THE ALEKSANDROV PROBLEM IN QUASI CONVEX 2-NORMED LINEAR SPACES

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Abstract. In this paper, we prove that the Aleksandrov problem holds without the condition ”2-Lipschitz mapping” in quasi convex 2-normed linear spaces. Moreover, we show that the Mazur-Ulam theorem holds in quasi convex 2-normed linear spaces.

Keywords: Aleksandrov problem; Mazur-Ulam theorem; Quasi Convex 2-normed spaces.

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1. Introduction

Let $E$ and $F$ be metric spaces. A mapping $f : E \to F$ is called an isometry if $f$ satisfies

$$d_F(f(x), f(y)) = d_E(x, y)$$

for all $x, y \in E$, where $d_E(\cdot, \cdot)$ and $d_F(\cdot, \cdot)$ denote the metric in the space $E$ and $F$, respectively. For some fixed number $r > 0$, suppose that $f$ preserves distance $r$; i.e., for all $x, y \in E$ with $d_E(x, y) = r$, we have $d_F(f(x), f(y)) = r$. Then $r$ is called a conservative distance for the mapping $f$. The classical Mazur-Ulam theorem states that every surjective isometry between normed spaces is

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a linear mapping up to translation. In 1970, Aleksandrov [1] posed the following question: "Whether or not a mapping with distance one preserving property is an isometry? " It is called the Aleksandrov problem. The Aleksandrov problem has been investigated in several papers[4]-[13].

Recently, Chu et al. [4] begin to consider the Aleksandrov problem in linear 2-normed space. They introduced the concept of 2-isometry, which is suitable to represent the notion of area preserving mappings in appropriate spaces as 2-normed spaces. Chu [2] proved the Mazur-Ulam theorem holds in 2-normed spaces via this 2-isometry. However, this ideal cannot be used to prove the Mazur-Ulam theorem in quasi convex 2-normed linear space, since the triangle inequality fails in quasi convex 2-normed linear space. Chu et al.[4] proved also that the Rassias and Šemrl theorem holds under some conditions in linear 2-normed spaces as follows:

**Theorem 1.1.**[4] Let f be a 2-Lipschitz mapping with the 2-Lipschitz constant \( K \leq 1 \). Assume that if \( x, y \) and \( z \) are collinear, then \( f(x), f(y) \) and \( f(z) \) are collinear, and that \( f \) satisfies (DOPP). Then \( f \) is a 2-isometry.

In this paper, we consider generalized 2-isometries, which is suitable for representing the notion of distance preserving mappings in quasi convex 2-normed linear spaces. We show that every generalized 2-isometries is affine. Also we prove that a mapping preserving the one distance property and collinear between two quasi convex 2-normed linear spaces is an affine generalized 2-isometry.

### 2. Preliminaries

In the remainder of this introduction, we will recall some definitions and give some Lemmas about them in quasi convex 2-normed linear space.

**Definition 2.1.**[12] Let \( E \) be a real linear space with \( \dim E > 1 \) and \( \|\cdot,\cdot\| \) be a function from \( E \times E \) into \( \mathbb{R} \). Then \((E,\|\cdot,\cdot\|)\) is called a quasi convex 2-normed linear space if

- (a) \( \|x,y\| = 0 \iff x \) and \( y \) are linearly dependent,
- (b) \( \|x,y\| = \|y,x\| \),
- (c) \( \|\alpha x,y\| = |\alpha| \|x,y\| \),
(d) \( \|tx + (1-t)y, z\| \leq \max\{\|x, z\|, \|y, z\|\} \),
for any \( \alpha \in \mathbb{R}, t \in [0,1] \) and \( x, y, z \in E \). The function \( \|\cdot, \cdot\| \) is called the quasi convex 2-norm on \( E \).

From now on, let \( E \) and \( F \) be quasi convex 2-normed linear space and the mapping \( f : E \to F \).

**Definition 2.2.**[13] A mapping \( f : E \to F \) is said to be a generalized 2-isometry if it satisfies

\[
\|f(x) - f(y), f(p) - f(q)\| = \|x, y, p, q\|.
\]

for every \( x, y, p, q \in E \). In particular, if \( y = q \), the mapping \( f \) is said to be a 2-isometry.

**Definition 2.3.**[13] A mapping \( f : E \to F \) satisfies the distance one preserving property (briefly DOPP), if \( \|x, y, p, q\| = 1 \) for all \( x, z, p, q \in Y \), it follows that

\[
\|f(x) - f(z), f(p) - f(q)\| = 1.
\]

**Definition 2.4.**[4] We call \( f \) a 2-Lipschitz mapping if there is a \( K \geq 0 \) such that

\[
\|f(x) - f(y), f(p) - f(q)\| \leq K\|x, y, p, q\|
\]

for all \( x, y, p, q \in E \). In this case, the constant \( k \) is called the 2-Lipschitz constant.

**Definition 2.5.**[7] A mapping \( f : E \to F \) on two real normed space \( E \) and \( F \) is called an affine mapping if for all \( x, y \in E \) and \( \lambda \in [0,1] \) satisfies

\[
f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y).
\]

**Definition 2.6.**[4] The point \( x, y, z \) of \( E \) are said to be collinear if \( y - z = t(x - z) \) for some real number \( t \). We say that a mapping \( f : E \to F \) preserves collinearity, if \( x, y, z \in E \) are collinear, then \( f(x), f(y), f(z) \) are collinear.

**Remark 2.7.** Each 2-Lipschitz mapping preserves collinear.

**Lemma 2.1.** Let \( E \) be a quasi convex 2-normed linear space with \( \text{dim} E \geq 1 \). For \( x, y, z \in E \), if \( x \) and \( y \) are linearly dependent, then \( \|x + y, z\| \leq \|x, z\| + \|y, z\| \).
Lemma 2.2. Let $E$ be a quasi convex 2-normed linear space with $\dim E > 1$, for $x_i, z \in E, t_i > 0, \sum_{i=1}^{n} t_i = 1(i = 1, 2, \cdots, n)$, we have

$$\| \sum_{i=1}^{n} t_i x_i, z \| \leq \max\{ \| x_i, z \| : i = 1, 2, \cdots, n \}.$$ 

Proof. If $n = 2$, then $\| t_1 x_1 + t_2 x_2, z \| \leq \max\{ \| x_1, z \|, \| x_2, z \| \}$. Assume that $\| \sum_{i=1}^{k-1} t_i x_i, z \| \leq \max\{ \| x_1, z \|, \| x_2, z \|, \cdots, \| x_{k-1}, z \| \}$. Let $n = k$, we can obtain

$$\| \sum_{i=1}^{k} t_i x_i, z \| = \| \sum_{i=1}^{k-1} t_i x_i + t_k x_k, z \|$$

$$= \| \sum_{i=1}^{k-1} t_i \left( \frac{\sum_{i=1}^{k-1} t_i x_i}{\sum_{i=1}^{k-1} t_i} \right) + t_k x_k, z \|$$

$$\leq \max\{ \| \sum_{i=1}^{k-1} t_i x_i, z \|, \| x_k, z \| \}$$

$$\leq \max\{ \| x_1, z \|, \| x_2, z \|, \cdots, \| x_{k-1}, z \|, \| x_k, z \| \}.$$ 

Therefore

$$\| \sum_{i=1}^{n} t_i x_i, z \| \leq \max\{ \| x_1, z \|, \| x_2, z \|, \cdots, \| x_{n-1}, z \|, \| x_n, z \| \}$$

i.e.

$$\| \sum_{i=1}^{n} t_i x_i, z \| \leq \max\{ \| x_i, z \| : i = 1, 2, \cdots, n \}.$$ 

The next result follows easily from [6, Lemma 8].

Lemma 2.3. Let $E$ be a quasi convex 2-normed linear space with $\dim E > 2$. Suppose $0 < \| x - y, p - q \| \leq 2r$, for any $r > 0$, and $x, y, p, q \in E$, then there exists $z \in E$ such that $\| x - z, p - q \| = \| z - y, p - q \| = r$.

3. Main results
In this section, let $E$ and $F$ be quasi convex 2-normed linear spaces with dimension greater than 1.

**Lemma 3.1.** Let $E$ and $F$ be two quasi convex 2-normed linear spaces. If $f : E \to F$ satisfies (DOPP) and preserves collinearity, then $f$ is injective and for any $x, y \in E$, we have

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}.$$ 

**Proof.** Let $z = \frac{x + y}{2}$ for distinct $x, y \in E$. Then $z - x = y - z = \frac{y - x}{2} \neq 0$. We can choose $p, q \in E$ such that $\|x - y, p - q\| = 1$. Since the mapping $f$ satisfies (DOPP), we have

$$\|f(x) - f(y), f(p) - f(q)\| = 1.$$ 

This implies $f(x) \neq f(y)$, and thus $f$ is injective. On the other hand,

$$\|z - y, 2p - 2q\| = \|z - x, 2p - 2q\| = 1.$$ 

Then

$$\|f(z) - f(y), f(2p) - f(2q)\| = \|f(z) - f(x), f(2p) - f(2q)\| = 1. \quad (1)$$ 

Since $f$ preserves collinearity, there exists a real number $t$ such that

$$f(z) - f(y) = t(f(z) - f(x)).$$

Because $f$ is injective, and it follows from the equation (1) we conclude that $t = -1$. Thus $f(z) - f(y) = f(x) - f(z)$ and

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}.$$ 

**Theorem 3.1.** Let $E$ and $F$ be two quasi convex 2-normed linear spaces. If $f : E \to F$ is a generalized 2-isometry, then $f$ is affine.

**Proof.** Assume that $x, y$ and $z$ are colinear, then $f$ preserves collinearity by the condition that $\|x - z, y - z\| = 0$ implies $\|f(x) - f(z), f(y) - f(z)\| = 0$. Let $g(x) = f(x) - f(0)$. It suffices to prove that the mapping $g$ is linear. Since $g$ satisfies (DOPP) and $g(0) = 0$. From Lemma 3.1,
the mapping $g$ is $\mathbb{Q}$-linear. Let $\xi \in \mathbb{R}^+$ with $\xi \neq 1$ and $x \in E$. Since $0, x, \xi x$ are collinear, $g$ preserves collinearity and also $g(0) = 0$, so there exists a real number $\eta$ such that

$$g(\xi x) = \eta g(x).$$

For any $x \in E$ with $x \neq 0$, there exists $y \in E$ such that $\|x, y\| = 1$. Hence we obtain

$$\xi = \|\xi x, y\| = \|g(\xi x), g(y)\| = \|\eta g(x), g(y)\| = |\eta| \|g(x), g(y)\| = |\eta|.$$

Thus $\eta = \pm \xi$. While $\eta = -\xi$, that is to say $g(\xi x) = -\xi g(x)$, it deduces that

$$|1 - \xi| = \|x - \xi x, y\| = \|g(x) - g(\xi x), g(y)\| = \|g(x) + \xi g(x), g(y)\| = (1 + \xi) \|g(x), g(y)\| = 1 + \xi.$$

So $\xi = 0$, while it conflict with $\xi \in \mathbb{R}^+$. Hence we get $\xi = \eta$, that is to say $g(\xi x) = \xi g(x)$. This completes the proof.

**Lemma 3.2.** Let $E$ and $F$ be two quasi convex 2-normed linear spaces, if $f : E \to F$ satisfies (DOPP) and preserves collinearity, then $f$ preserves distance $\frac{m}{\xi}$, for each $m, k \in \mathbb{N}$.

**Proof.** We first prove $f$ preserves distance $1/k$. Let $\|x - y, p - q\| = \frac{1}{k}$ with $x, y, p, q \in E$, we define

$$\omega_i = x + i(y - x) \quad \forall i = 0, 1, \cdots, k.$$

Then

$$\omega_i = \frac{\omega_{i-1} + \omega_{i+1}}{2}, \quad \forall i = 1, \cdots, k - 1.$$

According to Lemma 3.1, we have

$$f(\omega_i) = \frac{f(\omega_{i-1}) + f(\omega_{i+1})}{2}, \quad \forall i = 1, \cdots, k - 1.$$

That is

$$f(\omega_{i+1}) - f(\omega_i) = f(\omega_i) - f(\omega_{i-1}), \quad \forall i = 1, \cdots, k - 1.$$
Hence
\[ f(\omega_k) - f(x) = f(\omega_k) - f(\omega_{k-1}) + f(\omega_{k-1}) - f(\omega_{k-2}) + \cdots + f(\omega_1) - f(\omega_0) \]
\[ = k(f(\omega_1) - f(\omega_0)) = k(f(y) - f(x)). \]

Since \[ \|\omega_k - x, p - q\| = 1, \]
\[ k\|f(y) - f(x), f(p) - f(q)\| = \|f(\omega_k) - f(x), f(p) - f(q)\| = 1. \]

Therefore \[ \|f(y) - f(x), f(p) - f(q)\| = \frac{1}{k}. \]

Next, we shall show that \( f \) preserves distance \( \frac{m}{k} \) for integers \( m, k \). Let \( \|x - y, p - q\| = \frac{m}{k} \) with \( x, y, p, q \in E \). We define
\[ z_i := x + \frac{i}{m}(y - x), \quad \forall i = 0, 1, \cdots, k. \]

Then
\[ z_i = \frac{z_{i-1} + z_{i+1}}{2}, \quad \forall i = 1, \cdots, k - 1. \]

By the same method as above,
\[ f(y) - f(x) = f(z_m) - f(z_0) = m(f(z_1) - f(z_0)). \]

Note that \( \|z_1 - z_0, p - q\| = \frac{1}{k} \) and \( f \) preserves distance \( \frac{1}{k} \),
\[ \|f(y) - f(x), f(p) - f(q)\| = \|m(f(z_1) - f(z_0)), f(p) - f(q)\| = \frac{m}{k}. \]

This completes the proof.

**Theorem 3.2.** Let \( E \) and \( F \) be two quasi convex 2-normed linear spaces with \( \dim E > 2 \). If \( f : E \to F \) satisfies (DOPP) and preserves collinearity, then \( f \) is an affine generalized 2-isometry.

**Proof.** We first prove that \( f \) is a 2-Lipschitz mapping with the constant \( K = 1 \). That is, for any \( x, y, p, q \in E \),
\[ \|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|. \]
If $\|x - y, p - q\| = 0$ for some $x, y, p, q \in E$. Then we have $x - y = t(p - q)$ for some real number $t$. Let $g(x) = f(x) - f(0)$. It follows from Lemma 3.1 that $g$ is additive and preserves collinearity. Thus

$$\|f(x) - f(y), f(p) - f(q)\| = \|g(x) - g(y), g(p) - g(q)\| = 0.$$  

On the other hand, let $x, y, p, q \in E$ and $k, m \in N$, such that

$$\frac{m - 1}{k} < \|x - y, p - q\| \leq \frac{m}{k}$$

Set

$$\omega_i = x + \frac{i}{k} \frac{y - x}{\|x - y, p - q\|}, \quad i = 0, 1, \cdots, m - 2$$

and also define $\omega_m = y$. Then

$$\|\omega_i - \omega_{i+1}, p - q\| = \frac{1}{k}, \quad i = 1, \cdots, m - 2.$$  

Moreover,

$$0 < \|\omega_m - \omega_{m-2}, p - q\| = \left\| \frac{m - 2}{k} \frac{y - x}{\|x - y, p - q\|} + (x - y), p - q \right\|$$

$$= \|x - y, p - q\| - \frac{m - 2}{k}$$

$$\leq \frac{m}{k} - \frac{m - 2}{k} = \frac{2}{k}.$$  

From Lemma 2.3, we can choose $\omega_{m-1} \in E$, such that

$$\|\omega_{m-1} - \omega_{m-2}, p - q\| = \|\omega_{m-1} - \omega_m, p - q\| = \frac{1}{k}$$

By Lemma 3.2, $f$ preserves $\frac{1}{k}$ distance. Therefore, for $i = 0, 1, \cdots, m$, we have

$$\|f(\omega_i) - f(\omega_{i-1}), f(p) - f(q)\| = \frac{1}{k}.$$
From Lemma 2.2,
\[ \| f(x) - f(y), f(p) - f(q) \| = \| f(\omega_0) - f(\omega_m), f(p) - f(q) \| \]
\[ = \| m \sum_{i=0}^{m-1} (f(\omega_i) - f(\omega_{i+1})), f(p) - f(q) \| \]
\[ = m \| m \sum_{i=0}^{m-1} \frac{1}{m} (f(\omega_i) - f(\omega_{i+1})), f(p) - f(q) \| \]
\[ \leq m \max\{ \| f(\omega_i) - f(\omega_{i+1}), f(p) - f(q) \| : i = 0, 1, \ldots, m - 1 \} \]
\[ \leq \frac{m}{k}. \]

Hence \[ \| f(x) - f(y), f(p) - f(q) \| \leq \| x - y, p - q \|. \]

Next, we will show that \( f \) is a generalized 2-isometry. Otherwise, there exists \( x, y, p, q \in E \) and \( m \in \mathbb{N} \) such that \( 0 < \| x - y, p - q \| < m \) and
\[ \| f(x) - f(y), f(p) - f(q) \| < \| x - y, p - q \|. \]

Set \( z := x + \frac{m(y - x)}{\| x - y, p - q \|} \). Then we obtain that
\[ \| z - x, p - q \| = m \]
\[ \| z - y, p - q \| = m - \| x - y, p - q \|. \]

Since \( f \) preserves collinearity, there exists a real number \( t \) such that
\[ f(z) - f(x) = t(f(y) - f(x)). \]

Then \( f(z) - f(y) = (t - 1)(f(y) - f(x)) \). By Lemma 3.2, \( f \) preserves distance \( m \). So we have
\[ m = \| f(z) - f(x), f(p) - f(q) \| \]
\[ = |t| \| f(x) - f(y), f(p) - f(q) \| \]
\[ \leq |t - 1| \| f(x) - f(y), f(p) - f(q) \| + \| f(x) - f(y), f(p) - f(q) \| \]
\[ = \| f(z) - f(y), f(p) - f(q) \| + \| f(x) - f(y), f(p) - f(q) \| \]
\[ < m - \| x - y, p - q \| + \| x - y, p - q \| = m, \]

which is a contraction. By Theorem 3.1, the proof of the theorem is finished.
Conflict of Interests

The authors declare that there is no conflict of interests.

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