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GRAPH CONVERGENCE FOR $H(\cdot,\cdot)$-COCOERCIVE OPERATOR IN q-UNIFORMLY SMOOTH BANACH SPACES WITH AN APPLICATION

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Abstract. In this paper, we consider a class of $H(.,.)$-cocoercive operator, which generalizes many existing monotone operators. Further, we introduce a concept of graph convergence concerned with the $H(.,.)$-cocoercive operator in q-uniformly smooth Banach spaces and given an equivalence theorem between graph-convergence and resolvent operator convergence for the $H(.,.)$-cocoercive operator. As an application, a perturbed algorithm for solving a class of variational inclusion involving $H(.,.)$-cocoercive operator is constructed. Furthermore, under some suitable conditions, the existence of the solution for the variational inclusion and the convergence of iterative sequence generated by perturbed algorithm are given.

Keywords: $H(\cdot,\cdot)$-cocoercive operator; graph convergence; resolvent operator; variational inclusion; iterative algorithm.

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1. Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years, see for example [1,2,4,5-12,14,15,19,23,25]. Various iterative schemes have been proposed for solving variational inequalities, but the convergence proof required the

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underlying mapping to be strongly monotone over a feasible set. Cocoercivity is a weaker concept than strong monotonicity. A strongly monotone and Lipschitz continuous mapping must be cocoercive but conversely, a cocoercive mapping is monotone but not necessarily strongly monotone or even strictly monotone. Tseng [20] and Marcotte and Wu [18] studied the convergence of iterative processes when the underlying mapping is affine and cocoercive. Zhu and Marcotte [24] investigated iterative schemes for solving non-linear variational inequalities under the cocoercivity assumptions.

Recently, many authors have studied the perturbed algorithms for solving variational inequalities involving maximal monotone mapping in Hilbert space. Using the concept of graph-convergence for maximal monotone mappings and the equivalence between graph-convergence and resolvent operator convergence considered by Attouch [3], they constructed some perturbed algorithms for variational inequality and proved the convergence of sequences generated by perturbed algorithms under some suitable conditions, see for example [1,5,7,10,11,14,16].

On the other hand, in 2001, Huang and Fang [13] were the first to introduce the generalized m-accretive mapping and given the definition of the resolvent operator for the generalized m-accretive mapping in Banach spaces. They also showed some properties of the resolvent operator for the generalized m-accretive mappings in Banach spaces. Recently, many authors have studied and investigated several generalized operators such as H-monotone, H-accretive, (H,η)-monotone, (H,η)-accretive, $H(\cdot,\cdot)$-accretive and $H(\cdot,\cdot)$-cocoercive operators in Hilbert and Banach spaces, see for example [2,5-9,12-14,23-25]. They studied some properties of these operators and defined resolvent operators associated with these operators. They also studied the existence of solutions for some classes of variational inclusions using the resolvent operator technique. The resolvent operator technique is used to established an equivalence between variational inequalities (inclusions) and resolvent operator equations for finding powerful and implementable numerical techniques for solving various variational inequalities (inclusions). For details, see [1,2,4,5-12,14,15,19,23,25].

Recently, Zou and Huang [25], Fang and Huang [9], Xu and Wang [23] and Ahmad *et al.* [2] introduced and studied the concept of $H(\cdot,\cdot)$-accretive, $H(\cdot,\cdot)$-η-accretive and $H(\cdot,\cdot)$-cocoercive operators in Banach spaces. Very, recently Li and Huang [22] studied the concept of graph-convergence for the $H(\cdot,\cdot)$-accretive operator in Banach spaces and given an application for solving a class of variational inclusions.

Inspired by the work above, in this paper, we introduce a concept of graph convergence for the $H(.,.)$-cocoercive operator in q-uniifromly smooth Banach spaces and given an equivalence theorem between graph-convergence and resolvent operator convergence for the $H(.,.)$-cocoercive operator. As an application, a perturbed algorithm for solving a class of variational inclusion involving $H(.,.)$-cocoercive operator is constructed. Furthermore, under some suitable conditions, the existence of the solution for the variational inclusion and the convergence of iterative sequence generated by perturbed algorithm are given. The theorems presented in this paper generalize, improve and unify the results given in [1,5,7,10,11,14,16].

2. Preliminaries

Let E be a real Banach space equipped with norm $||\cdot||$ and E^* be the topological dual space of E. Let $\langle .,. \rangle$ be the dual pair between E and $E^\star$ and 2^E be the family of all the nonempty subsets of E.

Definition 2.1 [21]. For $q > 1$, a mapping $J_q : E \to 2^{E^*}$ is said to be *generalized duality mapping* defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^q,\ ||x||^{q-1} = ||f^*||\},\ \forall x \in E. \tag{2.1}$$

In particular, J_2 is the usual *normalized duality mapping* on E. It is well known (see e.g., [21]) that $J_q(x) = ||x||^{q-2}J_2(x),\ \forall\ x(\neq 0) \in E$ and J_q is single-valued if E^* is strictly convex. In the sequel, we always assume that E is a real Banach space such that J_q is single-valued. If $E \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on H.

Definition 2.2 [6]. A Banach space E is called *smooth* if, for every $x \in E$ with $||x|| = 1$, there exists a unique $f \in E^*$ such that $||f|| = f(x) = 1$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$, defined by

$$\rho_E(\tau) = \sup\Big\{\frac{(||x+y|| + ||x-y||)}{2} - 1 :\ ||x|| = 1, ||y|| = \tau,\ \forall x, y \in E\Big\}.$$

Definition 2.3 [22]. The Banach space E is said to be

(i) *uniformly smooth,* if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0;$$

(ii) *q-uniformly smooth*, for $q>1$, if there exists a constant $c>0$ such that

$$\rho_E(\tau)\leq c\tau^q,\ \tau\in[0,\infty). \tag{2.2}$$

It is well known (see, e.g., [21]) that

$$L_q\ (\text{or } l_q) \text{ is} \begin{cases} q-\textit{uniformly smooth}, \text{ if } 1<q\leq 2 \\ \\ 2-\textit{uniformly smooth}, \text{ if } q\geq 2. \end{cases}$$

Note that, if E is uniformly smooth, J_q becomes single-valued. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [21] established the following lemma.

Lemma 2.1. *Let $q>1$ be a real number and let E be a smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q>0$ such that for every $x,y\in E$,*

$$||x+y||^q\leq ||x||^q+q\langle y,J_q(x)\rangle+c_q||y||^q. \tag{2.3}$$

From Lemma 2 of Liu [17], it is easy to have the following lemma.

Lemma 2.2. *Let $\{a_n\}$ and $\{b_n\}$ be two real sequences of nonnegative real numbers satisfying*

$$a_{n+1}\leq ka_n+b_n$$

with $0<k<1$ and $b_n\to 0$. Then $\lim_{n\to\infty}a_n=0$.

3. $H(.,.)$-cocoercive operator

Throughout the rest of the paper unless otherwise stated, we assume that E is q-uniformly smooth Banach space. In this section, we give some properties of $H(.,.)$-cocoercive operator.

Definition 3.1. Let $M:E\to E$ be a single-valued mappings. Then:

(i) M is *accretive,* if

$$\langle Mx-My,J_q(x-y)\rangle\geq 0,\ \forall x,y\in E;$$

(ii) M is *strictly accretive,* if

$$\langle Mx - My, J_q(x-y)\rangle > 0, \ \forall x, y \in E,$$

and equality holds if and only if $x = y$;

(iii) M is *r-strongly accretive,* if there exists a constant $r > 0$, such that

$$\langle Mx - My, J_q(x-y)\rangle \geq r||x-y||^q, \ \forall x, y \in E;$$

(iv) M is *s-relaxed accretive,* if there exists a constant $s > 0$, such that

$$\langle Mx - My, J_q(x-y)\rangle \geq -s||x-y||^q, \ \forall x, y \in E;$$

(v) M is *μ-strongly cocoercive* if there exists a constant $\mu > 0$, such that

$$\langle Mx - My, J_q(x-y)\rangle \geq \mu||Mx-My||^q, \ \forall x, y \in E;$$

(vi) M is *γ-relaxed cocoercive,* if there exists a constant $\gamma > 0$, such that

$$\langle Mx - My, J_q(x-y)\rangle \geq -\gamma||Mx-My||^q, \ \forall x, y \in E.$$

Definition 3.2. Let $A, B : E \to E, \ H : E \times E \to E$ be three single-valued mappings.

(i) $H(A, \cdot)$ is said to be *cocoercive* with respect to A, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), \ J_q(x-y)\rangle \geq \mu\|Ax - Ay\|^q, \ \forall \ x, y, u \in E;$$

(ii) $H(\cdot, B)$ is said to be *relaxed cocoercive* with respect to B, if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By), \ J_q(x-y)\rangle \geq (-\gamma)\|Bx - By\|^q, \ \forall \ x, y, u \in E;$$

(iii) $H(A, \cdot)$ is said to be r_1-*Lipschitz continuous* with respect to A, if there exists a constant $r_1 > 0$ such that

$$\|H(Ax, u) - H(Ay, u)\| \leq r_1\|x - y\|, \ \forall \ x, y, u \in E;$$

(iv) $H(\cdot,B)$ is said to be r_2-*Lipschitz continuous* with respect to B, if there exists a constant $r_2 > 0$ such that

$$\|H(u,Bx) - H(u,By)\| \le r_2\|x-y\|,\ \forall\, x,y,u \in E;$$

(v) B is *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|B(x) - B(y)\| \le \beta\|x-y\|,\ \forall\, x,y \in E;$$

(vi) A is α-*expansive*, if there exists a constant $\alpha > 0$ such that

$$\|A(x) - A(y)\| \ge \alpha\|x-y\|,\ \forall\, x,y \in E.$$

If $\alpha = 1$, then it is expansive.

Definition 3.3. A set-valued mapping $M : E \to 2^E$ is said to be cocoercive, if there exists a constant $\mu > 0$ such that

$$\langle u-v,\ J_q(x-y)\rangle \ge \mu\|u-v\|^2,\ \forall\, x,y \in X, u \in M(x), v \in M(y).$$

Definition 3.4. Let $A,B : E \to E$, $H : E \times E \to E$ be three single-valued mappings. Let $M : E \to 2^E$ be a set-valued mapping. M is said to be $H(\cdot,\cdot)$-*cocoercive* with respect to A and B if M is cocoercive and $(H(A,B)+\lambda M)(E) = E$, for every $\lambda > 0$.

Example 3.1. *Let $E = \mathbb{R}$ be a real Banach space and let $A,B : E \to E$ such that $A(x) = \frac{x}{2}$ and $B(x) = -x$. Suppose that $H(A,B) : E \times E \to E$ is defined by*

$$H(A,B)(x) = H(Ax,Bx) = Ax + Bx.$$

Then $H(A,B)$ is 2-cocoercive with respect to A and 1-relaxed cocoercive with respect to B. Let $M : E \to E$ be such that $Mx = 2x$. Then M is $\frac{1}{2}$-cocoercive and

$$(H(A,B) + \lambda M)(E) = E,\ \text{for } \lambda > 0,$$

which means that M is $H(\cdot,\cdot)$-cocoercive with respect to A and B.

Theorem 3.1 [2]. *Let $H(A,B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let*

$M: E \to 2^E$ be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then the operator $(H(A,B)+\lambda M)^{-1}$ is single-valued.

Definition 3.5 [2]. Let $H(A,B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M}: E \to E$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M}(x) = (H(A,B)+\lambda M)^{-1}(x),\ \forall\, x \in E. \tag{3.1}$$

Theorem 3.2 [2]. *Let $H(A,B)$ be μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let M be $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M}: E \to E$ is $\frac{1}{\mu\alpha^q - \gamma\beta^q}$-Lipschitz continuous, that is*

$$\|R^{H(\cdot,\cdot)}_{\lambda,M}(x) - R^{H(\cdot,\cdot)}_{\lambda,M}(y)\| \le \frac{1}{\mu\alpha^q - \gamma\beta^q}\|x-y\|,\ \forall\, x,y \in E. \tag{3.2}$$

4. Graph convergence for $H(\cdot,\cdot)$-cocoercive operator

In this section, we introduce the graph convergence for the $H(\cdot,\cdot)$-cocoercive operator. Since $H(\cdot,\cdot)$-cocoercive operators are more general than maximal monotone operators, we give the following characterization of $H(\cdot,\cdot)$-cocoercive operators.

Proposition 4.1 [2]. Let $H(A,B)$ is μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, A is α-expansive, B is β-Lipschitz continuous and $\mu > \gamma$, $\alpha > \beta$. Let $M: E \to 2^E$ be $H(\cdot,\cdot)$-cocoercive operator. If the following inequality

$$\langle u-v, x-y\rangle \ge 0 \tag{4.1}$$

holds for all $(y,v) \in \text{graph}(M)$, then $u \in M(x)$, where

$$\text{Graph}(M) = \{(x,u) \in E \times E\ :\ u \in M(x)\}. \tag{4.2}$$

Definition 4.1 [16]. Let $A,B : E \to E$ and $H : E\times E \to E$ be three single-valued mappings. Let $M_n, M : E \to 2^E$ be $H(\cdot,\cdot)$-cocoercive operators for $n = 0,1,2,....$ The sequence $\{M_n\}$ is said to be graph-convergence to M, denoted by $M_n \to^G M$, if for every $(x,y) \in \text{graph}(M)$ there exists a sequence $(x_n, y_n) \in \text{graph}(M_n)$ such that

$$x_n \to x,\ y_n \to y \text{ as } n \to \infty.$$

Theorem 4.1. *Let $A,B : E \to E$ be single-valued mappings and let $M_n, M : E \to 2^E$ be $H(\cdot,\cdot)$-cocoercive operators for $n = 0,1,2,....$ Assume that $H : E\times E \to E$ is a single-valued mapping such that*

(i) $H(A,B)$ is μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B and $\mu > \gamma$;

(ii) $H(A,B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B;

(iii) A is α-expansive and B is β-Lipschitz continuous.

Then $M_n \to^G M$ if and only if $R^{H(\cdot,\cdot)}_{M_n,\lambda}(x) \longrightarrow R^{H(\cdot,\cdot)}_{M,\lambda}(x),\ \forall x \in E,\ \lambda > 0,$ where $R^{H(\cdot,\cdot)}_{M_n,\lambda} = (H(A,B)+\lambda M_n)^{-1}$, $R^{H(\cdot,\cdot)}_{M,\lambda} = (H(A,B)+\lambda M)^{-1}$.

Proof. It follows from Theorem 3.2 that $R^{H(\cdot,\cdot)}_{M_n,\lambda}$ and $R^{H(\cdot,\cdot)}_{M,\lambda}$ are both $\frac{1}{\mu\alpha^q-\gamma\beta^q}$-Lipschitz continuous.

Necessity: Suppose that $M_n \to^G M$. For any $x \in E$, let $z_n = R^{H(\cdot,\cdot)}_{M_n,\lambda}(x),\ z = R^{H(\cdot,\cdot)}_{M,\lambda}(x)$. Then $\frac{1}{\lambda}[x - H(Az,Bz)] \in M(z)$ and therefore $\left(z, \frac{1}{\lambda}[x - H(Az,Bz)]\right) \in \text{graph}(M)$. By Definition 4.1, we know there exists a sequence $(z'_n, y'_n) \in \text{graph}(M_n)$ such that

$$z'_n \to z,\ y'_n \longrightarrow \frac{1}{\lambda}[x - H(Az,Bz)] \text{ as } n \to \infty. \tag{4.3}$$

Since $y'_n \in M_n(z'_n)$, we have $H(Az'_n, Bz'_n) + \lambda y'_n \in [H(A,B) + \lambda M_n](z'_n)$. It follows that $z'_n = R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Az'_n, Bz'_n) + \lambda y'_n]$. By the Lipschitz continuity of M_n, we have

$$\begin{aligned}
\|z_n - z\| &\le \|z_n - z'_n\| + \|z'_n - z\| \\
&= \|R^{H(\cdot,\cdot)}_{M_n,\lambda}(x) - R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Az'_n, Bz'_n) + \lambda y'_n]\| + \|z'_n - z\| \\
&\le \frac{1}{\mu\alpha^q - \gamma\beta^q}\|x - H(Az'_n, Bz'_n) - \lambda y'_n\| + \|z'_n - z\| \\
&\le \frac{1}{\mu\alpha^q - \gamma\beta^q}\Big(\|x - H(Az, Bz) - \lambda y'_n\| + \|H(Az, Bz) - H(Az'_n, Bz'_n)\|\Big) + \|z'_n - z\|.
\end{aligned} \tag{4.4}$$

By (ii) of Theorem 4.1, we have

$$\begin{aligned}
\|H(Az, Bz) - H(Az'_n, Bz'_n)\| &\le \|H(Az, Bz) - H(Az, Bz'_n)\| + \|H(Az, Bz'_n) - H(Az'_n, Bz'_n)\| \\
&\le (r_1 + r_2)\|z'_n - z\|.
\end{aligned} \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\|z_n - z\| \le \frac{1}{\mu\alpha^q - \gamma\beta^q}\|x - H(Az, Bz) - \lambda y'_n\| + \Big(1 + \frac{1}{\mu\alpha^q - \gamma\beta^q}(r_1 + r_2)\Big)\|z'_n - z\|.$$

By (4.3), we have

$$\|z'_n - z\| \to 0, \ \left\|\frac{1}{\lambda}[x - H(Az, Bz)] - y'_n\right\| \to 0, \text{ and so } \|z_n - z\| \to 0, \text{ as } n \to \infty.$$

Sufficiency: Suppose that $R^{H(\cdot,\cdot)}_{M_n,\lambda}(x) \longrightarrow R^{H(\cdot,\cdot)}_{M,\lambda}(x), \ \forall x \in E, \ \lambda > 0$. For any given $(x,y) \in \text{graph}(M)$, we have

$$H(Ax, Bx) + \lambda y \in [H(A,B) + \lambda M](x), \text{ and so } x = R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax, Bx) + \lambda y].$$

Let $x_n = R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax, Bx) + \lambda y]$. Then $\frac{1}{\lambda}[H(Ax, Bx) - H(Ax_n, Bx_n) + \lambda y] \in M_n(x_n)$. Let

$$y_n = \frac{1}{\lambda}[H(Ax, Bx) - H(Ax_n, Bx_n) + \lambda y] \in M_n(x_n).$$

It follows from (4.5) that

$$\begin{aligned}
\|y_n - y\| = \left\|\frac{1}{\lambda}[H(Ax, Bx) - H(Ax_n, Bx_n) + \lambda y] - y\right\| &= \frac{1}{\lambda}\|H(Ax, Bx) - H(Ax_n, Bx_n)\| \\
&\le \frac{1}{\lambda}(r_1 + r_2)\|x_n - x\|.
\end{aligned} \tag{4.6}$$

Since $R^{H(\cdot,\cdot)}_{M_n,\lambda}(x) \longrightarrow R^{H(\cdot,\cdot)}_{M,\lambda}(x)$ for any x in E, we know that $\|x_n - x\| \to 0$. Now (4.6) implies that $y_n \to y$ as $n \to \infty$ and so $M_n \to^G M$. This completes the proof.

5. Applications

In this section, by using the concept of graph convergence of $H(\cdot,\cdot)$-ccoocercive operator, we apply $H(\cdot,\cdot)$-cococercive operator for solving the variational inclusion in q-uniformly smooth Banach spaces.

We consider the perturbed algorithm for solving the following variational inclusion problem of finding $x \in E$ such that

$$0 \in T(x) + M(x), \tag{5.1}$$

where $T : E \to E$ is a single-valued mapping and $M : E \to 2^E$ is a $H(\cdot,\cdot)$-cocoercive operator. The problem (5.1) is called a variational inclusion (in short, VI) which includes many variational inequalities (inclusions) as special cases, see for example [1,2,4,5-12,14,15,19,23,25]. From the definition of resolvent operator $R^{H(\cdot,\cdot)}_{M,\lambda}$, we have an equivalence bteween resolvent equation and solution of VI (5.1).

Lemma 5.1. *$x \in E$ is a solution of VI (5.1) if and only if x satisfies*

$$x = R^{H(\cdot,\cdot)}_{\lambda,M}[H(A(x),B(x)) - \lambda T(x)], \tag{5.2}$$

where $\lambda > 0$ is a constant.

Proof. By using the definition of resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M}$, the conclusion follows directly.

Let $M_n : E \to 2^E$ be $H(\cdot,\cdot)$-cocoercive operators for $n = 0,1,2,...$ Based on (5.2), we can construct the following perturbed algorithm.

Algorithm 5.1. For any given $x_0 \in E$, compute $\{x_n\} \subset E$ as follows:

$$x_{n+1} = R^{H(\cdot,\cdot)}_{M_n,\lambda}\Big[H(Ax_n, Bx_n) - \lambda T(x_n)\Big], \tag{5.3}$$

where $\lambda > 0$ is a constant and $n = 0,1,2,....$

Now, we prove the existence of solutions of VI (5.1) and analyze the convergence of iterative sequence generated by Algorithm 5.1.

Theorem 5.1. *Let $T,A,B : E \to E$ be three single-valued mappings. Let $H : E \times E \to E$ be a single-valued mapping and $M_n, M : E \to 2^E$ be $H(\cdot,\cdot)$-cocoercive set-valued operators such that $M_n \to^G M$. Assume that*

(i) T is τ-Lipschitz continuous and δ-strongly accretive;

(ii) $H(A,B)$ is μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B and $\mu > \gamma$;

(iii) $H(A,B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B;

(iv) A is α-expansive and B is β-Lipschitz continuous.

Suppose that there exists a constant $\lambda > 0$ such that the following condition is satisfied.

$$\frac{1}{\mu\alpha^q - \gamma\beta^q}\left[\sqrt[q]{1+(r_1+r_2)^q c_q - q(\mu\alpha^q - \gamma\beta^q)} + \sqrt[q]{1 + c_q\lambda^q\tau^q - q\lambda\delta}\,\right] < 1. \tag{5.4}$$

Then VI (5.1) has a unique solution $x \in E$ and the iterative sequence $\{x_n\}$ generated by Algorithm 5.1 converges strongly to x.

Proof. Let $F : E \to E$ be defined as follows:

$$F(x) = R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax,Bx) - \lambda T(x)],\ \forall x \in E. \tag{5.5}$$

For any $x,y \in E$, it follows from (5.2) and Theorem 3.2 that

$$\begin{aligned}\|F(x)-F(y)\| &= \|R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax,Bx)-\lambda T(x)] - R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ay,By)-\lambda T(y)]\| \\ &\le \frac{1}{\mu\alpha^q - \gamma\beta^q}\|H(Ax,Bx)-H(Ay,By)-\lambda(T(x)-T(y))\| \\ &\le \frac{1}{\mu\alpha^q - \gamma\beta^q}\Big(\|H(Ax,Bx)-H(Ay,By)-(x-y)\| + \|x-y-\lambda(T(x)-T(y))\|\Big).\end{aligned} \tag{5.6}$$

By assumptions and Lemma 2.1, we have

$$\begin{aligned}\|H(Ax,Bx)-H(Ay,By)-(x-y)\|^q &\le \|x-y\|^q - q\langle H(Ax,Bx)-H(Ay,By), J_q(x-y)\rangle \\ &\quad + c_q\|H(Ax,Bx)-H(Ay,By)\|^q.\end{aligned} \tag{5.7}$$

Since $H(A,B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B, we have

$$\begin{aligned}\|H(Ax,Bx)-H(Ay,By)\| &\le \|H(Ax,Bx)-H(Ay,Bx)\|+\|H(Ay,Bx)-H(Ay,By)\| \\ &\le (r_1+r_2)\|x-y\|.\end{aligned} \tag{5.8}$$

Also $H(A,B)$ is μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B; A is α-expansive; B is β-Lipschitz continuous, we can obtain

$$\begin{aligned}&\langle H(Ax,Bx)-H(Ay,By),J_q(x-y)\rangle \\ &=\langle H(Ax,Bx)-H(Ay,Bx),J_q(x-y)\rangle+\langle H(Ay,Bx)-H(Ay,By),J_q(x-y)\rangle \\ &\ge \mu\|Ax-Ay\|^q-\gamma\|Bx-By\|^q \\ &\ge (\mu\alpha^q-\gamma\beta^q)\|x-y\|^q.\end{aligned} \tag{5.9}$$

Now, from (5.7)-(5.9), we have

$$\|H(Ax,Bx)-H(Ay,By)-(x-y)\|^q \le \Big(1+(r_1+r_2)^q c_q - q(\mu\alpha^q-\gamma\beta^q)\Big)\|x-y\|^q. \tag{5.10}$$

From τ-Lipschitz continuity and accretivity of T, we have

$$\|x-y-\lambda(T(x)-T(y))\|^q \le \|x-y\|^q - q\lambda\langle T(x)-T(y),J_q(x-y)\rangle + c_q\lambda^q\|T(x)-T(y)\|^q$$

$$\le (1+c_q\lambda^q\tau^q-q\lambda\delta)\|x-y\|^q. \tag{5.11}$$

From (5.6),(5.10) and (5.11), we have

$$\|F(x)-F(y)\| \le k\|x-y\|, \tag{5.12}$$

where

$$k := \frac{1}{\mu\alpha^q-\gamma\beta^q}\Big[\sqrt[q]{1+(r_1+r_2)^q c_q - q(\mu\alpha^q-\gamma\beta^q)}+\sqrt[q]{1+c_q\lambda^q\tau^q-q\lambda\delta}\,\Big]. \tag{5.13}$$

By assumption we know that $0<k<1$ and so (5.12) implies that $F=R_{M,\lambda}^{H(\cdot,\cdot)}[H(A,B)+\lambda T]$ has a unique fixed point $x\in E$. Thus, x is a unique solution of VI (5.1).

Now we prove that $\{x_n\}$ converges strongly to x. In fact, it follows from (5.2) and (5.3) that

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax_n,Bx_n)-\lambda T(x_n)] - R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax,Bx)-\lambda T(x)]\| \\ &\leq \|R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax_n,Bx_n)-\lambda T(x_n)] - \|R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax,Bx)-\lambda T(x)]\| \\ &\quad + \|R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax,Bx)-\lambda T(x)] - R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax,Bx)-\lambda T(x)]\|. \end{aligned} \tag{5.14}$$

Similarly from (5.6)-(5.12), we can obtain

$$\|R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax_n,Bx_n)-\lambda T(x_n)] - R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax,Bx)-\lambda T(x)]\| \leq k\|x_n - x\|. \tag{5.15}$$

By Theorem 4.1, we have

$R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax,Bx)-\lambda T(x)] \longrightarrow R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax,Bx)-\lambda T(x)]$.

Let $b_n = \|R^{H(\cdot,\cdot)}_{M_n,\lambda}[H(Ax,Bx)-\lambda T(x)] - R^{H(\cdot,\cdot)}_{M,\lambda}[H(Ax,Bx)-\lambda T(x)]\|$.

Then $b_n \to 0$ as $n \to \infty$. It follows that

$$\|x_{n+1}-x_n\| \leq k\|x_n - x\| + b_n.$$

Now, Lemma 2.2 implies that $\|x_{n+1}-x_n\| \to 0$. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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