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STABILITY OF CUBIC TYPE FUNCTIONAL EQUATION IN ORTHOGONAL SPACE

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Abstract: In this paper, we investigate the Hyers -Ulam -Rassias stability of cubic type functional equation $f(2x\pm y) = 2f(x\pm y) + 12f(x)$ for the mapping f from orthogonal linear space into Banach space.

Keywords: Hyers - Ulam - Rassias stability; Cubic functional equation; orthogonal space.

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1. INTRODUCTION AND PRELIMINARIES

The first problem on the stability of group homomorphism was given by S. M. Ulam [22] in 1940. He discussed the number of unsolved problems before the Mathematics club of the University of Wisconsin. One of the interesting problem related to homomorphism was as follows:

Let $(G_1, *)$ be a group and let $(G_2, \square, d)$ be a metric group with the metric 'd'. Given ε > 0, does there exists a δ > 0 such that if a mapping h: $G_1 \to G_2$ satisfies the following inequality $d(h(x*y), h(x)\square h(y)) < \delta$, for all x, y $\in G_1$, then there is a homomorphism H: $G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$, for all $x \in G_1$? If the answer is affirmative, we would say that equations of homomorphism H(x y) = H(x) H(y) is stable.

In 1941, D. H. Hyers [2] gave the first affirmative answer of the Ulam's problem for additive mapping f(x+y) = f(x)+f(y) on Banach space. A generalized version of Hyers [2] was given by Th. M. Rassias [24]. In 1978, he allows Cauchy difference to be unbounded .The generalizations given by Th. M. Rassias [24] is called the Hyers-Ulam-Rassias stability. In 1994,

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P. Gavruta [16] provided a further generalization of Th. M. Rassias [24] theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function $\phi(x, y)$ for the existence of unique linear mapping. The Hyers-Ulam-Rassias stability of various functional equations have been extensively introduced by a number of Mathematicians.

In 1975, the orthogonally additive functional equation $f(x+y) = f(x)+f(y)$, $x \perp y$, Where $\perp$ is the orthogonality symbol was investigated by S. Gudder and D. Strawther [23]. Later on, Ger and Sikorska [21] established the orthogonal stability of above additive functional equations in the sense of J. Ratz [8] for the mapping $f : X \to Y$, where X is orthogonal linear space and Y is a Banach space. This result was also generalized by M. S. Moslehian [13] in the framework of Banach modules. The orthogonal quadratic functional equations $f(x + y) = 2f(x) + 2f(y)$, where $x \perp y$ means the Hilbert space . Orthogonality was first investigated by F. Vajzovice [5] .Later on, the result of Vajzovice [5] was generalized by F. Drljevice [3], M. Fochi [11] and G. Szabo [7]. For further detailed study of stability of orthogonal functional equations one may also refer to [4, 9, 12, 14, 15, 17, 19]. The functional equations

$$D(f(x, y)) = f(2x \pm y) - 2f(x \pm y) - 12f(x) \qquad (1.0)$$

is called Cubic functional equation. In 2008, W. Townanlong and P. Nakmahalasiant [25] established the general solution and proved the Hyers -Ulam -Rassias stability of the above functional equation. In this paper, We investigate the orthogonal stability of Cubic funtional equations (1.0). There are several orthogonality concepts on a arbitrary real normed space given by many famous mathematicians such as G. Birkhoff [6], R. C. James [18], C. R. Diminni [1], G. Szabo [7], J. Ratz [8]. Further, Ashish et. al. [10, 20] also studied the orthogonality of cubic and quadratic functional equations. In 1985 J. Ratz [8] presented the following definitions of orthogonality:

Definition 1.1. Suppose X is a real vector space with dim≥2 and $\perp$ is a binary relation on X with the following properties:

(O1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$.

(O2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent.

(O3) homogeneity if , x, y ϵX , $x \perp y$, then $\alpha x \perp \beta y$ for $\alpha, \beta \in X$

(O4) the Thalesian property: if P is a 2- dimensional subspace of X and $\lambda \in R$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$. The pair (X , $\perp$) is called an orthogonality space.

By an orthogonality space we mean an orthogonality space equipped with a norm. The relation $\perp$ is called symmetric if $x \perp y$ and $y \perp x$ for all x, $y \in X$.

Definition 1.2. Let X be an orthogonality normed space and Y be a real Banach space. A mapping $f: X \to Y$ is said to orthogonally quadratic if it satisfies the so-called orthogonally quadratic functional equation (1.0) for all x, $y \in X$ with $x \perp y$.

2. MAIN RESULTS

Throughout, this section, let $(X, \perp)$ denotes an orthogonality normal space with norm $\|\cdot\|_x$ and $(Y, \|\cdot\|_Y)$ is a Banach space.

Theorem 2.1 Let ε and p (p < 3) be non-negative real numbers. Suppose that $f : X \to Y$ is a cubic mapping satisfying the inequality

$$\| D(f(x,y)) \|_Y \leq \varepsilon\left(\| x \|_X^p + \| y \|_X^p \right) \qquad 2.1$$

for all x, $y \in X$ with $x \perp y$. then there exist a unique orthogonally cubic mapping $J : X \to Y$ such that

$$\| f(x) - J(x) \|_Y \leq \frac{\varepsilon}{2\left(2^3 - 2^p\right)} \| x \|_X^p \qquad 2.2$$

for all $x \in X$.

Proof. To prove this theorem, we have to solve the following steps :

a) $\left\{ f\left(2^n x\right) / 2^{3n} \right\}$ is a Cauchy sequence for every fixed $x \in X$ such that $x \perp 0$

b) There exist a cubic mapping J: $X \to Y$ defined by $J(x) = \lim_{n \to \infty} \left\{ f\left(2^n x\right) / 2^{3n} \right\}$

c) The mapping J: $X \to Y$ satisfies $\| f(x) - J(x) \|_Y \leq \frac{\varepsilon}{2\left(2^3 - 2^p\right)} \|x\|_X^p$ for $p < 3$

d) The mapping $J : X \to Y$ is unique.

To prove (a) Let us take y = 0 in (2.1), we have

$$\| 2f(2x) - 4f(x) - 12f(x) \|_Y \leq \varepsilon\left(\| x \|_X^p + \| y \|_X^p \right)$$

$$\| 2f(2x) - 16f(x) \|_Y \leq \varepsilon\left(\|x\|_X^p \right)$$

$$\| \frac{f(2x)}{2^3} - f(x) \|_Y \leq \frac{\varepsilon}{2.2^3} \left(\| x \|_X^p \right) \qquad 2.3$$

For all x, y $\in$ X with x $\perp$ 0. Now replacing x with 2*x* and diving by 2^3 in (2.3) and then adding the resulting equation with (2.3) we get

$$\| \frac{f(2^2 x)}{2^6} - f(x) \| \leq \frac{\varepsilon}{2^4} \left(\| x \|_X^p \right) + \frac{\varepsilon}{2^7} \left(\| 2x \|_X^p \right)$$

$$\| \frac{f(2^2 x)}{2^6} - f(x) \| \leq \frac{\varepsilon}{2^4} \left(1 + \frac{\varepsilon}{2^3} .2^p \right) \| x \|_X^p$$

$$\| \frac{f(2^2 x)}{2^6} - f(x) \| \leq \frac{\varepsilon}{2.2^3} \left(1 + \frac{\varepsilon}{2^3} .2^p \right) \| x \|_X^p \qquad 2.4$$

By using induction on n, we get

$$\| \frac{f(2^n x)}{2^{3n}} - f(x) \| \leq \frac{\varepsilon}{2.2^3} \sum_{k=0}^{n-1} \frac{2^{pk}}{2^{3k}} \| x \|_X^p \qquad 2.5$$

For all x, y $\in$ X with x $\perp$ 0 and n $\geq$ 1. To show that $\left\{ f(2^n x) / 2^{3n} \right\}$ is a Cauchy sequence, replacing x with $2^m x$ and dividing by 2^{3m} in equation (2.5), we get for n, m > 0

$$\| \frac{f(2^{n+m} x)}{2^{3n+3m}} - \frac{f(2^m x)}{2^{3m}} \| \leq \frac{\varepsilon}{2.2^3 .2^{3m}} \sum_{k=0}^{n-1} \frac{2^{pk}}{2^{3k}} \| 2^m x \|_X^p$$

$$\| \frac{f(2^{n+m} x)}{2^{3n+3m}} - \frac{f(2^m x)}{2^{3m}} \| \leq \frac{\varepsilon}{2.2^3} \sum_{k=0}^{n-1} \frac{2^{p(k+m)}}{2^{3(k+m)}} \| x \|_X^p$$

$$\frac{1}{2^{3m}} \| \frac{f(2^{n+m} x)}{2^{3m}} - f(2^m x) \| \leq \frac{\varepsilon}{2.2^3 2^{3m-pm}} \sum_{k=0}^{n-1} \frac{2^{pk}}{2^{3k}} \| x \|_X^p \qquad 2.6$$

As we obtain for p < 3 and m $\to \infty$ than equation (2.6) tends to zero for all x $\in$ X. Thus the sequence $\left\{ f(2^n x) / 2^{3n} \right\}$ is convergent in Y. Since Y is complete normed liner space where the sequence $\left\{ f(2^n x) / 2^{3n} \right\}$ is a cauchy sequence for every x $\in$ X then there exists a othogonality cubic mapping J : X $\to$ Y such that

$$J(x) = \lim_{n \to \infty} \left\{ f(2^n x) / 2^{3n} \right\} \text{ for all } x \in X \qquad 2.7$$

(b) Now, we claim that the mapping J : X $\rightarrow$ Y is cubic that is it satisfies the equation (1.0). Substituting $2^n x$ and 2^ny at the place of x and y in (2.1) respectively and dividing by 2^{3n}. We have

$$\left\| \frac{D\left(f\left(2^n x, 2^n y\right)\right)}{2^{3n}} \right\| \leq \frac{\varepsilon}{2^{3n}}\left(\| 2^n x \|_X^p + \| 2^n y \|_X^p\right) \qquad 2.8$$

Taking n $\rightarrow \infty$ in (2.8), we find

$$\| J(2x \pm y) - 2J(x \pm y) - 12J(x) \|_y \leq 0$$

$$\| J(2x \pm y) \| \leq 2J(x \pm y) + 12J(x)$$

for all x, y $\in$ X with x $\perp$ y. Which proves that the mapping J : X$\rightarrow$ Y is orthogonally cubic mapping.

(c) By taking n $\rightarrow \infty$ in the equation (2.5) we obtain the following

$$\| f(x) - J(x) \|_Y \leq \frac{\varepsilon}{2\left(2^2 - 2^p\right)} \| x \|_X^p \text{ for all } x \in X.$$

(d) Now, we shall prove the uniqeness of orthogonally cubic mapping J : X $\rightarrow$ Y. We consider another orthogonally cubic mapping J': X $\rightarrow$ y satisfying the equation (1.0).

Hence, $\| J(x) - J'(x) \|_Y \leq \frac{1}{2^{3n}}\left\{\left\| J(2^n x) - f(2^n x) \right\|_Y + \left\| f(2^n x) - J'(2^n x) \right\|_Y\right\}$

$$\| J(x) - J'(x) \|_Y \leq \frac{\varepsilon}{2\left(2^3 - 2^p\right) 2^{n(3-p)}} \| x \|_X^p \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } x \in X.$$

Which proves that J(x) = J' (x) that means the orthogonally cubic mapping J is unique. This complete the proof of theorem.

Theorem 2.2 Let ε and p (p < 3) be non-negative real numbers. Suppose that f : X $\rightarrow$ Y is a cubic mapping satisfying the inequality (2.2) for all x, y $\in$ x with x $\perp$y. Then there exist a unique orthogonally cubic mapping J : X $\rightarrow$ Y such that

$$\| f(x) - J(x) \|_Y \leq \frac{\varepsilon}{2\left(2^p - 2^3\right)} \| x \|_X^p \text{ for all } x \in X. \qquad 2.9$$

Proof. Replacing x with x/2 and multiplying 2^3in (2.3), we have

$$\left\| f(x) - 2^3 f\left(\frac{x}{2}\right) \right\|_Y \leq \frac{\varepsilon}{2} \left\| \frac{x}{2} \right\|_X^p$$

$$\| f(x) - 2^3 f\left(\frac{x}{2}\right) \|_Y \leq \frac{\varepsilon}{2.2^p} \| \frac{x}{2} \|_Y^p \quad 2.10$$

for all $x \in X$ with $x \perp 0$. Again replacing x with x/2 and multiplying by 2^3 in (2.10) and then adding the resulting equation with (2.10), we have

$$\| f(x) - 2^6 f\left(\frac{x}{2^2}\right) \|_Y \leq \frac{\varepsilon}{2.2^p}\left(1 + \frac{2^3}{2^p}\right) \| x \|_X^p \quad 2.11$$

By using induction on n, we find

$$\| f(x) - 2^{3n} f\left(\frac{x}{2^n}\right) \|_Y \leq \frac{\varepsilon}{2.2^p} \sum_{k=0}^{n-1} \frac{2^{3k}}{2^{pk}} \| x \|_X^p \quad 2.12$$

for all $x \in X$ with $x \perp 0$ and $n \geq 1$. Now to show that convergence of $\left\{ f\left(2^n x\right) / 2^{3n} \right\}$ replacing x with $x/2^m$ and multiplying by 2^{3m} in (2.12) we obtain for n, m > 0.

$$\| 2^{3m} f\left(\frac{x}{2^m}\right) - 2^{3m+3n} f\left(\frac{x}{2^{n+m}}\right) \|_Y \leq \frac{\varepsilon}{2.2^{m(p-3)}} \sum_{k=0}^{\infty} \frac{2^{3k}}{2^{p(k+1)}} \| x \|_X^p \quad 2.13$$

For $p > 3$ the right hand side of (2.13) tends to zero as $m \rightarrow 0$ for all $x \epsilon X$. Thus the sequence $\left\{ 2^{3n} f\left(x/2^n\right) \right\}$ is convergent in Y. Since Y is complete normed space, hence the sequence $\left\{ 2^{3n} f\left(x/2^n\right) \right\}$ is a Cauchy sequence for every $x \in X$. Then there exist a orthogonal cubic mapping $J : X \rightarrow Y$ such that

$$J(x) = \lim_{x \to \infty} \left\{ 2^{3n} f\left(x/2^n\right) \right\} \text{ for all } x \in X \quad 2.14$$

By making $n \rightarrow \infty$ in the equation (2.13) and using (2.14), we obtain the required result.

Further, to prove the orthogonal cubic mapping is unique the proof is similar to that of Theorem 2.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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