{3}\{\phi(0,t)u(0,t) + 4\sum_{i=1}^{\frac{N+1}{2}} \phi((2i-1)h,t)u((2i-1)h,t) \\ &+ 2\sum_{i=1}^{\frac{N+1}{2}} \phi(2ih,t)u(2ih,t) + \phi((N+1)h,t)u((N+1)h,t)\} \\ &+ g_1(t) + O(h^5) \end{aligned} \tag{5}$$

$$\begin{aligned} u(1,t) &= \frac{h}{3}\{\psi(0,t)u(0,t) + 4\sum_{i=1}^{\frac{N+1}{2}} \psi((2i-1)h,t)u((2i-1)h,t) \\ &+ 2\sum_{i=1}^{\frac{N+1}{2}} \psi(2ih,t)u(2ih,t) + \psi((N+1)h,t)u((N+1)h,t)\} \\ &+ g_2(t) + O(h^5). \end{aligned} \tag{6}$$

Solving (5) and (6) simultaneously for U_0 and U_{N+1} and using their values in above the system we have a system of N linear ordinary differential equations which can be written in vector matrix form as

$$\frac{d\mathbf{U}(t)}{dt} = A\mathbf{U}(t) + \mathbf{v}(t), \ \ t > 0 \tag{7}$$

with initial distribution

$$\mathbf{U}(0) = \mathbf{f} \tag{8}$$

in which $\mathbf{U}(t) = [U_1(t), U_2(t), ..., U_N(t)]^T$, $\mathbf{f} = [f(x_1), f(x_2), ..., f(x_N)]^T$, T denoting transpose and matrix $\mathbf{A}$ is given by

$$A = \frac{1}{180h^2} B$$

and

$$B = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots & & \alpha_{N-1} & \alpha_N \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \cdots & & \beta_{N-1} & \beta_N \\ -13 & 228 & -420 & 200 & 15 & -12 & 2 & & & \\ & -13 & 228 & -420 & 200 & 15 & -12 & 2 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & -13 & 228 & -420 & 200 & 15 & -12 & 2 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \cdots & & \gamma_{N-1} & \gamma_N \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \cdots & & \delta_{N-1} & \delta_N \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \cdots & & \zeta_{N-1} & \zeta_N \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \cdots & & \eta_{N-1} & \eta_N \end{bmatrix}_{N \times N}$$

where

$\alpha_1 = 124m_1 - 56$, $\alpha_2 = 124m_2 - 528$, $\alpha_3 = 124m_3 + 925$, $\alpha_4 = 124m_4 - 740$, $\alpha_5 = 124m_5 + 366$, $\alpha_6 = 124m_6 - 104$, $\alpha_7 = 124m_7 + 13$ and $\alpha_i = 124m_i$ for $i \geq 8$

$\beta_1 = -13m_1 + 228$, $\beta_2 = -13m_2 - 420$, $\beta_3 = -13m_3 + 200$, $\beta_4 = -13m_4 + 15$, $\beta_5 = -13m_5 - 12$, $\beta_6 = -13m_6 + 2$ and $\beta_i = -13m_i$ for $i \geq 7$

$\gamma_{N-5} = 2n_{N-5} - 13$, $\gamma_{N-4} = 2n_{N-4} + 228$, $\gamma_{N-3} = 2n_{N-3} - 420$, $\gamma_{N-2} = 2n_{N-2} + 200$, $\gamma_{N-1} = 2n_{N-1} + 15$, $\gamma_N = 2n_N - 12$ and $\gamma_i = 2n_i$ for $1 \leq i \leq N-6$

$\delta_{N-6} = 4n_{N-6} - 2$, $\delta_{N-5} = 4n_{N-5} + 16$, $\delta_{N-4} = 4n_{N-4} - 69$, $\delta_{N-3} = 4n_{N-3} + 340$, $\delta_{N-2} = 4n_{N-2} - 560$, $\delta_{N-1} = 4n_{N-1} + 312$, $\delta_N = 4n_N - 41$, $\delta_i = 4n_i$, for $1 \leq i \leq N-7$

$\zeta_{N-6} = -9n_{N-6} - 4$, $\zeta_{N-5} = -9n_{N-5} + 30$, $\zeta_{N-4} = -9n_{N-4} - 96$, $\zeta_{N-3} = -9n_{N-3} + 155$, $\zeta_{N-2} = -9n_{N-2} + 60$, $\zeta_{N-1} = -9n_{N-1} - 336$, $\zeta_N = -9n_N + 200$, $\zeta_i = -9n_i$ for $1 \leq i \leq N-7$

$\eta_{N-6} = 128n_{N-6} + 9$, $\eta_{N-5} = 128n_{N-5} - 76$, $\eta_{N-4} = 128n_{N-4} + 282$, $\eta_{N-3} = 128n_{N-3} - 600$, $\eta_{N-2} = 128n_{N-2} + 785$, $\eta_{N-1} = 128n_{N-1} - 444$, $\eta_N = 128n_N - 84$ and $\eta_i = 128n_i$ for $1 \leq i \leq$

$N-7$

in which

$$m_i = \begin{cases} \frac{4\frac{h}{3}(c_4\phi_i - c_2\psi_i)}{c_1c_4 - c_2c_3} & \text{for i=1,3,5,...,N,} \\ \frac{2\frac{h}{3}(c_4\phi_i - c_2\psi_i)}{c_1c_4 - c_2c_3} & \text{for i=2,4,6,...,N-1,} \end{cases}$$

and

$$n_i = \begin{cases} \frac{4\frac{h}{3}(c_3\phi_i - c_1\psi_i)}{c_2c_3 - c_1c_4} & \text{for i=1,3,5,...,N,} \\ \frac{2\frac{h}{3}(c_3\phi_i - c_1\psi_i)}{c_2c_3 - c_1c_4} & \text{for i=2,4,6,...,N-1.} \end{cases}$$

Here $c_1 = 1 - \frac{h}{3}\phi_0$, $c_2 = -\frac{h}{3}\phi_{N+1}$, $c_3 = -\frac{h}{3}\psi_0$ and $c_4 = 1 - \frac{h}{3}\psi_{N+1}$ also $\phi_i = \phi(ih,t)$ and $\psi_i = \psi(ih,t)$. The column vector $\mathbf{v}(t)$ contains the contribution from the functions $q(x,t)$, $g_1(t)$ and $g_2(t)$ and is given by

$\mathbf{v}(t) = [\frac{124l_1}{180h^2} + q_1, \frac{-13l_1}{180h^2} + q_2, q_3, q_4, \ldots, q_{N-4}, \frac{2l_2}{180h^2} + q_{N-3}, \frac{4l_2}{180h^2} + q_{N-2}, \frac{-9l_2}{180h^2} + q_{N-1}, \frac{128l_2}{180h^2} + q_N]^T$, where $l_1 = \frac{c_4g_1(t) - c_2g_2(t)}{c_1c_4 - c_2c_3}$ and $l_2 = \frac{c_1g_2(t) - c_2g_2(t)}{c_1c_4 - c_2c_3}$. The solution of the system (7) and (8) is given by

$$\mathbf{U}(t) = exp(tA)\mathbf{f} + \int_0^t exp[A(t+l-s)], \mathbf{v}(s)ds \tag{9}$$

which satisfies the recurrence relation

$$\mathbf{U}(t) = exp(lA)\mathbf{U}(t) + \int_t^{t+l} exp[A(t+l-s)]\mathbf{v}(s)ds, \quad t = 0, l, 2l, \ldots \tag{10}$$

To approximate the matrix exponential function in (10), a rational approximation

$$E_5(\theta) = \frac{1 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4}{1 - a_1\theta + a_2\theta^2 - a_3\theta^3 + a_4\theta^4 - a_5\theta^5}, \tag{11}$$

where

$$a_5 = \sum_{K=0}^{4} (-1)^K \frac{a_K}{(5-K)!} \tag{12}$$

and

$$b_K = \sum_{i=0}^{K} (-1)^i \frac{a_i}{(K-i)!}, \; K = 0,1,2,3,4 \tag{13}$$

is used. Choosing the values of parameters a_1, a_2, a_3, a_4, a_5 as $\frac{91}{20}, \frac{481}{120}, \frac{107}{80}, \frac{691}{3600}$ and $\frac{1}{100}$ so that the method uses only real arithmetic. The quadrature term appearing in (10) is approximated as

$$\begin{aligned}\int_t^{t+l} exp((t+l-s)A)\mathbf{v}(s)ds = \quad & W_1\mathbf{v}(s_1)+W_2\mathbf{v}(s_2)+W_3\mathbf{v}(s_3)+\\ & W_4\mathbf{v}(s_4)+W_5\mathbf{v}(s_5).\end{aligned} \tag{14}$$

The values of W_1, W_2, W_3, W_4 and W_5 are given by [11].

3. Numerical experiments

In this section the numerical methods described in this paper will be applied to four problems from literature and result are obtained will be compared with exact solution as well as with the results existing in literature.

Example 1

$$f(x) = x^2, \quad 0 < x < 1$$
$$g_1(t) = \frac{-1}{4(t+1)^2}, \quad 0 < t < 1$$
$$g_2(t) = \frac{3}{4(t+1)^2}, \quad 0 < t < 1$$
$$\phi(x,t) = x, \quad 0 < x < 1$$
$$\psi(x,t) = x, \quad 0 < x < 1$$
$$q(x,t) = \frac{-2(x^2+t+1)}{(t+1)^3}, \quad 0 < t \leq 1,\ 0 < x < 1,$$

which has theoretical solution $u(x,t) = (\frac{x}{t+1})^2$ [5].

For comparison purpose the problem is solved for $h = l = 0.05$, 0.025, 0.01, 0.005, 0.0025, 0.001 at $x = 0.6$ and $t = 1$. The relative errors obtained by the new scheme are given in Table 1 and the results are compared with different schemes, BTCS implicit scheme, Crandall method, FTCS scheme and Dufort Frankel scheme given by [5]. From the table we can see that the results of the new scheme are for batter than those of the schemes given in [5].

Example 2

$$f(x) = exp(x), \quad 0 < x < 1$$
$$g_1(t) = 0, \quad 0 < t < 1$$

TABLE 1. Relative errors at various spatial lengths at t=1

Spatial length	BTCS	Crandall	FTCS	Dufort-Frankel	New scheme
$h = 0.0500$	7.3×10^{-02}	3.8×10^{-03}	7.5×10^{-02}	7.8×10^{-02}	6.9×10^{-08}
$h = 0.0250$	1.8×10^{-02}	2.1×10^{-04}	1.9×10^{-02}	1.9×10^{-02}	2.9×10^{-09}
$h = 0.0100$	4.4×10^{-03}	1.2×10^{-05}	4.0×10^{-03}	3.9×10^{-03}	3.3×10^{-11}
$h = 0.0050$	1.2×10^{-02}	7.1×10^{-07}	1.0×10^{-03}	1.0×10^{-03}	5.6×10^{-12}
$h = 0.0025$	3.0×10^{-04}	4.3×10^{-08}	2.5×10^{-04}	2.4×10^{-04}	1.2×10^{-11}
$h = 0.0010$	7.5×10^{-05}	2.5×10^{-09}	6.1×10^{-05}	6.0×10^{-05}	2.5×10^{-11}

$$g_2(t) = 0, \quad 0 < t < 1$$

$$\phi(x,t) = ax, \quad 0 < x < 1$$

$$\psi(x,t) = bcos(x), \quad 0 < x < 1$$

$$q(x,t) = -exp[-x(x+sint)](1+cost), \quad 0 < t \leq 1, \ 0 < x < 1$$

where $a = \frac{e}{e-2}$ and $b = \frac{2}{(sin(1)-cos(1)+e)}$ [5].

Which has theoretical solution $u(x,t) = exp(-(x+sint))$. For example 2 results are given in Table 3 and Table 3. In Table 2 the results are compared for $h = l = 0.05$, 0.025, 0.01, 0.005, 0.0025, 0.001 at $x = 0.6$ and $t = 1$. The relative errors obtained by the new scheme are given in Table 2 and the results are compared with different schemes, BTCS implicit scheme, Crandall method, FTCS scheme and Dufort Frankel scheme given by [5]. From the table it is cleared that the results are in good agrement as compared with the exact ones as well as better than other schemes. Moreover the new scheme is fifth-order accurate except for very small values of h and l when the accumulating error is developed due to large number of arithmetic operations.

Example 3

$$f(x) = sin(\pi x) + cos(\pi x), \quad 0 < x < 1$$

$$g_1(t) = 0, \quad 0 < t < 1$$

$$g_2(t) = 0, \quad 0 < t < 1$$

$$\phi(x,t) = 2sin(\pi x), \quad 0 < x < 1$$

$$\psi(x,t) = -2cos(\pi x), \quad 0 < x < 1$$

TABLE 2. Relative errors at various spatial lengths at t=1

Spatial length	BTCS	Crandall	FTCS	Dufort-Frankel	New scheme
$h = 0.0500$	6.3×10^{-02}	3.9×10^{-03}	6.4×10^{-02}	6.8×10^{-02}	1.1×10^{-07}
$h = 0.0250$	1.5×10^{-02}	2.4×10^{-04}	1.6×10^{-02}	1.7×10^{-02}	6.9×10^{-09}
$h = 0.0100$	4.0×10^{-03}	1.5×10^{-05}	4.1×10^{-03}	4.1×10^{-03}	1.8×10^{-11}
$h = 0.0050$	1.0×10^{-03}	1.0×10^{-06}	1.0×10^{-03}	1.0×10^{-03}	1.4×10^{-12}
$h = 0.0025$	2.4×10^{-04}	6.4×10^{-08}	2.5×10^{-04}	2.6×10^{-04}	2.1×10^{-11}
$h = 0.0010$	6.1×10^{-05}	4.0×10^{-09}	4.0×10^{-05}	3.9×10^{-05}	2.5×10^{-11}

TABLE 3. Results for u at different values of t

t	Exact u	Error Crank-Nicolson	The Implicit	The Parallel	New scheme
$h = 0.1$	0.7048055	6.0×10^{-05}	5.2×10^{-05}	3.8×10^{-06}	1.8×10^{-10}
$h = 0.2$	0.6384772	5.2×10^{-05}	4.1×10^{-05}	3.7×10^{-06}	4.1×10^{-10}
$h = 0.3$	0.5795403	9.7×10^{-05}	7.1×10^{-05}	4.6×10^{-06}	6.1×10^{-10}
$h = 0.4$	0.5275993	8.0×10^{-05}	6.5×10^{-05}	5.5×10^{-06}	8.0×10^{-10}
$h = 0.5$	0.4821859	1.2×10^{-05}	8.9×10^{-05}	2.3×10^{-06}	9.8×10^{-10}
$h = 0.6$	0.4427977	1.1×10^{-05}	9.8×10^{-05}	1.0×10^{-06}	1.2×10^{-09}
$h = 0.7$	0.4089274	2.5×10^{-05}	1.4×10^{-05}	1.1×10^{-06}	1.4×10^{-09}
$h = 0.8$	0.3800687	3.8×10^{-05}	2.6×10^{-05}	1.0×10^{-06}	1.6×10^{-09}
$h = 0.9$	0.3558213	5.8×10^{-05}	4.4×10^{-05}	2.1×10^{-06}	1.9×10^{-09}
$h = 1.0$	0.3357223	7.1×10^{-05}	6.4×10^{-05}	1.9×10^{-06}	2.1×10^{-09}

$$q(x,t) = (\pi^2 - 1)exp(-t)\{sin(\pi x) + cos(\pi x)\}, \quad 0 < t \leq 1, \; 0 < x < 1$$

Which has theoretical solution $u(x,t) = exp(-t)\{sin(\pi x) + cos(\pi x)\}$ [4].In this problem the results are computed for $h = l = 0.01$ for different values of t at x=0.25 and the results are presented in Table 4. Table 4 shows that the scheme developed in this paper gives superior results to other schemes, namely the Crank Nicolson finite difference method [15] and the parallel techniques [4].

TABLE 4. Results for u at different values of t

t	Exact u	Error Crank-Nicolson	The Implicit	The Parallel	New scheme
$h = 0.1$	1.2796330	5.2×10^{-05}	4.3×10^{-05}	4.8×10^{-06}	1.9×10^{-10}
$h = 0.2$	1.1578600	6.2×10^{-05}	6.0×10^{-05}	4.7×10^{-06}	3.4×10^{-10}
$h = 0.3$	1.0476750	6.5×10^{-05}	6.4×10^{-05}	3.9×10^{-06}	3.8×10^{-10}
$h = 0.4$	0.9479756	6.4×10^{-05}	6.3×10^{-05}	4.8×10^{-06}	3.7×10^{-10}
$h = 0.5$	0.8577639	6.2×10^{-05}	5.9×10^{-05}	5.3×10^{-06}	3.5×10^{-10}
$h = 0.6$	0.7761369	5.6×10^{-05}	4.8×10^{-05}	3.7×10^{-06}	3.2×10^{-10}
$h = 0.7$	0.7022777	5.0×10^{-05}	4.9×10^{-05}	2.3×10^{-06}	2.9×10^{-10}
$h = 0.8$	0.6354471	1.6×10^{-05}	1.5×10^{-05}	1.6×10^{-06}	2.6×10^{-10}
$h = 0.9$	0.5749763	4.1×10^{-05}	3.3×10^{-05}	1.1×10^{-06}	2.4×10^{-10}
$h = 1.0$	0.5202601	5.0×10^{-05}	4.7×10^{-05}	1.0×10^{-06}	2.2×10^{-10}

Example 4

$$f(x) = x(x-1) + \frac{\delta}{6(1+\delta)}, \quad 0 < x < 1$$

$$g_1(t) = 0, \quad 0 < t < 1$$

$$g_2(t) = 0, \quad 0 < t < 1$$

$$\phi(x,t) = -\delta, \quad 0 < x < 1$$

$$\psi(x,t) = -\delta, \quad 0 < x < 1$$

$$q(x,t) = (\pi^2 - 1)exp(-t)\{sin(\pi x) + cos(\pi x)\}, \quad 0 < t \leq 1, \ 0 < x < 1$$

Which has theoretical solution $u(x,t) = u(x,t) = [x(x-1) + \delta/(6(1+\delta))]exp(-t)$ Where $\delta = 0.0144$ [3].

In Example 4 results computed are given in Table 5 and Table 6. In Table 5 results are calculated for $h = l = 0.01$ and for different values of t at x=1. From the table it is clear that the numerical solution calculated by using the scheme developed in this paper is good agrement with the exact ones. Also solution converges towards exact solution as t increases. In Table 6 results are given for $t = 1$ with $h = l = 0.01$, 0.05, 0.025, 0.0125, 0.0625, at $x = 0.5$ and $t = 1$. CPU time taken for the new scheme developed in this paper is also given in the table.

TABLE 5. Results for h=0.01 at x=1

t	Exact u	Numerical Solution	Absolute Error
$h = 0.1$	-0.00681710790377	-0.00681710791411	6.4×10^{-15}
$h = 0.2$	-0.00061683743112	-0.00616837431411	8.2×10^{-15}
$h = 0.3$	-0.00558137588787	-0.00558137588786	9.3×10^{-15}
$h = 0.4$	-0.00505023774747	-0.00505023774746	8.3×10^{-15}
$h = 0.5$	-0.00456894408389	-0.00456964408388	7.4×10^{-15}
$h = 0.6$	-0.00413478495421	-0.00413478495420	7.4×10^{-15}
$h = 0.7$	-0.00374130814210	-0.00374130814209	5.7×10^{-15}
$h = 0.8$	-0.00338527559937	-0.00338527559937	5.5×10^{-15}
$h = 0.9$	-0.00306312403268	-0.00306312403267	5.6×10^{-15}
$h = 1.0$	-0.00277162924085	-0.00277162924085	4.8×10^{-15}

TABLE 6. Results for different spatial lengths at t=1

Spatial length	Absolute Errors at x=0.5	Absolute Errors at x=1.0	CPU Time in Seconds
$h = 0.100000$	1.2×10^{-09}	3.8×10^{-09}	0.032
$h = 0.050000$	6.8×10^{-11}	1.0×10^{-11}	0.047
$h = 0.025000$	3.0×10^{-12}	1.7×10^{-13}	0.281
$h = 0.012500$	8.0×10^{-13}	4.0×10^{-15}	1.1547
$h = 0.006250$	9.0×10^{-13}	2.7×10^{-14}	11.141
$h = 0.003125$	1.1×10^{-14}	3.4×10^{-14}	114.266

4. Conclusion

It is observed that the results obtained using new scheme are highly accurate as compared to those of other schemes. This method do not use complex arithmatics which is unavoidable while using higher order pade approximations. The use of complex arithmetics needs more storage capacity and increase CPU time. The scheme developed is fifth-order accurate in space and time as well L-acceptable. This technique can also be coded easily on serial and parallel computers.

Conflict of Interests

The authors declare that there is no conflict of interests.

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