ISOMETRIES OF P-NUCLEAR TYPE OPERATORS

R. KHALIL*, I. ADARAWI

Department of Mathematics, the University of Jordan, Amman, Jordan

Copyright © 2015 Khalil and Adarawi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. Let $X$ be a Banach space and let $C_p(\ell^p^*, X) = \{ T : \ell^p^* \to X : \| T \|_{C_p} = \sup(\sum_{n=1}^{\infty} \| T \theta_n \|^p)^{\frac{1}{p}} < \infty \}$, where the supremum is taken over all $p^*$–orthonormal sequences in $\ell^p^*$. The object of this paper is to study the isometries of $C_p(\ell^p^*, X)$. We give full characterization of certain classes of onto isometries of $C_p(\ell^p^*, X)$ for some Banach spaces $X$.

Keywords: Banach space; Isometries; P-nuclear operators.

2010 AMS Subject Classification: 46B04, 46B20.

1. Introduction

Let $X$ be a Banach space and $T$ be a bounded linear operator on $X$. $T$ is called an isometry if $\| Tx \| = \| x \|$ for all $x \in X$. The characterization of onto isometries on $X$ has been an important topic in analysis. Isometries is a main tool to study the Geometry of Banach spaces like extreme points, smooth points and exposed points of the unit ball a Banach space. In [1], Kadison characterized the isometries of $L(H)$, the space of bounded linear operators on a Hilbert space $H$. The isometries $C(I, X)$ were characterized by Lau [2]. The isometries of $\varphi$–nuclear operators on general Banach spaces were characterized by Khalil and Salih [3]. Isometries

*Corresponding author

Received September 19, 2014
of $L(\ell^p)$ $1 \leq p \neq 2 < \infty$, was an open problem since 1951. Khalil and Saleh [4] gave a full characterization of such isometries. The isometries of the $p-$nuclear operators $N_p(\ell^p, X)$ were characterized by Yousef and Khalil [5].

In this paper we study the onto isometries of $p$-nuclear type operators, to be denoted by $C_p(\ell^{p^*}, X)$. These are Schatten type classes. We give full characterization of some classes of onto isometries of $C_p(\ell^{p^*}, X)$. We refer to [2], [3], [6] and [7] for the basic facts on tensor product of Banach spaces and functional analysis.

2. The Space $C_p(\ell^{p^*}, X)$

In this section, we introduce our space of $p$-nuclear type operators.

**Definition 2.1.** Let $X$ be a Banach space, and $(x_n)$ be a sequence in $X$. The sequence $(x_n)$ is called $p$-orthogonal if $\|\sum \lambda_n x_n\| = (\sum |\lambda_n|^p \|x_n\|^p)^{\frac{1}{p}}$. It is called $p$-orthonormal if $\|x_n\| = 1$.

One can easily show that in $\ell^p-$spaces, $(x_n)$ is $p$-orthogonal if and only if the $x_n$ have disjoint support.

Now, we introduce our space.

**Definition 2.2.** For a Banach space $X$, we set

$$C_p(\ell^{p^*}, X) = \{T : \ell^{p^*} \to X : \|T\|_{C(p)} = \sup \left(\sum_{n=1}^{\infty} \|T \theta_n\|^p \right)^{\frac{1}{p}} < \infty\},$$

where the supremum is taken over all $p^*$-orthonormal sequences in $\ell^{p^*}$. One can easily see that $\left(C_p(\ell^{p^*}, X), \|\cdot\|_{C(p)}\right)$ is a normed space.

Further, we have

**Theorem 2.3.** If $X$ is Banach space, then $\left(C_p(\ell^{p^*}, X), \|\cdot\|_{C(p)}\right)$ is a Banach space.

**Proof.** We claim that every absolutely convergent series is convergent. So let $T_n \in C_p(\ell^{p^*}, X)$ be a sequence such that $\sum_{n=1}^{\infty} \|T_n\|_{C(p)} < \infty$. We claim $\sum_{n=1}^{\infty} T_n \in C_p(\ell^{p^*}, X)$. Define $T : \ell^{p^*} \to X$ as
\( T(x) = \sum_{n=1}^{\infty} T_n(x) \). Clearly, \( T \) is bounded and \( \| T \| \leq \| T \|_{C(p)} \). Further, we have

\[
\| T \|_{C(p)} = \sup \left( \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \| T_n(\theta_k) \| \right)^p \right)^{\frac{1}{p}}
\]

\[
= \sup \left( \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \| T_n(\theta_k) \| \right)^p \right)^{\frac{1}{p}}
\]

\[
\leq \sup \left( \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \| T_n(\theta_k) \|^p \right)^{\frac{1}{p}}
\]

\[
\leq \sum_{n=1}^{\infty} \| T_n \|_{C(p)} < \infty.
\]

Hence, we have \( T \in C_p(\ell^p, X) \). Remains to prove that \( \sum_{n=1}^{\infty} T_n \) converge to \( T \). So we claim

\[
\| T - S_n \|_{C(p)} \to 0.
\]

\[
\| T - S_n \|_{C(p)} = \sup \left( \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \| T \theta_k - S_n \theta_k \| \right)^p \right)^{\frac{1}{p}}
\]

\[
= \sup \left( \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \| T \theta_k - S_n \theta_k \| \right)^p \right)^{\frac{1}{p}}
\]

\[
\leq \sup \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \| T \theta_k \| \right)^p \frac{1}{p}
\]

\[
\leq \sum_{n=1}^{\infty} \| T_n \|_{C(p)}.
\]

But this goes to zero since it is the tail of a convergent series. Hence, \( \left( C_p(\ell^p, X), \| \cdot \|_{C(p)} \right) \) is a Banach space.

**Theorem 2.4.** Let \( X \) be a Banach space. Then the followings are equivalent:

(i) \( T \in C_p(\ell^p, X) \)

(ii) There exist \( (\lambda_n) \in \ell^p \), and \( g_n \in X \), such that \( \| g_n \| = 1 \), and \( T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n \).

Further, \( \| T \|_{C(p)} = \| (\lambda_n) \|_p \).

**Proof.** First, we show \( i \Rightarrow ii \). Let \( T \in C_p(\ell^p, X) \) and \( (\delta_n) \) be the natural basis in \( \ell^p \). Then...
Hence, for any $p\|T\delta_n\|$, (where $x = (a_1, a_2, \ldots)$)

$$Tx = T(\sum_{n=1}^{\infty} a_n\delta_n), \quad \text{(since $T$ is bounded linear operator)}$$

$$= \sum_{n=1}^{\infty} a_nT\delta_n \quad \text{(where $g_n = \frac{T\delta_n}{\|T\delta_n\|}$ and $\lambda_n = \|T\delta_n\|$)}$$

$$= \sum_{n=1}^{\infty} \lambda_n a_n g_n, \quad \text{(where $g_n = \frac{T\delta_n}{\|T\delta_n\|}$ and $\lambda_n = \|T\delta_n\|$)}$$

$$= \sum_{n=1}^{\infty} \lambda_n \langle \delta_n, x \rangle g_n.$$ 

$(a_n = \langle \delta_n, x \rangle$, and $(\lambda_n) \in \ell^p$ since $T \in C_p(\ell^p, X)$. Consequently, $T = \sum_{n=1}^{\infty} \lambda_n\delta_n \otimes g_n$. Remains to prove that $\|T\|_{c(p)} = \|\lambda_n\|_p$. Let $T \in C_p(\ell^p, X), 1 < p < \infty$, and $T = \sum_{n=1}^{\infty} \lambda_n\delta_n \otimes g_n$. Further, let $(\theta_k)$ be any $p^*$ - orthonormal sequence in $\ell^p$. Then $T\theta_k = \sum_{n=1}^{\infty} \lambda_n \langle \delta_n, \theta_k \rangle g_n$ and

$$\left( \sum_{k=1}^{\infty} \|T\theta_k\|^p \right)^{\frac{1}{p}} = \left( \sum_{k=1}^{\infty} \left\| \sum_{n=1}^{\infty} \lambda_n \langle \delta_n, \theta_k \rangle g_n \right\|^p \right)^{\frac{1}{p}}$$

$$\leq \left( \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \lambda_n \left\| \langle \delta_n, \theta_k \rangle \right\|^p \right) \right)^{\frac{1}{p}}$$

$$= \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \eta_k \lambda_n \left( \|\delta_n\|, \|\theta_k\| \right) \right),$$

where $\|\eta_k\|_{p^*} = 1$ (By Hahn Banach Theorem and the fact that $(\ell^p)^* = \ell^{p^*}$). Now, if $(e_n)$ is $p$ - orthonormal in $\ell^p$, then $(|e_n|)$ is $p$ - orthonormal. Hence, $x = \sum_{n=1}^{\infty} |\lambda_n| |\delta_n| \in \ell^p$, $\|x\|_p = \|(\lambda_n)\|_p$, and $y = \sum_{k=1}^{\infty} |\eta_k| |\theta_k| \in \ell^{p^*}, \|y\|_{p^*} = 1$. Now,

$$\left( \sum_{k=1}^{\infty} \|T\theta_k\|^p \right)^{\frac{1}{p}} = \left| \sum_{n,k=1}^{\infty} |\lambda_n| |\eta_k| \langle \delta_n, \theta_k \rangle \right| = \left| \sum_{n,k=1}^{\infty} |\lambda_n| \langle \delta_n, \theta_k \rangle |\eta_k| \langle \delta_n, \theta_k \rangle \right|$$

$$= \|\langle x, y \rangle\| \leq \|x\|_p \|y\|_{p^*} \leq \left( \sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{p}}, \quad \left( \|y\|_{p^*} = 1 \right).$$

Hence, for any $p$ - orthonormal sequence $(\theta_k)$, we have

$$\left( \sum_{k=1}^{\infty} \|T\theta_k\|^p \right)^{\frac{1}{p}} \leq \left( \sum_{n=1}^{\infty} \|T\delta_n\|^p \right)^{\frac{1}{p}} = \left( \sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{p}}.$$

So, sup $\left( \sum_{k=1}^{\infty} \|T\theta_k\|^p \right)^{\frac{1}{p}} = \|(\lambda_n)\|_p$. 
Next, we show (ii) $\Rightarrow$ (i). Let $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$, where $(\lambda_n) \in \ell^p$ and $\|g_n\| = 1$. Then as in case (i $\Rightarrow$ ii) we get
\[
\|T\|_{C(p)} = \left( \sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{p}} < \infty, \text{ and } T \in C_p(\ell^p^*, X).
\]
This ends the proof of the Theorem.

**Theorem 2.5.** Every operator $T \in C_p(\ell^p^*, X)$ has a representation for which the supremum is attained.

**Proof.** From Theorem 2.4, we find the desired conclusion immediately.

3. The isometries

In this section, we study the isometric onto operators of $C_p(\ell^p^*, X)$.

**Theorem 3.1.** Let $A$ be an isometric onto operator of $\ell^p^*$, and $B$ be an isometric onto operator on $X$. Then the map defined by $F : C_p(\ell^p^*, X) \to C_p(\ell^p^*, X)$, $F(T) = BTA$ is an isometric onto operator of $C_p(\ell^p^*, X)$.

**Proof.** Let $x \in \ell^p^*$ and let $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$ be an element in $C_p(\ell^p^*, X)$. Since $B$ is an isometry, we have
\[
F(T)x = BTAx = \sum_{n=1}^{\infty} \lambda_n \langle \delta_n, Ax \rangle Bg_n
\]
\[
= \sum_{n=1}^{\infty} \lambda_n \langle A^* \delta_n, x \rangle Bg_n = \sum_{n=1}^{\infty} \lambda_n \langle A^* \delta_n, x \rangle \hat{g}_n,
\]
where $\|\hat{g}_n\| = 1$. Further, Since $A^*$ is an isometric onto operator on $\ell^p$, then $A^* \delta_n = \delta_{\varphi(n)}$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is $(1-1)$ and onto map on the set of natural numbers. Thus $F(T) = \sum_{n=1}^{\infty} \lambda_n \delta_{\varphi(n)} \otimes \hat{g}_n = T$, say. Now, $\|\hat{T}\| = \left( \sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{p}}$, and $F$ is an isometry by Theorem 3.3.4.

To show that $F$ is onto, let $S = \sum_{n=1}^{\infty} a_n \delta_n \otimes g_n \in C_p(\ell^p^*, X)$, and $\hat{S} = \sum_{n=1}^{\infty} a_n \delta_{\varphi^{-1}(n)} \otimes B^{-1}g_n$. Clearly $F(S) = \sum_{n=1}^{\infty} a_n \delta_n \otimes g_n = S$. Then $F$ is onto. This completes the proof.

**Definition 3.2.** A basic atom in $C_p(\ell^p^*, X)$ is an operator of the form $\delta_k \otimes h$ for some $k \in \mathbb{N}$ and $h \in X$. 
**Theorem 3.3.** Let $F : C_p(\ell^p, X) \to C_p(\ell^p, X)$. If $F$ preserves basic atoms and if $F$ preserves rank, then the following are equivalent:

(i) $F$ is an isometric onto operator.

(ii) There exist two isometric onto operators: $A : \ell^p \to \ell^p$ and $B : X \to X$, and a sequence $(a_n), |a_n| = 1$ for all $n$ with $F(\sum \delta_n \otimes x_n) = \sum A(\delta_n) \otimes a_n Bx_n$.

**Proof.** First, we show $(i) \Rightarrow (ii)$. Let $F$ be an isometric onto operator. We divide the proof into steps:

**Step (i).** Let $X_1 = \delta_1 \otimes X = \{ \delta_1 \otimes x : x \in X \}$.

Then $F(X_1) = \{ \delta_k \otimes y : y \in X, \text{ for fixed } k, \forall y \in X \}$.

Claim. Let $x_1, x_2 \in X$. If possible assume $F(\delta_1 \otimes x_1) = \delta_k \otimes \hat{x}_1$ and $F(\delta_1 \otimes x_2) = \delta_k \otimes \hat{x}_2$, and $\delta_k \neq \delta_k$. Then, $\delta_1 \otimes x_1 + \delta_1 \otimes x_2 = \delta_1 \otimes (x_1 + x_2)$ is a basic atom. Since $F$ preserves basic atoms then $F(\delta_1 \otimes x_1 + \delta_1 \otimes x_2) = \delta_j \otimes y$ for some $y \in X$ and $j \in N$. Hence, $\delta_j \otimes y = \delta_k \otimes \hat{x}_1 + \delta_k \otimes \hat{x}_2$, which is a contradiction since $\delta_k \neq \delta_k$. So $\delta_k \otimes \hat{x}_1 + \delta_k \otimes \hat{x}_2$ is not a basic atom. So $F(\delta_1 \otimes x) = \delta_k \otimes y$, for fixed $k \in N$. Similarly for $\delta_2, \delta_3, \ldots$.

**Step (ii).** Define $A : \ell^p \to \ell^p, A \delta_1 = \delta_k$, where $F(\delta_1 \otimes X) = \delta_k \otimes X$. Similarly for $\delta_2, \delta_3, \ldots$. Then $A$ is an isometric onto operator since it permutes the basis $(\delta_k)$ and $F$ is onto. So $A$ can be recognized as: $A \delta_n = \delta_{\varphi(n)}$, where $\varphi$ is a permutation on the set of natural numbers, $N$.

**Step (iii).** $F(\delta_1 \otimes x) = \delta_{\varphi(1)} \otimes x_1$, and $F(\delta_2 \otimes x) = \delta_{\varphi(2)} \otimes x_1$.

Claim. If possible assume $F(\delta_1 \otimes x) = \delta_{\varphi(1)} \otimes y$ and $F(\delta_2 \otimes x) = \delta_{\varphi(2)} \otimes z$, $y \neq z$. Now, $\delta_1 \otimes x + \delta_2 \otimes x = (\delta_1 + \delta_2) \otimes x$, which is a 1-rank operator. But $F((\delta_1 + \delta_2) \otimes x) = \delta_{\varphi(1)} \otimes y + \delta_{\varphi(2)} \otimes z$. Now, since $||y|| = ||z|| = ||x||$, (since $F$ is an isometry ) then either $y, z$ are independent or $y = \pm z$. If $y, z$ are independent, then $F((\delta_1 + \delta_2) \otimes x) = \delta_{\varphi(1)} \otimes y + \delta_{\varphi(2)} \otimes z$ is a two rank operator which is a contradiction, since $F$ preserves rank. Hence, $F(\delta_1 \otimes x) = \delta_{\varphi(1)} \otimes a_1 y$, and $F(\delta_2 \otimes x) = \delta_{\varphi(2)} \otimes a_2 y$, with $|a_i| = 1$.

In a similar way one can prove $\{\delta_{\varphi(k)} \otimes a_k y : |a_k| = 1, k \in N\}$.

**Step (iv).** Define $B : X \to X, B(x) = y$, where $F(\delta_k \otimes x) = a_k \delta_{\varphi(k)} \otimes y$, and $|a_k| = 1$. Then $B$ is well-defined linear maps. To prove the linearity of $B$, let $x_1, x_2 \in X$, and $\beta \in \mathbb{R}$. Then, $F(\delta_k \otimes (\beta x_1 + x_2)) = F(\beta \delta_k \otimes x_1 + \delta_k \otimes x_2) = \beta F(\delta_k \otimes x_1) + F(\delta_k \otimes x_2) = \beta a_k \delta_{\varphi(k)} \otimes x_1 + a_k \delta_{\varphi(k)} \otimes x_2 = a_k \delta_{\varphi(k)} \otimes (\beta x_1 + x_2) = a_k \delta_{\varphi(k)} \otimes (\beta B(x_1) + B(x_2))$, where $|a_k| = 1$. That is
$B(\beta x_1 + x_2) = \beta B(x_1) + B(x_2)$. Since $F$ is an isometric operator, we have $\|x\| = \|\delta_k \otimes x\| = \|F(\delta_k \otimes x)\| = \|a_k \delta_{\varphi(k)} \otimes y\| = \|y\| = \|B(x)\|$. Hence $B$ is an isometry. Finally, let $y \in X$. Then $\delta_k \otimes y = F(\delta_{\varphi^{-1}(k)} \otimes x)$, since $F$ is onto. Therefore, $y = B(x)$ for some $x \in X$. Thus $B$ is an isometric operator.

Now, we want to show that $F(T) = \sum_{n=1}^{\infty} A(\delta_n) \otimes a_n Bx_n$. Indeed, we have $F(\sum_{n=1}^{\infty} \delta_n \otimes x_n) = \sum_{n=1}^{\infty} F(\delta_n \otimes x_n) = \sum_{n=1}^{\infty} \delta_{\varphi(n)} \otimes a_n y_n = \sum_{n=1}^{\infty} A(\delta_n) \otimes a_n Bx_n$.

Now, we are in a position to show $(ii) \Rightarrow (i)$. Let $T = \sum_{n=1}^{\infty} \delta_n \otimes x_n$ be an element in $C_\varphi(\ell^p, X)$. Since $B$ is an isometry, we have $F(T) = \sum_{n=1}^{\infty} (A(\delta_n) \otimes a_n B(x_n)) = \sum_{n=1}^{\infty} (\delta_{\varphi(n)} \otimes a_n y_n) = T$, where $\|y_n\| = \|x_n\|$. Hence, $F(T) = T$

Now,

$$\|F(T)\| = \|T\| = \left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{\frac{1}{p}} = \|T\|.$$

Hence $F$ is an isometry by Theorem 3.1. To show that $F$ is onto, let $S = \sum_{n=1}^{\infty} \delta_n \otimes y_n \in C_\varphi(\ell^p, X)$, where $y_n = a_n x_n$ such that $|a_n| = 1$. Let $\hat{S} = \sum_{n=1}^{\infty} \delta_{\varphi^{-1}(n)} \otimes B^{-1} x_n$. Clearly $F(\hat{S}) = \sum_{n=1}^{\infty} \delta_n \otimes a_n x_n = \sum_{n=1}^{\infty} \delta_n \otimes y_n = S$. Then $F$ is onto. This ends the proof.

**Theorem 3.4.** Let $T = \delta_k \otimes x \in C_1(\ell^\infty, X)$ with $\|T\| = 1$. Then $T$ is an extreme point of $C_1(\ell^\infty, X)$ if and only if $x$ is extreme in $B[X]$.

**Proof.** Let $x \in \text{ext}(B_1[X])$. We claim that $T = \delta_k \otimes x$ is an extreme point of $C_1(\ell^\infty, X)$. Without loss of generality, assume $T = \delta_1 \otimes x$ and assume that $T$ is not an extreme point. Hence, there exist $T_1 = \sum_{n=1}^{\infty} \delta_n \otimes x_n$ and $T_2 = \sum_{n=1}^{\infty} \delta_n \otimes y_n \in C_1(\ell^\infty, X)$ such that $\delta_1 \otimes x = \frac{1}{2} (T_1 + T_2)$ and $\|T_1\| = \|T_2\| = 1$. Thus $\delta_1 \otimes x = \frac{1}{2} (T_1 + T_2)$. So, $(x, 0, 0, 0, \ldots) = \frac{1}{2} \sum_{n=1}^{\infty} \delta_n \otimes (x_n + y_n) = (\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, \ldots)$. Then $x = \frac{x_1 + y_1}{2}$ which is a contradiction, since $x \in \text{ext}(B_1[X])$. Hence, $\delta_k \otimes x$ is an extreme point of $C_1(\ell^\infty, X)$.

The Converse is clear. This ends the proof.

For $X = \ell^p$, $1 \leq p < \infty$, we have the following.
**Theorem 3.5.** Let $F : C_1 (\ell^\infty, \ell^p) \to C_1 (\ell^\infty, \ell^p)$ be an isometric onto operator. Then $F$ preserves basic atoms.

**Proof.** Let $F$ be an isometric onto operator of $C_1 (\ell^\infty, \ell^p)$. Then as is known, $F$ preserves the extreme points of the unit ball of $C_1 (\ell^\infty, \ell^p)$. Now, let $\delta_k \otimes h \in C_1 (\ell^\infty, \ell^p)$ be basic atom. Then by Theorem 3.3.8 $\delta_k \otimes \frac{h}{\|h\|} \in \text{ext } B_1 (C_1 (\ell^\infty, \ell^p))$. Hence $F(\delta_k \otimes \frac{h}{\|h\|}) = \delta_j \otimes g$ for some $g \in \text{ext } B_1 (\ell^p)$. Since $\|h\| \delta_k \otimes \frac{h}{\|h\|} = \delta_k \otimes h$, then $F(\delta_k \otimes h) = \|h\| \delta_j \otimes g$, where $\|g\| = \|h\|$.

**Theorem 3.6.** Let $F : C_1 (\ell^\infty, \ell^p) \to C_1 (\ell^\infty, \ell^p)$ be a linear operator that preserves rank. Then $F$ is an isometric onto operator, if and only if $F(\sum_{n=1}^{\infty} \delta_n \otimes x_n) = A(\delta_n) \otimes a_n B(x_n)$, where $A : \ell^1 \to \ell^1$ is an isometric onto operator, and $B : \ell^p \to \ell^p$ is an isometric onto operator, and $(a_n)$ is a sequence of reals such that $|a_n| = 1$.

**Proof.** By using Theorem 3.5, We see that $F$ preserves basic atoms, and by using Theorem 3.3m we can obtain the result immediately.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**


