NAIMARK-SACKER BIFURCATION OF A CERTAIN SECOND ORDER QUADRATIC FRACTIONAL DIFFERENCE EQUATION

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Abstract. We investigate the Naimark-Sacker Bifurcation of the equilibrium of some special cases of the difference equation

\[ x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n}, \]

where the parameters \( \beta, \gamma, \delta, B, C, D \) are nonnegative numbers which satisfy \( B + C + D > 0 \) and the initial conditions \( x_{-1} \) and \( x_0 \) are arbitrary nonnegative numbers such that \( B x_n x_{n-1} + C x_{n-1}^2 + D x_n > 0 \) for all \( n \geq 0 \).

Keywords: bifurcation; boundedness; difference equation; global attractivity; local stability; Naimark-Sacker

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1. Introduction and Preliminaries

In this paper we study the Naimark-Sacker bifurcation of the equilibrium of some special cases of the difference equation
\[ x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_n^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n}, \quad n = 0, 1, 2, \ldots, \quad (1.1) \]

where the parameters \( \beta, \gamma, \delta, B, C, D \) are nonnegative numbers which satisfy \( B + C + D > 0 \) and the initial conditions \( x_{-1} \) and \( x_0 \) are arbitrary nonnegative numbers such that \( B x_n x_{n-1} + C x_{n-1}^2 + D x_n > 0 \) for all \( n \geq 0 \).

Equation (1.1), which has been studied in [2, 3, 17], is a special case of a general second order quadratic fractional difference equation of the form

\[ x_{n+1} = \frac{A x_n^2 + B x_n x_{n-1} + C x_{n-1}^2 + D x_n + E x_{n-1} + F}{a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f}, \quad n = 0, 1, \ldots \quad (1.2) \]

with non-negative parameters and initial conditions such that \( A + B + C > 0, a + b + c + d + e + f > 0 \) and \( a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f > 0, n = 0, 1, \ldots \). Several global asymptotic results for some special cases of Eq.(1.2) were obtained in [9, 10, 11, 19].

The change of variable \( x_n = 1/u_n \) transforms Eq.(1.1) to the difference equation

\[ u_{n+1} = \frac{D u_{n-1}^2 + C u_n + B u_{n-1}}{\delta u_{n-1}^2 + \gamma u_n + \beta u_{n-1}}, \quad n = 0, 1, \ldots \quad (1.3) \]

where we assume that \( \delta + \beta + \gamma > 0 \) and that the non-negative initial conditions \( u_{-1}, u_0 \) are such that \( \delta u_{n-1}^2 + \gamma u_n + \beta u_{n-1} > 0 \) for all \( n \geq 0 \). Thus the results of this paper extends to Eq.(1.3).

The first systematic study of global dynamics of a special quadratic fractional case of Eq.(1.2) where \( A = C = D = a = c = d = 0 \) was performed in [2, 3]. Dynamics of some related quadratic fractional difference equations was considered in the papers [9, 10, 11, 19]. Complete linear stability analysis of the equilibrium and the period-two solution of Eq.(1.1) was presented in [17], where it was found that 10 special cases of Eq.(1.1) exhibit locally transition from local attractor to the local repeller by passing through the critical non-hyperbolic case where both eigenvalues of characteristic equation are complex conjugate numbers on the unit circle. See Table 1-5 for the list of all special cases of Eq.(1.2), which exhibits this kind of local stability character.

In this paper we will perform the Naimark-Sacker bifurcation analysis of some special cases of Eq.(1.1), with mentioned local behavior, which are obtained when one or more coefficients of equation are set to be zero.
Now we consider bifurcation of a fixed point of map associated to Eq.(1.1) in the case where
the eigenvalues are complex conjugates and of unit module. For the sake of convinience we
include Naimark-Sacker bifurcation theorem, known also as Poincaré-Andronov-Hopf bifurca-
tion theorem for maps, see [6, 8, 20]:

**Theorem 3.1.** Let

\[ F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to F(\lambda, x) \]

be a \( C^4 \) map depending on real parameter \( \lambda \) satisfying the following conditions:

(i) \( F(\lambda, 0) = 0 \) for \( \lambda \) near some fixed \( \lambda_0 \);

(ii) \( DF(\lambda, 0) \) has two non-real eigenvalues \( \mu(\lambda) \) and \( \bar{\mu}(\lambda) \) for \( \lambda \) near \( \lambda_0 \) with \( |\mu(\lambda_0)| = 1 \);

(iii) \( \frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) \neq 0 \) at \( \lambda = \lambda_0 \);

(iv) \( \mu^k(\lambda_0) \neq 1 \) for \( k = 1, 2, 3, 4 \).

Then there is a smooth \( \lambda \)-dependent change of coordinate bringing \( f \) into the form

\[ F(\lambda, x) = \mathcal{F}(\lambda, x) + O(\|x\|^5) \]

and there are smooth functions \( a(\lambda) \), \( b(\lambda) \), and \( \omega(\lambda) \) so that in polar coordinates the function \( \mathcal{F}(\lambda, x) \) is given by

\[
\begin{pmatrix}
    r \\
    \theta
\end{pmatrix} =
\begin{pmatrix}
    |\mu(\lambda)|r - a(\lambda)r^3 \\
    \theta + \omega(\lambda) + b(\lambda)r^2
\end{pmatrix}.
\]  \hspace{1cm} (1.5)

If \( a(\lambda_0) > 0 \) and \( d(\lambda_0) > 0 \) \((d(\lambda_0) < 0)\), then there is a neighborhood \( U \) of the origin and a \( \delta > 0 \) such that for \( |\lambda - \lambda_0| < \delta \) and \( x_0 \in U \), the \( \omega \)-limit set of \( x_0 \) is the origin if \( \lambda < \lambda_0 \) \((\lambda > \lambda_0)\) and belongs to a closed invariant \( C^1 \) curve \( \Gamma(\lambda) \) encircling the origin if \( \lambda > \lambda_0 \) \((\lambda < \lambda_0)\). Furthermore, \( \Gamma(\lambda_0) = 0 \).

If \( a(\lambda_0) < 0 \) and \( d(\lambda_0) > 0 \) \((d(\lambda_0) < 0)\), then there is a neighborhood \( U \) of the origin and a \( \delta > 0 \) such that for \( |\lambda - \lambda_0| < \delta \) and \( x_0 \in U \), then \( \alpha \)-limit set of \( x_0 \) is the origin if \( \lambda > \lambda_0 \) \((\lambda < \lambda_0)\) and belongs to a closed invariant \( C^1 \) curve \( \Gamma(\lambda) \) encircling the origin if \( \lambda < \lambda_0 \) \((\lambda > \lambda_0)\). Furthermore, \( \Gamma(\lambda_0) = 0 \).

Consider a general map \( F(\lambda, x) \) that has a fixed point at the origin with complex eigenvalues
\( \mu(\lambda) = \alpha(\lambda) + i\beta(\lambda) \) and \( \bar{\mu}(\lambda) = \alpha(\lambda) - i\beta(\lambda) \) satisfying \( \alpha(\lambda)^2 + \beta(\lambda)^2 = 1 \) and \( \beta(\lambda) \neq 0 \).
By putting the linear part of such a map into Jordan Canonical form, we may assume \( F \) to have the following form near the origin
\[
F(\lambda, x) = \begin{pmatrix}
\alpha(\lambda) & -\beta(\lambda) \\
\beta(\lambda) & \alpha(\lambda)
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(\lambda, x_1, x_2) \\ g_2(\lambda, x_1, x_2) \end{pmatrix}.
\]
(1.6)

Then the coefficient \( a(\lambda_0) \) of the cubic term in Eq.(1.4) in polar coordinates is equal to
\[
a(\lambda_0) = Re \left[ \frac{(1 - 2\mu(\lambda_0))\hat{\mu}_2(\lambda_0)}{1 - \mu(\lambda_0)} \xi_{11} \xi_{20} \right] + \frac{1}{2} |\xi_{11}|^2 + |\xi_{02}|^2 - Re(\hat{\mu}(\lambda_0)\xi_{21}),
\]
(1.7)

where
\[
\xi_{20} = \frac{1}{8} \left\{ (g_1)_x x_1 - (g_1)_x x_2 + 2(g_2)_x x_1 + i[(g_2)_x x_1 - (g_2)_x x_2 - 2(g_1)_x x_1] \right\},
\]
\[
\xi_{11} = \frac{1}{4} \left\{ (g_1)_x x_1 + (g_1)_x x_2 + i[(g_2)_x x_1 + (g_2)_x x_2] \right\},
\]
\[
\xi_{02} = \frac{1}{8} \left\{ (g_1)_x x_1 - (g_1)_x x_2 - 2(g_2)_x x_1 + i[(g_2)_x x_1 - (g_2)_x x_2 + 2(g_1)_x x_1] \right\},
\]
\[
\xi_{21} = \frac{1}{16} \left\{ (g_1)_x x_1 x_1 + (g_1)_x x_1 x_2 + (g_2)_x x_1 x_2 + (g_2)_x x_2 x_2
\]
\[
+i[(g_2)_x x_1 x_1 + (g_2)_x x_1 x_2 - (g_1)_x x_1 x_2 - (g_1)_x x_2 x_2] \right\}.
\]

**Table 1. Equations of type (1,2)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equilibrium point</th>
<th>Stability</th>
<th>The eigenvalues</th>
<th>( \mu(\delta_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{n+1} = \frac{\delta x_n}{\delta x_{n-1} + x_n} )</td>
<td>( \bar{x} = \sqrt{\frac{\delta}{\pi\delta}} )</td>
<td>a repeller for ( \delta &gt; 0 ). ( B &gt; 0 )</td>
<td>( \mu(\delta) = \frac{1 + i\sqrt{\delta B + 1}}{2(\delta B + 1)} ) ( \bar{x} )</td>
<td>(</td>
</tr>
<tr>
<td>( x_{n+1} = \frac{\delta x_n}{C_{n-1} + x_n} )</td>
<td>( \bar{x} = \frac{\sqrt{C_{n-1} + 1} - 1}{\delta} )</td>
<td>LAS for ( c\delta &lt; 2 ); ( a ) repeller for ( c\delta &gt; 2 ); a non-hyperbolic for ( c\delta = 2 )</td>
<td>( \mu(\delta) = \frac{C_{\delta}}{\delta + C_{\delta}} ) ( \bar{C}<em>{\delta} ) ( \bar{C}</em>{\delta} + 1 ) ( \bar{C}_{\delta} + 2 )</td>
<td>(</td>
</tr>
</tbody>
</table>

The rest of the paper is organized into one section with two subsections where Naimark-Sacker bifurcation analysis is performed for one equation of type (2,2) and one equation of type (1,3). It will turn out that indeed both considered equations undergo the Naimark-Sacker bifurcation. The bifurcation analysis of the remaining two equations of type (2,2) will be performed in the separate paper as these two equations exhibit both Naimark-Sacker bifurcation and also period-doubling bifurcation, see [14, 15, 16]. We believe that in the case when these equations undergo period-doubling bifurcation we can make our results global.
2. Naimark-Sacker Bifurcation for Maps

2.1. The case \( x_{n+1} = \frac{\beta x_n x_{n-1} + \delta x_n}{C x_n^2 + x_n}, \quad \delta, C > 0, \quad \beta \geq 0 \)

It is easy to see that equation

\[
x_{n+1} = \frac{\beta x_n x_{n-1} + \delta x_n}{C x_n^2 + x_n}, \quad n = 0, 1, \ldots
\]

(2.1)

has the equilibrium point

\[
\bar{x} = \frac{\sqrt{4C\delta + (\beta - 1)^2 + \beta - 1}}{2C}. \tag{2.2}
\]

In order to apply Theorem we make a change of variable \( y_n = x_n - \bar{x} \). Then, transformed equation is given by

\[
y_{n+1} = \frac{(\bar{x} + y_n)(\beta (\bar{x} + y_{n-1}) + \delta)}{C (\bar{x} + y_{n-1})^2 + \bar{x} + y_n} - \bar{x}. \tag{2.3}
\]

By using the substitution \( u_n = y_{n-1}, \quad v_n = y_n \) we write Eq.(2.3) in the equivalent form:

\[
u_{n+1} = v_n
\]

\[
v_{n+1} = \frac{(2\bar{x} + v_n)(2\beta \bar{x} + \beta u_n + \delta)}{C (2\bar{x} + u_n)^2 + 2\bar{x} + v_n} - 2\bar{x}. \tag{2.4}
\]

Let \( F \) be the function defined by:

\[
F(u, v) = \left( \frac{v}{(\bar{x} + v)(\beta (\bar{x} + u) + \delta)} \right). \tag{2.4}
\]
<table>
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<tr>
<th>Equation</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( x_{n+1} = \frac{\beta x_n + \delta x_n}{\delta x_n + \gamma x_n} x_{n+1} )</td>
<td>( x = \sqrt{4B + \beta^2 + 4\beta + \beta} )</td>
<td>LAS for ( \beta \beta^2 &gt; \delta )</td>
<td>( \mu(\delta) = \frac{\sqrt{4B + \beta^2 + 4\beta + \beta}}{2(B + 1)^2} ) ( \Delta = 4(\beta x + \gamma)^2 (\beta x + (B + 2) \delta) - (\beta x + \delta)^2 &gt; 0 ) ( \mu(\delta_i) = \frac{1+i\sqrt{2B + 2}}{2(B + 1)} ) ( \delta_i = \beta \beta^2 )</td>
<td></td>
</tr>
<tr>
<td>( x_{n+1} = \frac{\beta x_n + \delta x_n}{\delta x_n + \gamma x_n} x_{n+1} )</td>
<td>( x = \sqrt{4\beta \epsilon^2 + \beta^2 + \beta - 1} )</td>
<td>LAS for ( c \delta &lt; 2(1 + \beta) )</td>
<td>( \mu(\delta) = \frac{c \delta + \delta x_n}{2(\beta + 1)^2} ) ( \Delta = (c \beta + C \delta)^2 ) ( -4(C \epsilon + 1)^2 (C \epsilon - 1 + 2C \delta) ) ( \mu(\delta_i) = \frac{1+i\beta}{2(1+\beta)} ) ( \Gamma = (\beta + 3)(3 \beta + 5) ) ( \delta_0 = \frac{2(1+\beta)}{C} )</td>
<td></td>
</tr>
<tr>
<td>( x_{n+1} = \frac{\gamma x_n + \delta x_n}{\delta x_n + \gamma x_n} x_{n+1} )</td>
<td>( x = \sqrt{4\gamma + 1 + \gamma - 1} )</td>
<td>LAS for ( \frac{3(B - 1)^2}{2(B + 1)^2} &lt; 4 \delta ) or ( \delta &lt; 2(2B + 1)^2 ) ( 4(B + 2)^2 \delta = 3(2B - 1)^2 )</td>
<td>( \mu(\delta) = \frac{C \delta + \gamma x_n}{2(\beta + 1)^2} ) ( \Delta = (\gamma - C \delta)^2 ) ( -4(C \gamma x_n + 2C x + 1) (2C \delta - 2 \gamma) ) ( \Delta &lt; 0 ) for ( c \delta &gt; \gamma ) ( \mu(\delta_i) = \frac{1+i\sqrt{2}}{4} ) ( \delta_i = \frac{(\gamma - 2)(2 + y_1)}{C} )</td>
<td></td>
</tr>
<tr>
<td>( x_{n+1} = \frac{\gamma x_n + \delta x_n}{\delta x_n + \gamma x_n} x_{n+1} )</td>
<td>( x = \sqrt{4\beta \epsilon^2 + 1 + \gamma - 1} )</td>
<td>LAS for ( \frac{3(\gamma - 2)}{(2 + y_1)} &lt; \delta ) and ( \frac{3(\gamma - 2)}{(2 + y_1)} &lt; \delta )</td>
<td>( \mu(\delta) = \frac{C \delta + \gamma x_n}{2(\beta + 1)^2} ) ( \Delta = (\gamma - C \delta)^2 ) ( -4(C \gamma x_n + 2C x + 1) (2C \delta - 2 \gamma) ) ( \Delta &lt; 0 ) for ( c \delta &gt; \gamma ) ( \mu(\delta_i) = \frac{1+i\sqrt{2}}{4} ) ( \delta_i = \frac{(\gamma - 2)(2 + y_1)}{C} )</td>
<td></td>
</tr>
</tbody>
</table>

Then \( F \) has the unique fixed point \( (0, 0) \) and maps \( (-\bar{x}, \infty) \) into \( (-\bar{x}, \infty) \). The Jacobian matrix of \( F \) is given by

\[
Jac_F(u, v) = \begin{pmatrix}
0 & 1 \\
(C(x + \bar{x})^2 + \gamma + \bar{x} - 2C(x + \bar{x})(\delta + \beta(x + \bar{x}))) & \frac{C(x + \bar{x})^2 (\delta + \beta(x + \bar{x}))}{C(x + \bar{x})^2 + \gamma + \bar{x}}
\end{pmatrix}.
\]
The eigenvalues of (2.5) are \( \mu(\bar{\delta}) \) where
\[
\mu(\bar{\delta}) = \pm \sqrt{(C\bar{x} + C\delta)^2 - 4(C\bar{x} + 1)^2(\beta(C\bar{x} - 1) + 2C\delta) + C\beta\bar{x} + C\delta}.
\]

Then we have that
\[
F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-C\bar{x}\beta + \beta - 2C\delta}{(C\bar{x} + 1)^2} & C(\bar{x} + \beta \bar{x}) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(\bar{\delta}, u, v) \\ f_2(\bar{\delta}, u, v) \end{pmatrix},
\]
and
\[
f_1(\bar{\delta}, u, v) = 0,
\]
\[
f_2(\bar{\delta}, u, v) = \frac{(\bar{x} + v)(\beta(\bar{x} + u) + \delta)}{C(\bar{x} + u)^2 + \bar{x} + v} + \frac{u(\beta(C\bar{x} - 1) + 2C\delta)}{(C\bar{x} + 1)^2} - \frac{Cv(\beta \bar{x} + \delta)}{(C\bar{x} + 1)^2} - \bar{x}.
\]

Let
\[
\delta_0 = \frac{2(1 + \beta)}{C}.
\]
For $\delta = \delta_0$ we obtain
\[
\bar{x} = \frac{1+\beta}{C} \quad \text{and} \quad J_0 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{\beta+1}{\beta+2} \end{pmatrix}.
\]
The eigenvalues of $J_0$ are $\mu(\delta_0)$ and $\overline{\mu(\delta_0)}$ where
\[
\mu(\delta_0) = \frac{\beta + i\sqrt{(\beta+3)(3\beta+5)} + 1}{2(\beta+2)}.
\]
The eigenvectors corresponding to $\mu(\delta)$ and $\overline{\mu(\delta)}$ are $v(\delta_0)$ and $\overline{v(\delta_0)}$ where
\[
v(\delta_0) = \begin{pmatrix} \beta - i\sqrt{(\beta+3)(3\beta+5)} + 1 \\ 2(\beta+2) \end{pmatrix}.
\]
One can prove that
\[
|\mu(\delta_0)| = 1,
\]
\[
\mu^2(\delta_0) = -\frac{\beta(\beta+6) + 7}{2(\beta+2)^2} + \frac{i(\beta+1)\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)^2},
\]
\[
\mu^3(\delta_0) = -\frac{(\beta+1)(2\beta(\beta+5)+11)}{2(\beta+2)^3} - \frac{i\sqrt{(\beta+3)(3\beta+5)}(2\beta+3)}{2(\beta+2)^3},
\]
\[
\mu^4(\delta_0) = \frac{17 - (\beta-2)\beta(\beta+6)+10)}{2(\beta+2)^4} - \frac{i(\beta+1)\sqrt{(\beta+3)(3\beta+5)}(\beta(\beta+6)+7)}{2(\beta+2)^4},
\]
from which follows that $\mu^k(\delta_0) \neq 1$ for $k = 1,2,3,4$ and $\beta \geq 0$. Substituting $\delta = \delta_0$ and $\bar{x}$ into (2.6) we get
\[
F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{\beta+1}{\beta+2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix},
\]
and
\[
h_1(u,v) = f_1(\delta_0,u,v) = 0
\]
\[
h_2(u,v) = f_2(\delta_0,u,v) = -\frac{C(-Cu^2(\beta+2)+u^2(\beta+1)(Cv-\beta-2)+uv\beta(\beta+1)+v^2(\beta+1))}{(\beta+2)(C(u(Cu+2)+v)+2Cu\beta+\beta^2+3\beta+2)}.
\]
Hence, for $\delta = \delta_0$ system (2.4) is equivalent to
\[
\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{\beta+1}{\beta+2} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} h_1(u_n,v_n) \\ h_2(u_n,v_n) \end{pmatrix}.
\]
where
\[ P = \begin{pmatrix}
\frac{\beta+1}{2(\beta+2)} & \frac{\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)} \\
1 & 0
\end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix}
0 & 1 \\
\frac{2(\beta+2)}{\sqrt{(\beta+3)(3\beta+5)}} & -\frac{\beta+1}{\sqrt{(\beta+3)(3\beta+5)}}
\end{pmatrix}.\]

Then system (2.8) is equivalent to its normal form
\[
\begin{pmatrix}
\xi_{n+1} \\
\eta_{n+1}
\end{pmatrix} = \begin{pmatrix}
\frac{\beta+1}{2\beta+4} & -\frac{\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)} \\
\frac{\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)} & \frac{\beta+1}{2\beta+4}
\end{pmatrix}\begin{pmatrix}
\xi_n \\
\eta_n
\end{pmatrix} + G\begin{pmatrix}
\xi_n \\
\eta_n
\end{pmatrix},
\]

where
\[ H\begin{pmatrix}u \\ v\end{pmatrix} := \begin{pmatrix}h_1(u,v) \\ h_2(u,v)\end{pmatrix}.\]

Let
\[
G\begin{pmatrix}u \\ v\end{pmatrix} = \begin{pmatrix}g_1(u,v) \\ g_2(u,v)\end{pmatrix} = P^{-1}H\begin{pmatrix}u \\ v\end{pmatrix}.
\]

By straightforward calculation we obtain that
\[
g_1(u,v) = \frac{\Delta_1}{\Delta_2} - \frac{2u(\beta+1)}{2(\beta+2)},
\]
\[
g_2(u,v) = -\frac{\beta+1}{\sqrt{(\beta+3)(3\beta+5)}}g_1(u,v),
\]

where
\[
\Delta_1 = C\left(u\beta + u + v\sqrt{(\beta+3)(3\beta+5)} + 2(\beta+1)(\beta+2)\right) \\
\left(Cu^2(\beta+1)^2 + 2u(\beta+1)\left(C\sqrt{(\beta+3)(3\beta+5)} + 2\beta(\beta+4) + 8\right) + Cv^2(\beta+3)(3\beta+5)\right)
\]
\[
\Delta_2 = C^2\left(u^2(\beta+1)^2 + 2uv(\beta+1)\sqrt{(\beta+3)(3\beta+5)} + v^2(\beta+3)(3\beta+5)\right) \\
+ 4C(\beta+2)\left(u(\beta(\beta+3)+3) + v\sqrt{(\beta+3)(3\beta+5)}(\beta+1)\right) + 4(\beta+1)(\beta+2)^3.
\]
and furthermore

\[ \xi_{20}(0,0) = \frac{C\sqrt{(\beta + 3)(3\beta + 5)}(-\beta - 3) + iC(2\beta^2 + 2\beta - 1)}{4(\beta + 2)^2\sqrt{(\beta + 3)(3\beta + 5)}}, \]

\[ \xi_{11}(0,0) = \frac{C(\beta + 1)(\beta + 4)(-i\beta + \sqrt{(\beta + 3)(3\beta + 5)} - i)}{4(\beta + 2)^3\sqrt{(\beta + 3)(3\beta + 5)}}, \]

\[ \xi_{02}(0,0) = \frac{C\sqrt{(\beta + 3)(3\beta + 5)}(-\beta^2 - 4\beta - 5) + iC(3\beta^2 + 7\beta^2 + 17\beta + 13)}{4(\beta + 2)^3\sqrt{(\beta + 3)(3\beta + 5)}}, \]

\[ \xi_{21}(0,0) = \frac{C^2\beta\sqrt{(\beta + 3)(3\beta + 5)}(\beta^2 + 3\beta) + iC^2\beta(3\beta^3 + 24\beta^2 + 57\beta + 40)}{8(\beta + 1)(\beta + 2)^4\sqrt{(\beta + 3)(3\beta + 5)}}. \]

Since

\[ \text{Re} \left( \frac{1 - 2\mu(\lambda_0)}{1 - \mu(\lambda_0)} \xi_{11}(\delta_0) \xi_{20}(\delta_0) \right) = \frac{C^2(\beta(\beta + 5) + 5)}{4(\beta + 1)(\beta + 2)^4}, \]

\[ \text{Re} \left( \frac{(1 - 2\mu(\lambda_0))\mu(\lambda_0)}{1 - \mu(\lambda_0)} \xi_{11}(\delta_0) \xi_{20}(\delta_0) \right) = \frac{C^2(\beta(\beta + 4) + 2)}{4(\beta + 2)^4(\beta + 3)(3\beta + 5)}, \]

\[ \xi_{11}(\delta_0) \xi_{11}(\delta_0) = \frac{C^2(\beta(\beta + 5) + 5)}{4(\beta + 1)^2(\beta + 2)^4(\beta + 3)}, \]

then by using (2.9) and after lengthy calculation we obtain that

\[ a(\delta_0) = \frac{C^2(\beta(2\beta + 17) + 43) + 32}{8(\beta + 1)(\beta + 2)^4(\beta + 3)} > 0. \]

One can see that

\[ |\mu(\delta)|^2 = \mu(\delta)\overline{\mu(\delta)} = \frac{\beta(\sqrt{C\delta} - 1) + 2C\delta}{(\sqrt{C\delta} + 1)^2} = \frac{2(\beta \left( \frac{\sqrt{4C\delta + (\beta - 1)^2} + \beta - 3}{\sqrt{4C\delta + (\beta - 1)^2 + \beta + 1}} \right) + 4C\delta)}{\left( \sqrt{4C\delta + (\beta - 1)^2 + \beta + 1} \right)^2}, \]

from which we get

\[ \frac{d}{d\delta} |\mu(\delta)| = \frac{\sqrt{2C(\beta + 2)}}{\left( \sqrt{4C\delta + (\beta - 1)^2 + \beta + 1} \right) \sqrt{(4C\delta + (\beta - 1)^2)(\beta \left( \sqrt{4C\delta + (\beta - 1)^2 + \beta - 3} + 4C\delta \right))}, \]

which by substituting (2.7) simplifies to

\[ \frac{d|\mu(\delta)|}{d\delta} \bigg|_{\delta = \delta_0} = \frac{C}{2(\beta^2 + 5\beta + 6)} > 0. \]

Thus we have proved the following result:
Theorem 2.1. Let
\[ \delta_0 = \frac{2(1 + \beta)}{C} \quad \text{and} \quad \bar{x} = \frac{\sqrt{4C\delta + (\beta - 1)^2} + \beta - 1}{2C}. \]
Assume that \( C, \delta > 0 \) and \( \beta \geq 0 \). Then there is a neighborhood \( U \) of the equilibrium point \( \bar{x} \) and a \( \rho > 0 \) such that for \( |\delta - \delta_0| < \rho \) and \( x_0, x_{-1} \in U \), the \( \omega \)-limit set of solution of Eq.(2.1), with initial condition \( x_0, x_{-1} \) is the equilibrium point \( \bar{x} \) if \( \delta < \delta_0 \) and belongs to a closed invariant \( C^1 \) curve \( \Gamma(\delta) \) encircling \( \bar{x} \) if \( \delta > \delta_0 \). Furthermore, \( \Gamma(\delta_0) = 0 \).

The visual illustration of Theorem 2.1. is given in Figure 1.
2.2. The case \( x_{n+1} = \frac{x_n}{Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n^2}, \ B, C, D > 0 \)

Equation

\[
x_{n+1} = \frac{x_n}{Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n^2}, \quad n = 0, 1, \ldots
\]

has the equilibrium point \( \bar{x} = \frac{2}{\sqrt{4(B+C) + D^2}} \).

In order to apply Theorem we make a change of variable \( y_n = x_n - \bar{x} \). Then, new equation is given by

\[
y_{n+1} = \frac{\bar{x} + y_n}{B(\bar{x} + y_n)(\bar{x} + y_{n-1}) + C(\bar{x} + y_{n-1})^2 + D(\bar{x} + y_{n-1})^2 - \bar{x}}.
\]

By using the substitution \( u_n = y_{n-1}, v_n = y_n \) we write Eq.(2.11) in the equivalent form:

\[
\begin{align*}
u_{n+1} &= v_n \\
v_{n+1} &= \frac{\bar{x} + v_n}{B(\bar{x} + u_n)(\bar{x} + v_n) + C(\bar{x} + u_n)^2 + D(\bar{x} + v_n)^2 - \bar{x}}.
\end{align*}
\]

Define the map \( F \) as:

\[
F(u, v) = \begin{pmatrix} u \\ v \\ \frac{\bar{x} + v}{B(\bar{x} + u)(\bar{x} + v) + C(\bar{x} + u)^2 + D(\bar{x} + v)^2 - \bar{x}} \end{pmatrix}.
\]

Then \( F \) has the unique fixed point \((0, 0)\) and maps \((-\bar{x}, \infty)^2\) into \((-\bar{x}, \infty)^2\). The Jacobian matrix of \( F \) is given by

\[
Jac_F(u, v) = \begin{pmatrix} 0 & 1 \\ -\frac{(v + \bar{x})(2C(u + \bar{x}) + B(v + \bar{x}))}{(D(v + \bar{x}) + (u + \bar{x})(C(u + \bar{x}) + B(v + \bar{x}))^2} & \frac{1}{C(u + \bar{x})^2} \\ -\frac{B + 2C}{(D + (B+C)\bar{x})^2} & \frac{1}{(D + (B+C)\bar{x})^2} \end{pmatrix}.
\]

At \((0, 0)\), \( Jac_F(u, v) \) has the form

\[
J_0 = JacF(0, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{B + 2C}{(D + (B+C)\bar{x})^2} & \frac{1}{(D + (B+C)\bar{x})^2} \end{pmatrix}.
\]

The eigenvalues of (2.13) are \( \mu(B) \) and \( \overline{\mu(B)} \) where

\[
\mu(B) = \frac{C + i \sqrt{4(B + 2C)(\bar{x}(B + C) + D)^2 - C^2}}{2(\bar{x}(B + C) + D)^2},
\]

since

\[
C^2 - 4(B + 2C)(\bar{x}(B + C) + D)^2 < 0.
\]
Now we have
\[
F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{B+2C}{(D+(B+C)\bar{x})^2} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(\delta, u, v) \\ f_2(\delta, u, v) \end{pmatrix},
\]
(2.14)
and
\[
f_1(\delta, u, v) = 0
\]
\[
f_2(\delta, u, v) = \frac{\bar{x} + v}{(\bar{x} + u)(B(\bar{x} + v) + C(\bar{x} + u)) + D(\bar{x} + v)} + \frac{u(B+2C)}{(\bar{x}(B+C)+D)^2} - \frac{Cv}{(\bar{x}(B+C)+D)^2} - \bar{x}.
\]
Let
\[
B_0 = \frac{C(C-2D^2)}{D^2} \quad \text{and} \quad C - 2D^2 > 0.
\]
(2.15)
For \( B = B_0 \) we obtain
\[
\bar{x} = \frac{D}{C} \quad \text{and} \quad J_0 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{D^2}{C} \end{pmatrix}.
\]
The eigenvalues of \( J_0 \) are \( \mu(B_0) \) and \( \overline{\mu(B_0)} \) where
\[
\mu(B_0) = \frac{D^2 + i\sqrt{4C^2 - D^4}}{2C}.
\]
The eigenvectors corresponding to \( \mu(B) \) and \( \overline{\mu(B)} \) are \( v(B_0) \) and \( \overline{v(B_0)} \) where
\[
v(B_0) = \left( \frac{D^2 - i\sqrt{4C^2 - D^4}}{2C}, 1 \right).
\]
One can prove that
\[
|\mu(B_0)| = 1,
\]
\[
\mu^2(B_0) = \frac{D^4}{2C^2} + \frac{iD^2\sqrt{4C^2 - D^4}}{2C^2} - 1,
\]
\[
\mu^3(B_0) = \frac{D^6 - 3C^2D^2}{2C^3} - \frac{i(C^2 - D^4)\sqrt{4C^2 - D^4}}{2C^3},
\]
\[
\mu^4(B_0) = \frac{D^8}{2C^4} - \frac{2D^4}{C^2} + \frac{iD^2\sqrt{4C^2 - D^4}(D^4 - 2C^2)}{2C^4} + 1
\]
from which follows that \( \mu^k(B_0) \neq 1 \) for \( k = 1, 2, 3, 4 \).
Substituting \( B = B_0 \) and \( \bar{x} \) into (2.14) we obtain
\[
F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{D^2}{C} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_1(u, v) \\ h_2(u, v) \end{pmatrix},
\]
h_1(u,v) = f_1(B_0,u,v) = 0,

h_2(u,v) = f_2(B_0,u,v) = \frac{C^3 u^2 v + C^2 Du(Du^2 - 2Duv - Dv^2 + u) - CD^3 (u^2(Dv + 1) - 2Duv^2 + v^2) + D^5 v^2}{C(C^2 uv + D^2(Cu(u - 2v) + 1) + CD(u + v) + D^3(-v))}.

Hence, for \( B = B_0 \) system (2.12) is equivalent to

\[
\begin{pmatrix}
    u_{n+1} \\
    v_{n+1}
\end{pmatrix}
= \begin{pmatrix}
    0 & 1 \\
    -1 & \frac{D^2}{C}
\end{pmatrix}
\begin{pmatrix}
    u_n \\
    v_n
\end{pmatrix}
+ \begin{pmatrix}
    h_1(u_n,v_n) \\
    h_2(u_n,v_n)
\end{pmatrix},
\]  
(2.16)

Let

\[
\begin{pmatrix}
    u_n \\
    v_n
\end{pmatrix}
= P\begin{pmatrix}
    \xi_n \\
    \eta_n
\end{pmatrix}
\]

where

\[
P = \begin{pmatrix}
    \frac{D^2}{2C} & \frac{\sqrt{4C^2-D^4}}{2C} \\
    1 & 0
\end{pmatrix}
\quad \text{and} \quad
P^{-1} = \begin{pmatrix}
    0 & 1 \\
    \frac{2C}{\sqrt{4C^2-D^4}} & -\frac{D^2}{\sqrt{4C^2-D^4}}
\end{pmatrix}.
\]

Then system (2.16) is equivalent to its normal form

\[
\begin{pmatrix}
    \xi_{n+1} \\
    \eta_{n+1}
\end{pmatrix}
= \begin{pmatrix}
    \frac{D^2}{2C} & -\frac{\sqrt{4C^2-D^4}}{2C} \\
    \frac{\sqrt{4C^2-D^4}}{2C} & \frac{D^2}{2C}
\end{pmatrix}
\begin{pmatrix}
    \xi_n \\
    \eta_n
\end{pmatrix}
+ P^{-1} H \begin{pmatrix}
    P\begin{pmatrix}
    \xi_n \\
    \eta_n
    \end{pmatrix}
\end{pmatrix},
\]

where

\[
H \begin{pmatrix}
    u \\
    v
\end{pmatrix}
:= \begin{pmatrix}
    h_1(u,v) \\
    h_2(u,v)
\end{pmatrix}.
\]

Let

\[
G \begin{pmatrix}
    u \\
    v
\end{pmatrix}
= \begin{pmatrix}
    g_1(u,v) \\
    g_2(u,v)
\end{pmatrix}
= P^{-1} H \begin{pmatrix}
    P\begin{pmatrix}
    u \\
    v
    \end{pmatrix}
\end{pmatrix}.
\]

By a straightforward calculation we obtain that

\[
g_1(u,v) = \frac{\Lambda(u,v)}{\Upsilon(u,v)}
\]

\[
g_2(u,v) = -\frac{D^2 \Lambda(u,v)}{\Upsilon(u,v) \sqrt{4C^2-D^4}}.
\]
where

\[ \Lambda(u, v) = 8C^4uv^2 + 8C^3Dv^2(1 - 2Du) - 2C^2D^3(Du^3 - Du^2v + 4u^2 + 4v^2) \]

\[ - D^2v\sqrt{4C^2 - D^4}(-4C^2v^2 - 4CDu + D^3(Du^2 + Dv^2 + 4u)) \]

\[ + 2CD^5(2Du^3 + 2Duv^2 + 5u^2 - v^2) + D^7(-(Du^3 + Duv^2 + 2u^2 - 2v^2)) \],

\[ Y(u, v) = 2CD(2C^2(Du^2 - 2Du^2 + 2u) - 2CD(2D^2u^2 + Du - 2) + D^5(u^2 - v^2)) \]

\[ + 4C\sqrt{4C^2 - D^4}(C^2u + C(D - 2D^2u) + D^4u) \].

By Mathematica aided calculation we obtain that

\[ \xi_{20}(0, 0) = -\frac{(C - D^2)\left(C\sqrt{4C^2 - D^4} + D^2\sqrt{4C^2 - D^4} + iCD^2 - iD^4\right)}{4CD\sqrt{4C^2 - D^4}} \]

\[ \xi_{11}(0, 0) = \frac{(C - D^2)^2\left(\sqrt{4C^2 - D^4} - iD^2\right)}{2CD\sqrt{4C^2 - D^4}} \]

\[ \xi_{02}(0, 0) = -\frac{(C - D^2)\left(-D^4\sqrt{4C^2 - D^4} + CD^2\left(\sqrt{4C^2 - D^4} - iD^2\right) + C^2\left(\sqrt{4C^2 - D^4} - 3iD^2\right) + iD^6\right)}{4C^2D^2\sqrt{4C^2 - D^4}} \]

\[ \xi_{21}(0, 0) = \frac{(C - 2D^2)\left(6iC^3 - 2D^4\sqrt{4C^2 - D^4} + 3CD^2\sqrt{4C^2 - D^4} - 5iCD^4 + 2iD^6\right)}{8CD^2\sqrt{4C^2 - D^4}}. \]

Now we have

\[ \text{Re}\left(\frac{\mu(B_0)\xi_{21}(B_0)}{1 - \mu(\lambda_0)}\xi_{11}(\delta_0)\xi_{20}(\delta_0)\right) = \frac{(C - 2D^2)(3C^2 - D^4)}{8CD^2} \]

\[ \text{Re}\left(\frac{1 - 2\mu(B_0)}{1 - \mu(\lambda_0)}\xi_{11}(\delta_0)\xi_{20}(\delta_0)\right) = \frac{(C - D^2)^2(3C^2 + 4CD^2 - D^4)}{16C^3D^2 - 4CD^6} \]

\[ \xi_{11}(B_0)\xi_{11}(B_0) = \frac{(C - D^2)^4}{4C^2D^2 - D^6} \]

\[ \xi_{02}(B_0)\xi_{02}(B_0) = \frac{(C - D^2)^2(3C^3 + 2C^2D^2 + CD^4 - D^6)}{16C^3D^2 - 4CD^6}. \]

By using (2.17) after lengthy calculation we obtain that

\[ a(B_0) = \frac{C^3 - 10C^2D^2 + 9CD^4 - 2D^6}{8C(D^2 - 2C)} > 0. \]
One can see that
\[
|\mu(B)|^2 = \mu(B)\ov{\mu(B)} = \frac{B+2C}{(\ov{\mu}(B+C)+D)^2} = \frac{B+2C}{\left(\frac{2(B+C)}{\sqrt{4(B+C)+D^2}} + D\right)^2},
\]
from which we obtain
\[
\frac{d}{dB}|\mu(B)| = \frac{D^2 \left(\sqrt{4(B+C)+D^2+D}\right) - 2C\sqrt{4(B+C)+D^2+2BD}}{\sqrt{(B+2C)(4(B+C)+D^2)}} \left(\sqrt{4(B+C)+D^2+D}\right) \left(D \left(\sqrt{4(B+C)+D^2+D}\right) + 2B+2C\right),
\]
which in view of (2.15) yields
\[
\left.\frac{d|\mu(B)|}{dB}\right|_{B=B_0} = \frac{D^4}{2C^2(D^2-2C)} < 0.
\]
Thus we have proved the following.

**Figure 2.** Figures a) and b): Bifurcation diagram in \((B-x)\) plane for \(C = 3.6\) and \(D = 0.5\); Figure c): Phase portraits when \(B = 33.6, x_{-1} = 0.5, x_0 = 0.5\) (blue) \(x_{-1} = 1.5, x_0 = 1.0\) (red) Figure d): Phase portraits when \(B = 60.6, x_{-1} = 0.5, x_0 = 0.5\) (blue) \(x_{-1} = 1.5, x_0 = 1.0\) (red).
Theorem 2.2. Let

\[ B_0 = \frac{C(C - 2D^2)}{D^2}, \quad C - 2D^2 > 0 \quad \text{and} \quad \bar{x} = \frac{D}{C}. \]

Then there is a neighborhood \( U \) of the equilibrium point \( \bar{x} \) and a \( \rho > 0 \) such that for \( |B - B_0| < \rho \) and \( x_0, x_{-1} \in U \), the \( \omega \)-limit set of a solution of Eq.(2.10), with initial condition \( x_0, x_{-1} \) is the equilibrium point \( \bar{x} \) if \( B > B_0 \) and belongs to a closed invariant \( C^1 \) curve \( \Gamma(B) \) encircling \( \bar{x} \) if \( B < B_0 \). Furthermore, \( \Gamma(B_0) = 0 \).

Visual illustration of Theorems is given in Figure 2. Based on our simulations we pose the following.

Conjecture 2.1. In both considered equations the equilibrium is globally asymptotically stable whenever is locally stable.

3. Conclusion

In this paper we have found normal forms and performed bifurcation analysis for two special cases of Eq.(1.1) for which the local stability analysis indicates the possibility of Naimark-Sacker bifurcation. Indeed, we found the regions of parameters where Naimark-Sacker bifurcation occurs showing in such a way the presence of a locally stable periodic solution of unknown period. We conjectured that the equilibrium is globally asymptotically stable in the complement of the parametric region where Naimark-Sacker bifurcation occurs.

Conflict of Interests

The authors declare that they have no competing interests.

REFERENCES


