ON INTUITIONISTIC FUZZY $R_1$—SPACES

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Abstract: In this paper, we define some notions of the intuitionistic fuzzy $R_1$—spaces. We investigate some relations among them and we also investigate the relationship between intuitionistic fuzzy topological space and intuitionistic topological space. We show that $R_1$—spaces satisfy “good extensions” property. It is also shown that these notions are hereditary and projective.

Keywords: intuitionistic set; intuitionistic fuzzy set; intuitionistic topological space; intuitionistic fuzzy topological space; intuitionistic fuzzy $R_1$—spaces.

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I. INTRODUCTION

Atanassov [3] introduced the concepts of intuitionistic fuzzy sets which take into account both the degrees of membership and non-membership subject to the condition that their sum does not exceed 1. D. Coker subsequently initiated a study of intuitionistic fuzzy topological spaces. After then many topologists work in intuitionistic fuzzy topological spaces.

In this paper, we define some new notions of $R_1$—spaces using intuitionistic fuzzy sets and we investigate the properties of $R_1$—spaces.

Definition 1.1.[10] An intuitionistic set $A$ is an object having the form $A = (x, A_1, A_2)$, where $A_1$ and $A_2$ are subsets of $X$ satisfying $A_1 \cap A_2 = \phi$. The set $A_1$ is called the set of member of $A$ while $A_2$ is called the set of non-member of $A$.

Throughout this paper, we use the simpler notation $A = (A_1, A_2)$ for an intuitionistic set.

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Remark 1.2[10] Every subset \( A \) on a nonempty set \( X \) may obviously be regarded as an intuitionistic set having the form \( A' = (A, A^C) \), where \( A^C = X \setminus A \) is the complement of \( A \) in \( X \).

Definition 1.3[10] Let the intuitionistic sets \( A \) and \( B \) on \( X \) be of the forms \( A = (A_1, A_2) \) and \( B = (B_1, B_2) \) respectively. Furthermore, let \( \{A_j : j \in J\} \) be an arbitrary family of intuitionistic sets in \( X \), where \( A_j = (A_j^{(1)}, A_j^{(2)}) \). Then

(a) \( A \subseteq B \) if and only if \( A_1 \subseteq B_1 \) and \( A_2 \supseteq B_2 \).
(b) \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).
(c) \( \overline{A} = (A_2, A_1) \), denotes the complement of \( A \).
(d) \( \bigcap A_j = (\bigcap A_j^{(1)}, \bigcup A_j^{(2)}) \).
(e) \( \bigcup A_j = (\bigcup A_j^{(1)}, \bigcap A_j^{(2)}) \).
(f) \( \phi_\sim = (\phi, X) \) and \( X_\sim = (X, \phi) \).

Definition 1.4[8] An intuitionistic topology on a set \( X \) is a family \( \tau \) of intuitionistic sets in \( X \) satisfying the following axioms:

1. \( \phi_\sim, X_\sim \in \tau \).
2. \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \).
3. \( \bigcup G_i \in \tau \) for any arbitrary family \( G_i \in \tau \).

In this case, the pair \((X, \tau)\) is called an intuitionistic topological space (ITS, in short) and any intuitionistic set in \( \tau \) is known as an intuitionistic open set (IOS, in short) in \( X \).

Definition 1.5[3] Let \( X \) be a non empty set and \( I \) be the unit interval \([0, 1]\). An intuitionistic fuzzy set \( A \) (IFS, in short) in \( X \) is an object having the form \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \), where \( \mu_A : X \rightarrow I \) and \( \nu_A : X \rightarrow I \) denote the degree of membership and the degree of non-membership respectively, and \( \mu_A(x) + \nu_A(x) \leq 1 \).
Let \( I(X) \) denote the set of all intuitionistic fuzzy sets in \( X \). Obviously every fuzzy set \( \mu_A \) in \( X \) is an intuitionistic fuzzy set of the form \((\mu_A, 1 - \mu_A)\).
Throughout this paper, we use the simpler notation \( A = (\mu_A, \nu_A) \) instead of \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \).
**Definition 1.6.**[3] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets in $X$. Then

1. $A \subseteq B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $A^c = (\nu_A, \mu_A)$.
4. $A \cap B = (\mu_A \cap \mu_B; \nu_A \cup \nu_B)$.
5. $A \cup B = (\mu_A \cup \mu_B; \nu_A \cap \nu_B)$.
6. $0_\sim = (0^-, 1^-)$ and $1_\sim = (1^-, 0^-)$.

**Definition 1.7.**[8] Let $\{A_i : i \in J\}$ be an arbitrary family of IFSs in $X$. Then

(a) $\bigcap A_i = (\bigcap \mu_{A_i}, \bigcup \nu_{A_i})$.
(b) $\bigcup A_i = (\bigcup \mu_{A_i}, \bigcap \nu_{A_i})$.

**Definition 1.8.**[9] An intuitionistic fuzzy topology (IFT, in short) on $X$ is a family $t$ of IFS’s in $X$ which satisfies the following axioms:

1. $0_\sim, 1_\sim \in t$.
2. If $A_1, A_2 \in t$, then $A_1 \cap A_2 \in t$.
3. If $A_i \in t$ for each $i$, then $\bigcup A_i \in t$.

The pair $(X, t)$ is called an intuitionistic fuzzy topological space (IFTS, in short). Let $(X, t)$ be an IFTS. Then any member of $t$ is called an intuitionistic fuzzy open set (IFOS, in short) in $X$. The complement of an IFOS in $X$ is called an intuitionistic fuzzy closed set (IFCS, in short) in $X$.

**Definition 1.9.**[3] Let $X$ and $Y$ be two nonempty sets and $f : X \to Y$ be a function. If $B = \{(y, \mu_B(y), \nu_B(y)) / y \in Y\}$ is an IFS in $Y$, then the pre image of $B$ under $f$, denoted by $f^{-1}(B)$ is the IFS in $X$ defined by $f^{-1}(B) = \{(x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)) / x \in X\}$ and the image of $A$ under $f$, denoted by $f(A) = \{(y, f(\mu_A), f(\nu_A)) / y \in Y\}$ is an IFS of $Y$, where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

$$f(\nu_A)(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

**Definition 1.10.**[6] Let $A = (X, \mu_A, \nu_A)$ and $B = (Y, \mu_B, \nu_B)$ be IFSs of $X$ and $Y$ respectively. Then the product of intuitionistic fuzzy sets $A$ and $B$ denoted by $A \times B$ is defined by $A \times B = \{X \times Y, \mu_A \times \mu_B, \nu_A \times \nu_B\}$ where $(\mu_A \times \mu_B)(x, y) = \min(\mu_A(x), \mu_B(y))$ and $(\nu_A \times \nu_B)(x, y) = \sup(\nu_A(x), \nu_B(y))$. 


Definition 1.11. Let \((X_i, t_i), \ i = 1, 2\) be two IFTSs, and then the product \(t_1 \times t_2\) on \(X_1 \times X_2\) is defined as the IFT generated by \(\{\rho_i^{-1}(U_i) : U_i \in t_i, \ i = 1, 2\}\), where \(\rho_i : X_1 \times X_2 \to X_i, \ i = 1, 2\) are the projection maps and the IFTS \((X_1 \times X_2, t_1 \times t_2)\) is called product IFTS.

Theorem 1.12. Let \((X, \tau)\) be an intuitionistic topological space and let \(t = \{1_A : A \in \tau\}, 1_{(A_1, A_2)} = (1_{A_1}, 1_{A_2})\), then \((X, t)\) is the corresponding intuitionistic fuzzy topological space of \((X, \tau)\).

2. INTUITIONISTIC FUZZY R_1–SPACES

Definition 2.1 An intuitionistic fuzzy topological space \((X, t)\) is called

1. IF\(_R\)-R(i) if for all \(x, y \in X, x \neq y\) whenever \(\exists A = (\mu_A, \upsilon_A) \in t\) with \(A(x) \neq A(y)\), then \(\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t\) such that \(\mu_B(x) = 1, \upsilon_B(x) = 0; \mu_C(y) = 1, \upsilon_C(y) = 0\) and \(B \cap C = \emptyset\).

2. IF\(_R\)-R(ii) if for all \(x, y \in X, x \neq y\) whenever \(\exists A = (\mu_A, \upsilon_A) \in t\) with \(A(x) \neq A(y)\), then \(\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t\) such that \(\mu_B(x) = 1, \upsilon_B(x) = 0; \mu_C(y) > 0, \upsilon_C(y) = 0\) and \(B \cap C = (0^- , \gamma^-)\) where \(\gamma \in (0, 1]\).

3. IF\(_R\)-R(iii) if for all \(x, y \in X, x \neq y\) whenever \(\exists A = (\mu_A, \upsilon_A) \in t\) with \(A(x) \neq A(y)\), then \(\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t\) such that \(\mu_B(x) > 0, \upsilon_B(x) = 0; \mu_C(y) = 1, \upsilon_C(y) = 0\) and \(B \cap C = (0^- , \gamma^-)\) where \(\gamma \in (0, 1]\).

4. IF\(_R\)-R(iv) if for all \(x, y \in X, x \neq y\) whenever \(\exists A = (\mu_A, \upsilon_A) \in t\) with \(A(x) \neq A(y)\), then \(\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t\) such that \(\mu_B(x) > 0, \upsilon_B(x) = 0; \mu_C(y) > 0, \upsilon_C(y) = 0\) and \(B \cap C = (0^- , \gamma^-)\) where \(\gamma \in (0, 1]\).

Definition 2.2. Let \(\alpha \in (0, 1)\). An intuitionistic fuzzy topological space \((X, t)\) is called
(a) $\alpha - \text{IF} - R_1(i)$ if for all $x, y \in X, x \neq y$ whenever $\exists A = (\mu_A, \upsilon_A) \in t$ with $A(x) \neq A(y)$, then $\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t$ such that $\mu_B(x) = 1, \upsilon_B(x) = 0; \mu_C(y) \geq \alpha, \upsilon_C(y) = 0$ and $B \cap C = 0$. 

(b) $\alpha - \text{IF} - R_1(ii)$ if for all $x, y \in X, x \neq y$ whenever $\exists A = (\mu_A, \upsilon_A) \in t$ with $A(x) \neq A(y)$, then $\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t$ such that $\mu_B(x) \geq \alpha, \upsilon_B(x) = 0; \mu_C(y) \geq \alpha, \upsilon_C(y) = 0$ and $B \cap C = (0^-, \gamma^-)$ where $\gamma \in (0, 1]$. 

(c) $\alpha - \text{IF} - R_1(iii)$ if for all $x, y \in X, x \neq y$ whenever $\exists A = (\mu_A, \upsilon_A) \in t$ with $A(x) \neq A(y)$, then $\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t$ such that $\mu_B(x) > 0, \upsilon_B(x) = 0; \mu_C(y) \geq \alpha, \upsilon_C(y) = 0$ and $B \cap C = (0^-, \gamma^-)$ where $\gamma \in (0, 1]$. 

Theorem 2.3. Let $(X, t)$ be an intutionistic fuzzy topological space. Then we have the following implication:

![Diagram](image)

**Proof:** Suppose $(X, t)$ is $\text{IF} - R_1(i)$ space. We shall prove that $(X, t)$ is $\text{IF} - R_1(ii)$. Let $x, y \in X, x \neq y$ and $A = (\mu_A, \upsilon_A) \in t$ with $A(x) \neq A(y)$. Since $(X, t)$ is $\text{IF} - R_1(i)$, then $\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t$ such that $\mu_B(x) = 1, \upsilon_B(x) = 0; \mu_C(y) = 1, \upsilon_C(y) = 0$ and $B \cap C = 0$ $\Rightarrow$ $\mu_B(x) = 1, \upsilon_B(x) = 0; \mu_C(y) > 0, \upsilon_C(y) = 0$ and $B \cap C = (0^-, \gamma^-)$ where $\gamma \in (0, 1]$. Which is $\text{IF} - R_1(ii)$. Hence $\text{IF} - R_1(i) \Rightarrow \text{IF} - R_1(ii)$.

Again, suppose $(X, t)$ is $\text{IF} - R_1(i)$. We shall prove that $(X, t)$ is $\text{IF} - R_1(iii)$,. Let $x, y \in X, x \neq y$ and $A = (\mu_A, \upsilon_A) \in t$ with $A(x) \neq A(y)$. Since $(X, t)$ is $\text{IF} - R_1(i)$, then $\exists B = (\mu_B, \upsilon_B), C = (\mu_C, \upsilon_C) \in t$ such that $\mu_B(x) = 1, \upsilon_B(x) = 0; \mu_C(y) = 1, \upsilon_C(y) = 0$ and $B \cap C = 0$ $\Rightarrow$ $\mu_B(x) > 0, \upsilon_B(x) = 0; \mu_C(y) = 1, \upsilon_C(y) = 0$ and $B \cap C = (0^-, \gamma^-)$ where $\gamma \in (0, 1]$. Which is $\text{IF} - R_1(iii)$. Hence $\text{IF} - R_1(i) \Rightarrow \text{IF} - R_1(iii)$.

Furthermore, it can verify that $\text{IF} - R_1(i) \Rightarrow \text{IF} - R_1(iv), \text{IF} - R_1(ii) \Rightarrow \text{IF} - R_1(iv)$ and $\text{IF} - R_1(iii) \Rightarrow \text{IF} - R_1(iv)$. 


None of the reverse implications is true in general which can be seen from the following examples.

**Example 2.3.1.** Let \( X = \{x, y\} \) and \( t \) be the intuitionistic fuzzy topology on \( X \) generated by \( \{A, B, C\} \) where \( A = \{(x, 0.4, 0), (y, 0, 0.2)\} \) and \( B = \{(x, 1, 0), (y, 0, 0.5)\} \), \( C = \{(x, 0, 0.6), (y, 0.7, 0)\} \). We see that the IFTS \((X, t)\) is IF\( R_1(\text{ii})\) but not IF\( R_1(\text{i})\).

**Example 2.3.2.** Let \( X = \{x, y\} \) and \( t \) be the intuitionistic fuzzy topology on \( X \) generated by \( \{A, B, C\} \) where \( A = \{(x, 0.3, 0), (y, 0, 0.2)\} \) and \( B = \{(x, 0.5, 0), (y, 0, 0.7)\} \), \( C = \{(x, 0, 0.6), (y, 1, 0)\} \). We see that the IFTS \((X, t)\) is IF\( R_1(\text{iii})\) but not IF\( R_1(\text{i})\).

**Example 2.3.3.** Let \( X = \{x, y\} \) and \( t \) be the intuitionistic fuzzy topology on \( X \) generated by \( \{A, B, C\} \) where \( A = \{(x, 0.1, 0), (y, 0, 0.7)\} \) and \( B = \{(x, 1, 0), (y, 0, 0.6)\} \), \( C = \{(x, 0, 0.3), (y, 0.9, 0)\} \). We see that the IFTS \((X, t)\) is IF\( R_1(\text{ii})\) but not IF\( R_1(\text{iii})\).

**Example 2.3.4.** Let \( X = \{x, y\} \) and \( t \) be the intuitionistic fuzzy topology on \( X \) generated by \( \{A, B, C\} \) where \( A = \{(x, 0.2, 0), (y, 0, 0.4)\} \) and \( B = \{(x, 0.6, 0), (y, 0, 0.3)\} \), \( C = \{(x, 0, 0.5), (y, 0.8, 0)\} \). We see that the IFTS \((X, t)\) is IF\( R_1(\text{iii})\) but not IF\( R_1(\text{ii})\).

**Theorem 2.4.** Let \((X, t)\) be an intuitionistic fuzzy topological space. Then we have the following implications:

\[
\alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii}) \quad \alpha \rightarrow \text{IF} \rightarrow R_1(\text{iii})
\]

**Proof:** Suppose \((X, t)\) is \(\alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii})\) space. We shall prove that \((X, t)\) is \(\alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii})\). Let \(\alpha \in \langle 0, 1 \rangle\). Again, let \(x, y \in X, x \neq y\) and \(A = (\mu_A, v_A) \in t\) with \(A(x) \neq A(y)\). Since \((X, t)\) is IF\( R_1(\text{ii})\), then \(\exists \ B = (\mu_B, v_B), \ C = (\mu_C, v_C) \in t\) such that \(\mu_B(x) = 1, v_B(x) = 0; \mu_C(y) \geq \alpha, v_C(y) = 0\) and \(B \cap C = 0\). \(\Rightarrow\) \(\mu_B(x) \geq \alpha, v_B(x) = 0; \mu_C(y) \geq \alpha, v_C(y) = 0\) for any \(\alpha \in \langle 0, 1 \rangle\) and \(B \cap C = (0^-, \gamma^-)\) where \(\gamma \in \langle 0, 1 \rangle\). Which is \(\alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii})\). Hence \(\alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii}) \Rightarrow \alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii})\).

Again, suppose \((X, t)\) is \(\alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii})\). We shall prove that \((X, t)\) is \(\alpha \rightarrow \text{IF} \rightarrow R_1(\text{iii})\). Let \(\alpha \in \langle 0, 1 \rangle\). Again, let \(x, y \in X, x \neq y\) and \(A = (\mu_A, v_A) \in t\) with \(A(x) \neq A(y)\). Since \((X, t)\) is \(\alpha \rightarrow \text{IF} \rightarrow R_1(\text{ii})\), then \(\exists \ B = (\mu_B, v_B), \ C = (\mu_C, v_C) \in t\) such that \(\mu_B(x) \geq \alpha, v_B(x) = 0; \mu_C(y) \geq \alpha, v_C(y) = 0\) and \(B \cap C = (0^-, \gamma^-)\) where \(\gamma \in \langle 0, 1 \rangle\) \(\Rightarrow\) \(\mu_B(x) > 0, v_B(x) = 0\);
\(\mu_c(y) \geq \alpha, \nu_c(y) = 0\) and \(B \cap C = (0^-, \gamma^-)\) where \(\gamma \in (0, 1]\). Which is \(\alpha - \text{IF}\text{-}\text{R}_1(\text{iii})\). Hence \(\alpha - \text{IF}\text{-}\text{R}_1(\text{ii}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{iii})\).

Furthermore, it can verify that \(\alpha - \text{IF}\text{-}\text{R}_1(\text{i}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{iii})\).

None of the reverse implications is true in general which can be seen from the following examples.

**Example 2.4.1.** Let \(X = \{x, y\} \) and \(t\) be the intuitionistic fuzzy topology on \(X\) generated by \(\{A, B, C\}\) where \(A = \{(x, 0.2, 0), (y, 0.3)\}\) and \(B = \{(x, 0.6, 0), (y, 0.4)\}\), \(C = \{(x, 0, 0.5), (y, 0.7, 0)\}\). For \(\alpha = 0.4\), we see that the IFTS \((X, t)\) is \(\alpha - \text{IF}\text{-}\text{R}_1(\text{ii})\) but not \(\alpha - \text{IF}\text{-}\text{R}_1(\text{i})\).

**Example 2.4.2.** Let \(X = \{x, y\} \) and \(t\) be the intuitionistic fuzzy topology on \(X\) generated by \(\{A, B, C\}\) where \(A = \{(x, 0.1, 0), (y, 0.4)\}\) and \(B = \{(x, 0.3, 0), (y, 0.7)\}, C = \{(x, 0, 0.6), (y, 0.5, 0)\}\). For \(\alpha = 0.5\), we see that the IFTS \((X, t)\) is \(\alpha - \text{IF}\text{-}\text{R}_1(\text{iii})\) but not \(\alpha - \text{IF}\text{-}\text{R}_1(\text{i})\).

**Theorem 2.5.** Let \((X, t)\) be an intuitionistic fuzzy topological space and \(0 < \alpha \leq \beta < 1\), then

1. \(\beta - \text{IF}\text{-}\text{R}_1(\text{i}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{i}).\)
2. \(\beta - \text{IF}\text{-}\text{R}_1(\text{ii}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{ii}).\)
3. \(\beta - \text{IF}\text{-}\text{R}_1(\text{iii}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{iii}).\)

**Proof:** Let \(\beta \in (0, 1)\). Suppose the intuitionistic fuzzy topological space \((X, t)\) is \(\beta - \text{IF}\text{-}\text{R}_1(\text{i})\). We shall prove that \((X, t)\) is \(\alpha - \text{IF}\text{-}\text{R}_1(\text{i})\). Let \(x, y \in X\), \(x \neq y\) and \(A = (\mu_A, \nu_A) \in t\) with \(A(x) \neq A(y)\). Since \((X, t)\) is \(\beta - \text{IF}\text{-}\text{R}_1(\text{i})\), then \(\exists B = (\mu_B, \nu_B)\), \(C = (\mu_C, \nu_C) \in t\) such that \(\mu_B(x) = 1, \nu_B(x) = 0; \mu_C(y) \geq \beta, \nu_C(y) = 0\) and \(B \cap C = 0^- \Rightarrow \mu_B(x) = 1, \nu_B(x) = 0; \mu_C(y) \geq \alpha, \nu_C(y) = 0\) and \(B \cap C = 0^-\) as \(0 < \alpha \leq \beta < 1\). Which is \(\alpha - \text{IF}\text{-}\text{R}_1(\text{i})\). Hence \(\beta - \text{IF}\text{-}\text{R}_1(\text{i}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{i})\).

Furthermore, it can verify that \(\beta - \text{IF}\text{-}\text{R}_1(\text{ii}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{ii})\) and \(\beta - \text{IF}\text{-}\text{R}_1(\text{iii}) \Rightarrow \alpha - \text{IF}\text{-}\text{R}_1(\text{iii})\).

None of the reverse implications is true in general which can be seen from the following examples.

**Example 2.5.1.** Let \(X = \{x, y\} \) and let \(t\) be the intuitionistic fuzzy topology on \(X\) generated by \(\{A, B, C\}\) where \(A = \{(x, 0.3, 0), (y, 0.7)\}\) and \(B = \{(x, 1, 0), (y, 0.4)\}, C = \{(x, 0, 0.5), (y, 0.6, 0)\}\). For \(\alpha = 0.5\) and \(\beta = 0.7\), it is clear that the IFTS \((X, t)\) is \(\alpha - \text{IF}\text{-}\text{R}_1(\text{i})\) but not \(\beta - \text{IF}\text{-}\text{R}_1(\text{i})\).
Example 2.5.2. Let $X = \{x, y\}$ and let $t$ be the intuitionistic fuzzy topology on $X$ generated by 
$\{A, B, C\}$ where $A = \{(x, 0.4, 0), (y, 0.2)\}$ and $B = \{(x, 0.7, 0), (y, 0.5)\}$, $C = \{(x, 0.3), (y, \beta, 0.6, 0)\}$. For $\alpha = 0.6$ and $\beta = 0.8$, it is clear that the IFTS $(X, t)$ is $\alpha - \text{IF} - R_1(\text{ii})$ but not $\beta - \text{IF} - R_1(\text{ii})$.

Example 2.5.3. Let $X = \{x, y\}$ and let $t$ be the intuitionistic fuzzy topology on $X$ generated by 
$\{A, B, C\}$ where $A = \{(x, 0.5, 0), (y, 0.1)\}$ and $B = \{(x, 0.4, 0), (y, 0.6)\}$, $C = \{(x, 0.3), (y, \beta, 0.5, 0)\}$. For $\alpha = 0.4$ and $\beta = 0.6$, it is clear that the IFTS $(X, t)$ is $\alpha - \text{IF} - R_1(\text{iii})$ but not $\beta - \text{IF} - R_1(\text{iii})$.

Theorem 2.6. Let $\alpha \in (0, 1)$ and let $(X, t)$ be an intuitionistic fuzzy topological space, $U \subseteq X$ and $t_U = \{A|U : A \in t\}$ then

(a) $(X, t)$ is $\text{IF} - R_1(\text{i}) \implies (U, t_U)$ is $\text{IF} - R_1(\text{i})$.

(b) $(X, t)$ is $\text{IF} - R_1(\text{ii}) \implies (U, t_U)$ is $\text{IF} - R_1(\text{ii})$.

(c) $(X, t)$ is $\text{IF} - R_1(\text{iii}) \implies (U, t_U)$ is $\text{IF} - R_1(\text{iii})$.

(d) $(X, t)$ is $\text{IF} - R_1(\text{iv}) \implies (U, t_U)$ is $\text{IF} - R_1(\text{iv})$.

(e) $(X, t)$ is $\alpha - \text{IF} - R_1(\text{i}) \implies (U, t_U)$ is $\alpha - \text{IF} - R_1(\text{i})$.

(f) $(X, t)$ is $\alpha - \text{IF} - R_1(\text{ii}) \implies (U, t_U)$ is $\alpha - \text{IF} - R_1(\text{ii})$.

(g) $(X, t)$ is $\alpha - \text{IF} - R_1(\text{iii}) \implies (U, t_U)$ is $\alpha - \text{IF} - R_1(\text{iii})$.

The proofs (a), (b), (c), (d), (e), (f), (g) are similar. As an example we proved (a).

Proof (a): Suppose $(X, t)$ is the intuitionistic fuzzy topological space and is also $\text{IF} - R_1(\text{i})$. We shall prove that $(U, t_U)$ is $\text{IF} - R_1(\text{i})$. Let $x, y \in U, x \neq y$ with $A_U = (\mu_{A_U}, \nu_{A_U}) \in t_U$ such that $A_U(x) \neq A_U(y)$. Since $x, y \in U \subseteq X$ then $x, y \in X, x \neq y$ as $U \subseteq X$. Suppose $A = (\mu_A, \nu_A) \in t$ is the extension IFS of $A_U$ on $X$, then $A(x) \neq A(y)$. Since $(X, t)$ is $\text{IF} - R_1(\text{i})$, then $\exists B = (\mu_B, \nu_B), C = (\mu_C, \nu_C) \in t$ such that $\mu_B(x) = 1, \nu_B(x) = 0; \mu_C(y) = 1, \nu_C(y) = 0$ and $B \cap C = 0_\sim \implies (\mu_B|U)(x) = 1, (\nu_B|U)(x) = 0; (\mu_C|U)(y) = 1, (\nu_C|U)(y) = 0$ and $(\mu_B|U, \nu_B|U) \cap (\mu_C|U, \nu_C|U) = 0_\sim$. Hence $\{(\mu_B|U, \nu_B|U), (\mu_C|U, \nu_C|U)\} \in t_U \implies (U, t_U)$ is $\text{IF} - R_1(\text{i})$.

Therefore $(U, t_U)$ is $\text{IF} - R_1(\text{i})$.

Definition 2.7 An intuitionistic topological space (ITS, in short) $(X, \tau)$ is called intuitionistic $R_1$-space (I$-R_1$ space) if for all $x, y \in X, x \neq y$ whenever $\exists P = (P_1, P_2) \in \tau$ with $(x \in P_1, y \in P_2)$ or $(y \in P_1, x \in P_2)$ then $\exists L = (L_1, L_2), M = (M_1, M_2) \in \tau$ such that $x \in L_1, x \notin L_2; y \in M_1, y \notin M_2$ and $L \cap M = \phi_\sim$. 

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Theorem 2.8. Let \((X, \tau)\) be an intuitionistic topological space and let \((X, t)\) be an intuitionistic fuzzy topological space. Then we have the following implications:

\[
\begin{align*}
&\text{IF} \rightarrow R_1(i) \\
&\text{IF} \rightarrow R_1(ii) \\
&\text{I} \rightarrow R_1(iii) \\
&\text{I} \rightarrow R_1(iv) \\
&\text{IF} \rightarrow R_1(\bar{g})
\end{align*}
\]

Proof: Let \((X, \tau)\) be \(I \rightarrow R_1\). We shall prove that \((X, t)\) is \(I \rightarrow R_1(i)\). Suppose \((X, \tau)\) is \(I \rightarrow R_1\). Let \(x, y \in X, x \neq y\) with \(A = (\mu_A, \nu_A) \in t\) such that \(A(x) \neq A(y)\). Since \(A(x) \neq A(y)\), then let \((1_c_1(x) = 1, 1_c_2(y) = 1)\) or \((1_c_1(y) = 1, 1_c_2(x) = 1)\) \(\Rightarrow (x \in C_1, y \in C_2)\) or \((y \in C_1, x \in C_2)\). Hence \((C_1, C_2) \in \tau\). Since \((X, \tau)\) is \(I \rightarrow R_1\), then \(\exists L = (L_1, L_2), M = (M_1, M_2) \in \tau\) such that \(x \in L_1, x \notin L_2; y \in M_1, y \notin M_2\) and \(L \cap M = \phi_\sim \Rightarrow 1_{L_1}(x) = 1, 1_{L_2}(x) = 0; 1_{M_1}(y) = 1, 1_{M_2}(y) = 0\) and \(L \cap M = \phi_\sim\). Let \(\mu_B = 1_{L_1}, \nu_B = 1_{L_2}, \mu_C = 1_{M_1}, \nu_C = 1_{M_2}\) where \(B = (\mu_B, \nu_B)\) and \(C = (\mu_C, \nu_C)\). Which implies for all \(x, y \in X, x \neq y\) and \(A = (\mu_A, \nu_A) \in t\) with \(A(x) \neq A(y)\), then \(\exists B = (\mu_B, \nu_B), C = (\mu_C, \nu_C) \in t\) such that \(\mu_B(x) = 1, \nu_B(x) = 0; \mu_C(y) = 1, \nu_C(y) = 0\) and \(B \cap C = \phi_\sim\). Which is \(I \rightarrow R_1(i)\).

Conversely, let \((X, t)\) be \(I \rightarrow R_1(i)\). We shall prove that \((X, \tau)\) is \(I \rightarrow R_1\). Suppose \((X, t)\) is \(I \rightarrow R_1(i)\). Let \(x, y \in X, x \neq y\) with \(P = (P_1, P_2) \in \tau\) such that \((x \in P_1, y \in P_2)\) or \((y \in P_1, x \in P_2)\). Since \((x \in P_1, y \in P_2)\) or \((y \in P_1, x \in P_2)\) \(\Rightarrow (1_{P_1}(x) = 1, 1_{P_2}(y) = 1)\) or \((1_{P_1}(y) = 1, 1_{P_2}(x) = 1)\). Hence \((1_{P_1}, 1_{P_2}) \in t\) and \((1_{P_1}, 1_{P_2}) x \neq (1_{P_1}, 1_{P_2}) y\). Since \((X, t)\) is \(I \rightarrow R_1(i)\), then \(\exists (1_{C_1}, 1_{C_2}), (1_{D_1}, 1_{D_2}) \in t\) such that \(1_{C_1}(x) = 1, 1_{C_2}(x) = 0; 1_{D_1}(y) = 1, 1_{D_2}(y) = 0\) and \((1_{C_1}, 1_{C_2}) \cap (1_{D_1}, 1_{D_2}) = \phi_\sim\). Which implies for all \(x, y \in X, x \neq y\) and \(A = (\mu_A, \nu_A) \in t\) with \(A(x) \neq A(y)\), then \(\exists B = (\mu_B, \nu_B), C = (\mu_C, \nu_C) \in t\) such that \(\mu_B(x) = 1, \nu_B(x) = 0; \mu_C(y) = 1, \nu_C(y) = 0\) and \(B \cap C = \phi_\sim\). Which is \(I \rightarrow R_1\). Therefore \(I \rightarrow R_1 \iff I \rightarrow R_1(i)\).

None of the reverse implications is true in general which can be seen from the following examples.
Example 2.8.1. Let $X = \{x, y\}$ and $t$ be the intuitionistic fuzzy topology on $X$ generated by \{A, B, C\} where $A = \{(x, 0.1, 0), (y, 0, 0.5)\}$, $B = \{(x, 1, 0), (y, 0, 0.7)\}$ and $C = \{(x, 0, 0.4), (y, 0.8, 0)\}$. We see that the IFTS $(X, t)$ is IF$-R_{1}(ii)$ but not corresponding I$-R_{1}$.

Example 2.8.2. Let $X = \{x, y\}$ and $t$ be the intuitionistic fuzzy topology on $X$ generated by \{A, B, C\} where $A = \{(x, 0.2, 0), (y, 0, 0.3)\}$, $B = \{(x, 0.3, 0), (y, 0, 0.6)\}$ and $C = \{(x, 0, 0.8), (y, 1, 0)\}$. We see that the IFTS $(X, t)$ is IF$-R_{1}(iii)$ but not corresponding I$-R_{1}$.

Example 2.8.3. Let $X = \{x, y\}$ and $t$ be the intuitionistic fuzzy topology on $X$ generated by \{A, B, C\} where $A = \{(x, 0.4, 0), (y, 0, 0.3)\}$ and $B = \{(x, 0.5, 0), (y, 0, 0.7)\}$, $C = \{(x, 0, 0.5), (y, 0.9, 0)\}$. We see that the IFTS $(X, t)$ is IF$-R_{1}(iv)$ but not corresponding I$-R_{1}$.

Theorem 2.9. Let $(X, \tau)$ be an intuitionistic topological space and let $(X, t)$ be an intuitionistic fuzzy topological space. Then we have the following implications:

\[ \alpha - IF - R_{1}(i) \Rightarrow \alpha - IF - R_{1}(ii) \Rightarrow \alpha - IF - R_{1}(iii) \]

Proof: Let $(X, \tau)$ be I$-R_{1}$. We shall prove that $(X, t)$ is IF$-R_{1}(i)$. Let $\alpha \in (0, 1)$. Suppose $(X, \tau)$ is I$-R_{1}$. Let $x, y \in X$, $x \neq y$ with $A = (\mu_{A}, \nu_{A}) \in t$ such that $A(x) \neq A(y)$. Since $A(x) \neq A(y)$, then let $(1_{C_{1}}(x) = 1, 1_{C_{2}}(y) = 1)$ or $(1_{C_{1}}(y) = 1, 1_{C_{2}}(x) = 1) \Rightarrow (x \in C_{1}$, $y \in C_{2})$ or $(y \in C_{1}, x \in C_{2})$. Hence $(C_{1}, C_{2}) \in \tau$. Since $(X, \tau)$ is I$-R_{1}$, then $\exists L = (L_{1}, L_{2})$, $M = (M_{1}, M_{2}) \in \tau$ such that $x \in L_{1}$, $x \notin L_{2}$; $y \in M_{1}$, $y \notin M_{2}$ and $L \cap M = \phi$. \Rightarrow $1_{L_{1}}(x) = 1$, $1_{L_{2}}(x) = 0$; $1_{M_{1}}(y) = 1$, $1_{M_{2}}(y) = 0$ and $L \cap M = \phi$. \Rightarrow $\mu_{B} = 1_{L_{1}}$, $\nu_{B} = 1_{L_{2}}$; $\mu_{C} = 1_{M_{1}}$, $\nu_{C} = 1_{M_{2}}$ where $B = (\mu_{B}, \nu_{B})$ and $C = (\mu_{C}, \nu_{C})$. Which implies for all $x, y \in X$, $x \neq y$ and $A = (\mu_{A}, \nu_{A}) \in t$ with $A(x) \neq A(y)$, then $\exists B = (\mu_{B}, \nu_{B}), C = (\mu_{C}, \nu_{C}) \in t$ such that $\mu_{B}(x) = 1, \nu_{B}(x) = 0$ or $B \cap C = 0_{\sim} \Rightarrow \mu_{B}(x) = 1, \nu_{B}(x) = 0$; $\mu_{C}(y) \geq \alpha$, $\nu_{C}(y) = 0$ and $B \cap C = 0_{\sim}$ for any $\alpha \in (0, 1)$. Which is $\alpha - IF - R_{1}(i)$.

None of the reverse implications is true in general which can be seen by the following examples.
Example 2.9.1. Let $X = \{x, y\}$ and $t$ be the intuitionistic fuzzy topology on $X$ generated by $\{A, B, C\}$ where $A = \{(x, 0.3, 0), (y, 0, 0.4)\}$, $B = \{(x, 1, 0), (y, 0, 0.4)\}$ and $C = \{(x, 0, 0.2), (y, 0.5, 0)\}$. For $\alpha = 0.5$, we see that the IFTS $(X, t)$ is $\alpha - \text{IF}\neg \neg R_1(i)$ but not corresponding $\text{I}\neg \neg R_1$.

Example 2.9.2. Let $X = \{x, y\}$ and $t$ be the intuitionistic fuzzy topology on $X$ generated by $\{A, B, C\}$ where $A = \{(x, 0.7, 0), (y, 0, 0.9)\}$, $B = \{(x, 0.6, 0), (y, 0, 0.4)\}$ and $C = \{(x, 0, 0.5), (y, 0.4, 0)\}$. For $\alpha = 0.3$, we see that the IFTS $(X, t)$ is $\alpha - \text{IF}\neg \neg R_1(ii)$ but not corresponding $\text{I}\neg \neg R_1$.

Example 2.9.3 Let $X = \{x, y\}$ and $t$ be the intuitionistic fuzzy topology on $X$ generated by $\{A, B, C\}$ where $A = \{(x, 0.2, 0), (y, 0, 0.6)\}$ and $B = \{(x, 0.5, 0), (y, 0, 0.4)\}$, $C = \{(x, 0, 0.3), (y, 0.8, 0)\}$. For $\alpha = 0.7$, we see that the IFTS $(X, t)$ is $\alpha - \text{IF}\neg \neg R_1(iii)$ but not corresponding $\text{I}\neg \neg R_1$.

Theorem 2.10. Let $(X, t)$ and $(Y, s)$ be two intuitionistic fuzzy topological spaces and $f: X \to Y$ be one-one, onto, continuous open mapping, then

1. $(X, t)$ is $\text{IF}\neg \neg R_1(i)$ $\iff$ $(Y, s)$ is $\text{IF}\neg \neg R_1(i)$.
2. $(X, t)$ is $\text{IF}\neg \neg R_1(ii)$ $\iff$ $(Y, s)$ is $\text{IF}\neg \neg R_1(ii)$.
3. $(X, t)$ is $\text{IF}\neg \neg R_1(iii)$ $\iff$ $(Y, s)$ is $\text{IF}\neg \neg R_1(iii)$.
4. $(X, t)$ is $\text{IF}\neg \neg R_1(iv)$ $\iff$ $(Y, s)$ is $\text{IF}\neg \neg R_1(iv)$.
5. $(X, t)$ is $\alpha - \text{IF}\neg \neg R_1(i)$ $\iff$ $(Y, s)$ is $\alpha - \text{IF}\neg \neg R_1(i)$.
6. $(X, t)$ is $\alpha - \text{IF}\neg \neg R_1(ii)$ $\iff$ $(Y, s)$ is $\alpha - \text{IF}\neg \neg R_1(ii)$.
7. $(X, t)$ is $\alpha - \text{IF}\neg \neg R_1(iii)$ $\iff$ $(Y, s)$ is $\alpha - \text{IF}\neg \neg R_1(iii)$.

Proof: Suppose the intuitionistic fuzzy topological space $(X, t)$ is $\text{IF}\neg \neg R_1(i)$. We shall prove that the intuitionistic fuzzy topological space $(Y, s)$ is $\text{IF}\neg \neg R_1(i)$. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$ and $W = (\mu_W, v_W) \in s$ such that $W(y_1) \neq W(y_2)$. Since $f$ is onto, then $\exists x_1, x_2 \in X$ such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Since $y_1 \neq y_2$, then $f^{-1}(y_1) \neq f^{-1}(y_2)$ as $f$ is one-one and onto. Hence $x_1 \neq x_2$. We have $A = (\mu_A, v_A) \in t$ such that $A = f^{-1}(W)$, that is $(\mu_A, v_A) = (f^{-1}(\mu_W), f^{-1}(v_W))$ as $f$ is $\text{IF}$-continuous. Now, $A(x_1) = (\mu_A(x_1) = (f^{-1}(\mu_W))(x_1) = \mu_W(f(x_1)) = \mu_W(y_1)$, $v_A(x_1) = (f^{-1}(v_W))(x_1) = v_W(f(x_1)) = v_W(y_1)$) and $A(x_2) = (\mu_A(x_2) = (f^{-1}(\mu_W))(x_2) = \mu_W(f(x_2)) = \mu_W(y_2)$, $v_A(x_2) = (f^{-1}(v_W))(x_2) = v_W(f(x_2)) = v_W(y_2)$). Hence $A(x_1) \neq A(x_2)$ as $W(y_1) \neq W(y_2)$. Therefore, since $(X, t)$ is $\text{IF}\neg \neg R_1(i)$, then $\exists B = (\mu_B, v_B)$, $C = (\mu_C, v_C) \in t$ such that $\mu_B(x_1) = 1, v_B(x_1) = 0$; $\mu_C(x_2) = 1, v_C(x_2) = 0$ and $B \cap C = 0$. Put $U = f(B)$ and $V = f(C)$ where $U = (\mu_U, v_U), V = (\mu_V, v_V) \in s$ as $f$ is $\text{IF}$-continuous. Now, $\{\mu_U(y_1) = (f(\mu_B))(y_1) = \mu_B(f^{-1}(y_1)) = \mu_B(x_1) = 1, v_U(y_1) = (f(v_B))(y_1) = v_B(f^{-1}(y_1)) = v_B(x_1) = 0 \}$.
0): \{ \mu_V(y_2) = (f(\mu_C))(y_2) = \mu_C(f^{-1}(y_2)) = \mu_C(x_2) = 1, \nu_V(y_2) = (f(\nu_C))(y_2) = \nu_C(f^{-1}(y_2)) = \nu_C(x_2) = 0 \} \text{ and } U \cap V = 0. \text{ Hence } (U, V) \in \mathcal{S}. \text{ Therefore } (Y, s) \text{ is IF}−\text{R}_1(i).

Conversely, Suppose the intuitionistic fuzzy topological space \((Y, s)\) is \text{IF}−\text{R}_1(i). We shall prove that the intuitionistic fuzzy topological space \((X, t)\) is \text{IF}−\text{R}_1(i). Let \(x_1, x_2 \in X, x_1 \neq x_2\) and \(A = (\mu_A, \nu_A) \in \mathcal{E}\) such that \(A(x_1) \neq A(x_2)\). Since \(f\) is one-one, then \(\exists y_1 \in s\) such that \(y_1 = f(x_1)\) and \(y_2 = f(x_2)\) and \(f(x_1) \neq f(x_2)\). That is, \(y_1 \neq y_2\). We have \(W = (\mu_W, \nu_W) \in \mathcal{S}\) such that \(W = f(A)\), that is \((\mu_W, \nu_W) = (f(\mu_A), f(\nu_A))\) as \(f\) is \text{IF}-continuous. Now, \(W(y_1) = \{(\mu_A(y_1)) = \mu_A(f^{-1}(y_1)) = \mu_A(x_1), (\nu_A(y_1)) = \nu_A(f^{-1}(y_1)) = \nu_A(x_1)\} \text{ and } W(y_2) = \{(\mu_A(y_2)) = \mu_A(f^{-1}(y_2)) = \mu_A(x_2), (\nu_A(y_2)) = \nu_A(f^{-1}(y_2)) = \nu_A(x_2)\}. \text{ Hence } W(y_1) \neq W(y_2) \text{ as } A(x_1) \neq A(x_2). \text{ Since } (Y, s) \text{ is IF}−\text{R}_1(i), \text{ then } \exists U = (\mu_U, \nu_U), V = (\mu_V, \nu_V) \in \mathcal{S}\) such that \(\mu_U(y_1) = 1, \nu_U(y_1) = 0; \mu_V(y_2) = 1, \nu_V(y_2) = 0 \text{ and } U \cap V = 0. \text{ Put } B = f^{-1}(U) \text{ and } C = f^{-1}(V) \text{ where } B = (\mu_B, \nu_B), C = (\mu_C, \nu_C) \in \mathcal{E}\text{ as } f \text{ is IF-continuous. Now,}\}

\{(f^{-1}(\mu_U))(x_1) = \mu_U(f(x_1)) = \mu_U(y_1) = 1, (f^{-1}(\nu_U))(x_1) = \nu_U(f(x_1)) = \nu_U(y_1) = 0\}; \{(f^{-1}(\mu_V))(x_2) = \mu_V(f(x_2)) = \mu_V(y_2) = 1, (f^{-1}(\nu_V))(x_2) = \nu_V(f(x_2)) = \nu_V(y_2) = 0\} \text{ and } B \cap C = 0. \text{ Hence } (B, C) \in \mathcal{E}\text{. Therefore } (X, t) \text{ is IF}−\text{R}_1(i).

**Theorem 2.11.** Let \(\{(X_m, t_m): m \in J\}\) be a finite family of intuitionistic fuzzy topological spaces and let \((X, t)\) be their product IFTS. Then each \((X_m, t_m)\) is IF−R_1(i) if the product IFTS \((\prod X_m, \prod t_m)\) is IF−R_1(i).

**Proof:** Suppose \((X, t)\) is IF−R_1(i). We shall prove that the intuitionistic fuzzy topological spaces \((X_m, t_m)\) is IF−R_1(i), for all \(m \in J\). Let for \(j \in J\), choose \(x_j, y_j \in X_j\), such that \(x_j \neq y_j\). Now consider \(x = \prod x_m, y = \prod y_m\) where \(x_m = y_m\) if \(m \neq j\) and the \(j\)th coordinate of \(x, y\) are \(x_j\) and \(y_j\), respectively. Then \(x \neq y\). Suppose for \(x_j, y_j \in X_j, x_j \neq y_j\) and \(A_j = (\mu_{A_j}, \nu_{A_j}) \in t_j\) such that \(A_j(x_j) \neq A_j(y_j)\). Let \(A_m = (1^-, 0^-)\), for \(m \neq j\), then \(A = \prod A_m \in \mathcal{E}\) and \(A(x) \neq A(y)\), where \(A = (\mu_A, \nu_A)\). Therefore, since \((X, t)\) is IF−R_1(i), then \(\exists B = (\mu_B, \nu_B), C = (\mu_C, \nu_C) \in \mathcal{E}\) such that \(\mu_B(y) = 1, \mu_C(y) = 1\) and \(B \cap C = 0. \text{ Now, } \mu_B(x) = 1 \Rightarrow \inf_{m \in J}\mu_{B_m}(x_m) = 1 \Rightarrow \mu_{B_m}(x_m) = 1\) and \(\mu_C(y) = 1 \Rightarrow \inf_{m \in J}\mu_{C_m}(y_m) = 1 \Rightarrow \mu_{C_m}(y_m) = 1\), for all \(m \in J\). Hence we have \(\mu_{B_j}(x_j) = 1\) and \(\mu_{C_j}(y_j) = 1\) and \(B_j \cap C_j = 0. \text{ Thus } (X_j, t_j) \text{ is IF}−\text{R}_1(i). \text{ Therefore } \{(X_m, t_m): m \in J\} \text{ is IF}−\text{R}_1(i)\).
For $n = ii, iii, iv$, we can prove that if suppose $\{(X_m, t_m) : m \in J\}$ be a finite family of intuitionistic fuzzy topological spaces and let $(X, t)$ be their product IFTS. Then each IFTS $(X_m, t_m)$ is IF$-R_1(n)$ if the product IFTS $(\prod X_m, \prod t_m)$ is IF$-R_1(n)$.

**Theorem: 2.12.** Let $\{(X_m, t_m) : m \in J\}$ is a finite family of intuitionistic fuzzy topological spaces. Let $(X, t)$ be their product. Then each IFTS $(X_m, t_m)$ is $\alpha - IF-R_1(n)$ if the product IFTS $(\prod X_m, \prod t_m)$ is $\alpha - IF-R_1(n)$. $n = i, ii, iii$.

**Proof:** The above theorem can be proved in the similar way.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**REFERENCES**


