Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 1126-1143
ISSN: 1927-5307

# HIGH ACCURACY CUBIC SPLINE APPROXIMATION ON A GEOMETRIC MESH FOR THE SOLUTION OF 1D NON-LINEAR WAVE EQUATIONS

SURUCHI SINGH[1], SWARN SINGH[2] AND R. K. MOHANTY[3,*]

[1]Department of Mathematics, Aditi Mahavidyalaya, University of Delhi, Delhi 110039, INDIA

[2]Department of Mathematics, Sri Venkateswara College, University of Delhi, Delhi 110021, INDIA

[3]Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110007, INDIA

**Abstract.** In this paper, we propose a new high order three-level implicit compact discretization based on cubic spline approximation on a non-uniform mesh in space-direction for the solution of 1D non-linear hyperbolic partial differential equation of the form $u_{tt} = u_{tt} + G(x, t, u, u_x, u_t)$ subject to appropriate initial and Dirichlet boundary conditions. We use only three grid points at each time level and describe the derivation procedure in details. We also show how our method is able to handle the wave equation in polar coordinates. Numerical results are provided to justify the usefulness of the proposed method.

**Keywords**: Variable mesh; Non-linear hyperbolic equation; High order method; Cubic spline approximation; Wave equation in polar coordinates; Vander Pol equation; Dissipative equation; Maximum absolute errors.

**Mathematics Subject Classification (2010):** 65M 06; 65M12

## 1. Introduction

*Corresponding author

E-mail addresses: ssuruchi2005@yahoo.co.in(S. Singh), sswarn2005@yahoo.co.in(S. Singh), rmohanty@maths.du.ac.in (R. K. Mohanty)

Received March 14, 2012

We consider the one-space dimensional non-linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + G\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \quad 0 < x < 1, \quad t > 0 \tag{1}$$

with the following initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad 0 \le x \le 1 \tag{2}$$

and the boundary conditions

$$u(0,t) = p_0(t), \quad u(1,t) = p_1(t), \quad t \ge 0 \tag{3}$$

We assume that the conditions (2) and (3) are given with sufficient smoothness to maintain the order of accuracy in the numerical method under consideration.

In past several numerical schemes have been developed for the solution of the linear and non-linear hyperbolic partial differential equation (1) (see [20-28]). First, Fyfe [5] and Bickley [32] have discussed the second order accurate cubic spline method for the solution of linear two point boundary value problems. Jain and Aziz [12] have derived fourth order cubic spline method for solving the non-linear two point boundary value problems with significant first derivative terms. Recently, Khan and Aziz [1], Kadalbajoo et al [15-16], and Kumar and Srivastava [17] have studied on the use of cubic spline technique for solving singular two point boundary value problems.

During last three decades, there has been much effort to develop stable numerical methods based on cubic spline approximations for the solution of time-dependent partial differential equations. In 1973, Papamichael and Whiteman [18] have used a cubic spline technique of lower order accuracy to solve one-dimensional heat conduction equation. Then using the same technique Raggett and Wilson [7] have solved one-dimensional wave equation. Later, Fleck Jr [9] has proposed a cubic spline method for solving the wave equation of non-linear optics. In recent years, Ding and Zhang [8] and Rashidinia et al [10] have discussed spline methods for the solution of linear hyperbolic equations with first derivative terms. Recently, Mohanty and Jain [19] have developed high accuracy cubic spline method for the solution of one-space dimensional quasi-linear parabolic

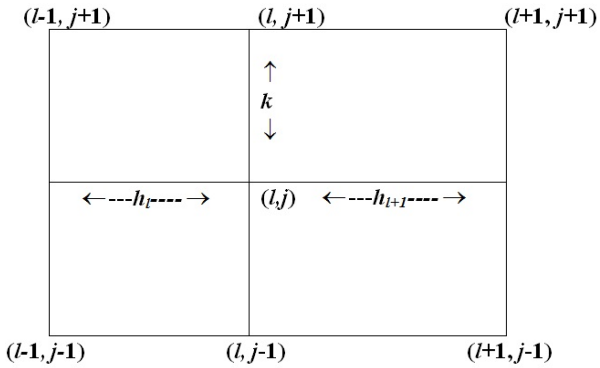


FIGURE 1. Schematic representation of three-level implicit scheme

equations. Twizell [6] has derived a new explicit difference method for the wave equation with extended stability range. Because of the stability restriction and instability due to non-uniform mesh, not many numerical methods for the solution of second order hyperbolic equations have been developed so far. Third order accurate variable mesh method for the solution of nonlinear two point boundary value problems have been discussed by several authors(see [4,14,28,29,30]). To the authors knowledge no high order methods on a variable mesh for the solution of 1D nonlinear hyperbolic equations have been discussed in the literature so far. In this paper, using nine-grid points (see Fig.1), we discuss a new three-level implicit cubic spline finite difference method of order two in time and three in space on a variable mesh for the solution of non-linear hyperbolic equation (1). In this method we require only three evaluation of function G. In next section, we discuss the cubic spline finite difference method. In section 3, we discuss the application of the proposed method to one dimensional wave equation in polar coordinates. In this section, we modify our technique in such a way that the solution retains its order and accuracy everywhere in the solution region. In section 4, we examine our method over a set of linear and nonlinear second order hyperbolic equations whose exact solutions are known and compare the results with the results of other known methods. Concluding remarks are given in section 5.

## 2. The numerical method based on cubic spline finite difference approximation

Let $k > 0$ be the mesh spacing in the time direction so that $t_j = jk$, $0 < j < J$, $J$ being a positive integers. Further, We discretize the unit interval [0,1] such that $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. Let $h_l = x_l - x_{l-1}$, $l = 1(1)N$ and $\sigma_l = (h_{l+1}/h_l) > 0$ be the mesh ratio parameter in the space direction. We replace the region $\Omega$ by a set of grid points $(x_l, t_j)$ denoted by $(l, j)$. The values of the exact solution $u(x,t)$ at the grid points $(l, j)$, are denoted by $U_l^j$. Let $u_l^j$ be the approximate solution at the same grid point.

For the derivation of the cubic spline finite difference method for the solution of differential equation (1), we follow the ideas given by Jain and Aziz [12]. We use the cubic spline approximations in $x$-direction and second order finite difference approximation in $t$-direction.

At the grid point $(x_l, t_j)$, we may write the differential equation (1) as

$$U_{ttl}^j - U_{xxl}^j = G(x_l, t_j, U_l^j, U_{xl}^j, U_{tl}^j) \equiv G_l^j, \quad \text{say}$$

We denote:

$$P_l = \sigma_l^2 + \sigma_l - 1, \qquad Q_l = (1+\sigma_l)(1+3\sigma_l+\sigma_l^2),$$
$$R_l = \sigma_l(1+\sigma_l-\sigma_l^2), \qquad S_l = \sigma_l(1+\sigma_l)$$

At the grid point $(l, j)$, we consider the following approximations:

(4a) $$\overline{U}_{tl}^j = (U_l^{j+1} - U_l^{j-1})/(2k) = U_{tl}^j + O(k^2)$$

(4b) $$\overline{U}_{tl+1}^j = (U_{l+1}^{j+1} - U_{l+1}^{j-1})/(2k) = U_{tl+1}^j + O(k^2 + k^2 h_l \sigma_l)$$

(4c) $$\overline{U}_{tl-1}^j = (U_{l-1}^{j+1} - U_{l-1}^{j-1})/(2k) = U_{tl-1}^j + O(k^2 - k^2 h_l)$$

(4d) $$\overline{U}_{ttl}^j = (U_l^{j+1} - 2U_l^j + U_l^{j-1})/(k^2) = U_{ttl}^j + O(k^2)$$

(4e) $$\overline{U}_{ttl+1}^j = (U_{l+1}^{j+1} - 2U_l^j + U_{l+1}^{j-1})/(k^2) = U_{ttl+1}^j + O(k^2 + k^2 h_l \sigma_l)$$

(4f) $$\overline{U}_{ttl-1}^j = (U_{l-1}^{j+1} - 2U_l^j + U_{l-1}^{j-1})/(k^2) = U_{ttl-1}^j + O(k^2 - k^2 h_l)$$

$$\overline{U}^j_{xl} = (U^j_{l+1} - (1-\sigma_l^2)U^j_l - \sigma_l^2 U^j_{l-1})/(h_1 S_l) = U^j_{xl} + \frac{h_l^2\sigma_l}{6}U_{30} + O(h_l^3\sigma_l^2 - h_l^3\sigma_l) \tag{5a}$$

$$\begin{aligned}\overline{U}^j_{xl+1} &= \Big((1+2\sigma_l)U^j_{l+1} - (1+\sigma_l)^2 U^j_l + \sigma_l^2 U^j_{l-1}\Big)/(h_1 S_l)\\ &= U^j_{xl+1} - \frac{h_l^2\sigma_l}{6}(1+\sigma_l)U_{30} + O(\sigma_l h_l^3 + \sigma_l^2 h_l^3 + \sigma_l^3 h_l^3)\end{aligned} \tag{5b}$$

$$\begin{aligned}\overline{U}^j_{xl-1} &= \Big(-U^j_{l+1} + (1+\sigma_l)^2 U^j_l - \sigma_l(2+\sigma_l)U^j_{l-1}\Big)/(h_1 S_l)\\ &= U^j_{xl-1} - \frac{h_l^2(1+\sigma_l)}{6}U_{30} + O(\sigma_l h_l^3 + \sigma_l^2 h_l^3 - \sigma_l^3 h_l^3)\end{aligned} \tag{5c}$$

$$\overline{U}^j_{xxl} = \frac{2[U^j_{l+1} - (1+\sigma_l)U^j_l + \sigma_l U^j_{l-1}]}{\sigma_l(1+\sigma_l)(h_l^2)} \tag{5d}$$

$$\overline{G}^j_{l+1} = G(x_{l+1}, t_j, u^j_{l+1}, \overline{U}^j_{xl+1}, \overline{U}^j_{tl+1}) \tag{6a}$$

$$\overline{G}^j_{l-1} = G(x_{l-1}, t_j, u^j_{l-1}, \overline{U}^j_{xl-1}, \overline{U}^j_{tl-1}) \tag{6b}$$

$$\overline{G}^j_{l} = G(x_{l}, t_j, u^j_{l}, \overline{U}^j_{xl}, \overline{U}^j_{tl}) \tag{6c}$$

Let $S_j(x)$ be the cubic spline interpolating polynomial of the function $u(x,t)$ in the interval $[x_{l-1}, x_l]$ and is given by

$$\begin{aligned}S_j(x) &= \frac{(x_l - x)^3}{6h_l}M^j_{l-1} + \frac{(x - x_{l-1})^3}{6h_l}M^j_l\\ &+ \left(U^j_{l-1} - \frac{h_l^2}{6}M^j_{l-1}\right)\left(\frac{x_l - x}{h_l}\right) + \left(U^j_l - \frac{h_l^2}{6}M^j_l\right)\left(\frac{x - x_{l-1}}{h_l}\right),\\ &x_{l-1} \le x \le x_l; l = 1,2,\ldots,N+1; j = 1,2,\ldots,J\end{aligned} \tag{7}$$

which satisfies at $j$th-level the following properties

(1) $S_j(x)$ coincides with a polynomial of degree three on each $[x_{l-1}, x_l]$, $l = 1,2,\ldots,N+1$; $j = 1,2,\ldots,J$,

(2) $S_j(x) \in C^2[0,1]$, and

(3) $S_j(x_l) = U^j_l$, $\quad l = 0,1,2,\ldots,N+1; \quad j = 1,2,\ldots,J.$

where

$$M_l^j = S_j^{"}(x_l) = U_{xxl}^j = U_{ttl}^j - G(x_l, t_j, U_l^j, U_{xl}^j, U_{tl}^j),\ l = 0, 1, \ldots, N+1; j = 1, 2, \ldots, J. \tag{8}$$

$$m_l^j = S_j^{'}(x_l) = U_{xl}^j \quad x_{l-1} \le x \le x_l \tag{9}$$

The derivations of cubic spline function $S_j(x)$ are given by

$$S_j^{'}(x) = \frac{-(x_l - x)^2}{2h_l} M_{l-1}^j + \frac{(x - x_{l-1})^2}{2h_l} M_l^j + \frac{U_l^j - U_{l-1}^j}{h_l} - \frac{h_l}{6}[M_l^j - M_{l-1}^j] \tag{10}$$

$$S_j^{"}(x) = \frac{(x_l - x)}{h_l} M_{l-1}^j + \frac{(x - x_{l-1})}{h_l} M_l^j \tag{11}$$

Also, we have

$$\begin{aligned} S_j(x) = {} & \frac{(x_{l+1} - x)^3}{6h_{l+1}} M_l^j + \frac{(x - x_l)^3}{6h_{l+1}} \\ & + \left(U_l^j - \frac{h_{l+1}^2}{6} M_l^j\right)\left(\frac{x_{l+1} - x}{h_{l+1}}\right) + \left(U_{l+1}^j - \frac{h_{l+1}^2}{6} M_{l+1}^j\right)\left(\frac{x - x_l}{h_{l+1}}\right), \\ & x_l \le x \le x_{l+1}; l = 0, 1, 2, \ldots, N; \quad j = 1, 2, \ldots, J \end{aligned} \tag{12}$$

$$S_l^{'}(x) = \frac{-(x_{l+1} - x)^2}{2h_{l+1}} M_l^j + \frac{(x - x_l)^2}{2h_{l+1}} M_{l+1}^j + \frac{U_{l+1}^j - U_l^j}{h_{l+1}} - \frac{h_{l+1}}{6}[M_{l+1}^j - M_l^j] \tag{13}$$

From continuity equation: $S'(x_{j^-}) = S'(x_{j^+})$

$$\frac{h_l}{6} M_{l-1}^j + \frac{h_{l+1}}{6} M_{l+1}^j + \frac{h_l + h_{l+1}}{3} M_l^j = \frac{U_{l+1}^j - U_l^j}{h_{l+1}} - \frac{U_l^j - U_{l-1}^j}{h_l} \tag{14}$$

Further, from (10), we have

$$m_{l-1}^j = S_j^{'}(x_{l-1}) = U_{xl-1}^j = \frac{U_l^j - U_{l-1}^j}{h_l} - \frac{h_l}{6}[M_l^j + 2M_{l-1}^j] \tag{15}$$

and from (13), we have

$$m_{l+1}^j = S_j^{'}(x_{l+1}) = U_{xl+1}^j = \frac{U_{l+1}^j - U_l^j}{h_{l+1}} + \frac{h_{l+1}}{6}[M_l^j + 2M_{l+1}^j] \tag{16}$$

Combining (10) and (13), we obtain

$$m_l^j = S_j^{'}(x_l) = U_{xl}^j = \overline{U}_{xl}^j - \frac{h_l \sigma_l}{6(1 + \sigma_l)}[M_{l+1}^j - M_{l-1}^j]. \tag{17}$$

Note that, (8), (15), (16) and (17) are important properties of the cubic spline function $S_j(x)$.

Further, at the grid point $(x_l, t_j)$, let us denote

$$\alpha_l^j = \left(\frac{\partial G}{\partial U_x}\right)_l^j, \quad \beta_l^j = \left(\frac{\partial G}{\partial U_t}\right)_l^j \tag{18}$$

Since the derivative values of $S_j(x)$ defined by (8), (15), (16) and (17) are not known at each grid point $(x_l, t_j)$, we use the following approximations for the derivatives of $S_j(x)$. Let

$$\overline{M}_l^j = \overline{U}_{ttl}^j - \overline{G}_l^j \tag{19a}$$

$$\overline{M}_{l+1}^j = \overline{U}_{ttl+1}^j - \overline{G}_{l+1}^j \tag{19b}$$

$$\overline{M}_{l-1}^j = \overline{U}_{ttl-1}^j - \overline{G}_{l-1}^j \tag{19c}$$

$$\widehat{m}_l^j = \overline{U}_{xl}^j - \frac{h_l \sigma_l}{6(1+\sigma_l)}\left[\overline{M}_{l+1}^j - M_{l-1}^j\right] \tag{20a}$$

$$\widehat{m}_{l+1}^j = \frac{U_{l+1}^j - U_l^j}{h_{l+1}} + \frac{h_{l+1}}{6}\left[\overline{M}_l^j + 2M_{l+1}^j\right] \tag{20b}$$

$$\widehat{m}_{l-1}^j = \frac{U_l^j - U_{l-1}^j}{h_l} - \frac{h_l}{6}\left[\overline{M}_l^j + 2M_{l-1}^j\right] \tag{20c}$$

Now we define the following approximations:

$$\widehat{G}_l^j = G(x_l, t_j, U_l^j, \widehat{m}_l^j, \overline{U}_{tl}^j), \tag{21a}$$

$$\widehat{G}_{l+1}^j = G(x_{l+1}, t_j, U_{l+1}^j, \widehat{m}_{l+1}^j, \overline{U}_{tl+1}^j), \tag{21b}$$

$$\widehat{G}_{l-1}^j = G(x_{l-1}, t_j, U_{l-1}^j, \widehat{m}_{l-1}^j, \overline{U}_{tl-1}^j), \tag{21c}$$

in which we use the cubic spline function $U_l^j = S_j(x_l)$, approximation of its first order space derivative defined by (20a)-(20c) in $x$-direction and central difference approximations of time derivative defined by (4a)-(4c) in $t$-direction.

With the help of the approximations (4a) and (5a), from (6a), we obtain

$$\overline{G}_l^j = G_l^j + \frac{h_l^2\sigma_l}{6}\ \frac{\partial^3 U_l^j}{\partial x^3}\alpha_l^j + O(h_l^3 + k^2) \tag{22a}$$

$$\overline{G}_{l+1}^j = G_{l+1}^j - \frac{h_l^2\sigma_l(1+\sigma_l)}{6}\ \frac{\partial^3 U_l^j}{\partial x^3}\alpha_l^j + O(h_l^3 + k^2) \tag{22b}$$

$$\overline{G}_{l-1}^j = G_{l-1}^j - \frac{h_l^2(1+\sigma_l)}{6}\ \frac{\partial^3 U_l^j}{\partial x^3}\alpha_l^j + O(h_l^3 + k^2) \tag{22c}$$

Now using the approximations (19a)-(19c), (22a)-(22c), and simplifying (20a)-(20c), we get

$$\widehat{m}_l^j = m_l^j + O(h_l^3 + k^2 h_l) \tag{23a}$$

$$\widehat{m}_{l+1}^j = m_{l+1}^j + O(h_l^3 + k^2 h_l) \tag{23b}$$

$$\widehat{m}_{l-1}^j = m_{l-1}^j + O(h_l^3 + k^2 h_l) \tag{23c}$$

Now, with the help of the approximations (4a) and (23a), from (21a), we obtain

$$\begin{aligned} \widehat{G}_l^j &= G(x_l, t_j, U_l^j, m_l^j + O(h_l^3 + k^2 h_l), U_{tl}^j + O(k^2)) \\ &= G(x_l, t_j, U_l^j, m_l^j, U_{tl}^j) + O(h_l^3 + k^2 h_l + k^2) \\ &= G_l^j + O(h_l^3 + k^2 h_l + k^2) \end{aligned} \tag{24a}$$

$$\widehat{G}_{l+1}^j = G_{l+1}^j + O(h_l^3 + k^2 h_l + k^2) \tag{24b}$$

$$\widehat{G}_{l-1}^j = G_{l-1}^j + O(h_l^3 + k^2 h_l + k^2) \tag{24c}$$

Then at each grid point $(x_l, t_j)$, a cubic spline finite difference method with accuracy of $O(h_l^3 + k^2 h_l + k^2)$ for the solution of differential equation (1) may be written as

$$\begin{aligned} [U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j] =& \frac{h_l^2}{12}[P_l \overline{U}_{ttl+1}^j + Q_l \overline{U}_{ttl}^j + R_l \overline{U}_{ttl-1}^j] \\ & - \frac{h_l^2}{12}[P_l \widehat{G}_{l+1}^j + Q_l \widehat{G}_l^j + R_l \widehat{G}_{l-1}^j] + \widehat{T}_l^j \end{aligned} \tag{25}$$

Using the approximations (4d)-(4f) and (24a)-(24c), from (25), we obtain the local truncation error

$$\widehat{T}_l^j = [U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j] - \frac{h_l^2}{12}[P_l U_{ttl+1}^j + Q_l U_{ttl}^j + R_l U_{ttl-1}^j] + \frac{h_l^2}{12}[P_l G_{l+1}^j + Q_l G_l^j + R_l G_{l-1}^j] + O(h_l^5 + k^2 h_l^3 + k^2 h_l^2) \tag{26}$$

Now substituting the values $G_l^j = U_{ttl}^j - U_{xxl}^j$ and $G_{l\pm1}^j = U_{ttl\pm1}^j - U_{xxl\pm1}^j$ in (26), and then using Taylor expansion of $U_{l\pm1}^j$, $U_{ttl\pm1}^j$ and $U_{xxl\pm1}^j$ at the grid point $(x_l, t_j)$, the local truncation error defined in (26) reduces to $\widehat{T}_l^j = O(h_l^5 + k^2 h_l^3 + k^2 h_l^2)$.

Note that, the initial and Dirichlet boundary conditions are given by (2) and (3), respectively. Incorporating the initial and boundary conditions, we can write the method (25) in a tri-diagonal matrix form. If the differential equation (1) is linear, we can solve the linear system using Gauss-elimination (tri-diagonal solver) method; in the non-linear case, we can use Newton-Raphson iterative method to solve the non-linear system (see Kelly [3], Hageman and Young [11], Varga [31] and Saad [33]).

## 3. Application to wave equation with singular coefficients

Consider the hyperbolic equation with singular coefficient.
Let us consider the equation of the form

$$u_{tt} = u_{rr} + D(r)u_r + E(r)u + f(r,t), \quad 0 < r < 1, \quad t > 0 \tag{27}$$

subject to appropriate initial and Dirichlet boundary conditions given by (2) and (3), respectively, where $D(r) = \alpha/r$, $E(r) = -\alpha/r^2$. For $\alpha = 0$, the equation above represents time dependent wave equation and for $\alpha = 1$ and 2, and replacing the variable $x$ by $r$, the equation (27) represents wave equation in cylindrical and spherical polar coordinates, respectively.

Applying the approximation (25) to the differential equation (27), and neglecting the local truncation error, we obtain the scheme

$$
\begin{aligned}
(28)\qquad & \frac{12}{h_l^2}[U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j] \\
&= [P_l\overline{U}_{ttl+1}^j + Q_l\overline{U}_{ttl}^j + R_l\overline{U}_{ttl-1}^j] - P_l[D_{l+1}\left(\frac{U_{l+1}^j - U_l^j}{h_l\sigma_l}\right)+E_{l+1}U_{l+1}^j + f_{l+1}^j] \\
&- R_l[D_{l-1}\left(\frac{U_l^j - U_{l-1}^j}{h_l}\right)+E_{l-1}U_{l-1}^j + f_{l-1}^j] \\
&+ \left(\frac{P_l D_{l+1} h_l \sigma_l}{6} - \frac{R_l D_{l-1} h_l}{6} - Q_l\right)[D_l\overline{U}_{xl}^j + E_l U_l^j + f_l^j] \\
&+ \left(\frac{2P_l D_{l+1} h_l \sigma_l}{6} - \frac{Q_l D_l h_l \sigma_l}{6(1+\sigma_l)}\right)[D_{l+1}\overline{U}_{xl+1}^j + E_{l+1}U_{l+1}^j + f_{l+1}^j] \\
&+ \left(\frac{Q_l D_l h_l \sigma_l}{6(1+\sigma_l)} - \frac{2R_l D_{l-1} h_l}{6}\right)[D_{l-1}\overline{U}_{xl-1}^j + E_{l-1}U_{l-1}^j + f_{l-1}^j] \\
&- \frac{P_l D_{l+1} h_l \sigma_l}{6}[\overline{U}_{ttl}^j + 2\overline{U}_{ttl+1}^j] + \frac{Q_l D_l h_l \sigma_l}{6(1+\sigma_l)}[\overline{U}_{ttl+1}^j - \overline{U}_{ttl-1}^j] \\
&+ \frac{2R_l D_{l-1} h_l}{6}[\overline{U}_{ttl}^j + 2\overline{U}_{ttl-1}^j] \qquad\qquad l = 1(1)N, \quad j = 0, 1, 2, \ldots
\end{aligned}
$$

where the values of $P_l$, $Q_l$, $R_l$ are already defined in the previous section and $D_l = (D_{r_l})$, $D_{l\pm1} = D(r_{l\pm1}), E_l = E(r_l), E_{l\pm1} = E(r_{\pm1}), f_l^j = f(r_l, t_j), f_{l\pm1}^j = f(r_{l\pm1}, t_j)$ etc.

Note that the linear variable mesh scheme (28) is of $O(k^2 + k^2 h_l + h_l^3)$ accuracy for the solution of the wave equation (27) with singular coefficients, however, the scheme fails to compute when the solution is to be determined at $l = 1$ due to zero division. We overcome this difficulty by using the following approximations.

Let

$$D_l \equiv D_0 \tag{29a}$$

$$D_{l+1} = D_l + \sigma_l h_l D_{rl} + \frac{\sigma_l^2 h_l^2}{2} D_{rrl} + O(h_l)^3 \equiv D_1 \tag{29b}$$

$$D_{l-1} = D_l - h_l D_{rl} + \frac{h_l^2}{2} D_{rrl} - O(h_l^3) \equiv D_2 \tag{29c}$$

$$E_l \equiv E_0 \tag{29d}$$

$$E_{l+1} = E_l + \sigma_l h_l E_{rl} + \frac{\sigma_l^2 h_l^2}{2} E_{rrl} + O(h_l)^3 \equiv E_1 \tag{29e}$$

$$E_{l-1} = E_l - h_l E_{rl} + \frac{h_l^2}{2} E_{rrl} - O(h_l^3) \equiv E_2 \tag{29f}$$

$$f_l^j = f(r_l, t) \equiv F_0 \tag{30a}$$

$$f_{l+1}^j = f_l^j + \sigma_l h_l f_{rl}^j + \frac{\sigma_l^2 h_l^2}{2} f_{rrl}^j + O(h_l)^3 \equiv F_1 \tag{30b}$$

$$f_{l-1}^j = f_l^j - h_l f_{rl}^j + \frac{h_l^2}{2}{}_{rrl} - O(h_l^3) \equiv F_2 \tag{30c}$$

where $D_{rl} = \frac{dD(r_l)}{dr}$, $D_{rrl} = \frac{d^2D(r_l)}{dr^2}$, $E_{rl} = \frac{dE(r_l)}{dr}$, $E_{rrl} = \frac{d^2E(r_l)}{dr^2}$, $f_{rl}^j = \frac{\partial f(r_l,t_j)}{\partial r}$, $f_{rrl}^j = \frac{\partial^2 f(r_l,t_j)}{\partial r^2}$, etc.

Now substituting the approximations (29a)-(29f) and (30a)-(30c) in (28) and neglecting high order terms, we obtain

$$
\begin{aligned}
(31)\quad & \frac{12}{h_l^2}[U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j] \\
& = [P_l \overline{U}_{ttl+1}^j + Q_l \overline{U}_{ttl}^j + R_l \overline{U}_{ttl-1}^j] - P_l[D_1\left(\frac{U_{l+1}^j - U_l^j}{h_l\sigma_l}\right) + E_1 U_{l+1}^j + F_1] \\
& - R_l[D_2\left(\frac{U_l^j - U_{l-1}^j}{h_l}\right) + E_2 U_{l-1}^j + F_2] \\
& + \left(\frac{P_l D_1 h_l \sigma_l}{6} - \frac{R_l D_2 h_l}{6} - Q_l\right)[D_0 \overline{U}_{xl}^j + E_0 U_l^j + F_0] \\
& + \left(\frac{2P_l D_1 h_l \sigma_l}{6} - \frac{Q_l D_0 h_l \sigma_l}{6(1+\sigma_l)}\right)[D_1 \overline{U}_{xl+1}^j + E_1 U_{l+1}^j + F_1] \\
& + \left(\frac{Q_l D_0 h_l \sigma_l}{6(1+\sigma_l)} - \frac{2R_l D_2 h_l}{6}\right)[D_2 \overline{U}_{xl-1}^j + E_2 U_{l-1}^j + F_2] \\
& - \frac{P_l D_1 h_l \sigma_l}{6}[\overline{U}_{ttl}^j + 2\overline{U}_{ttl+1}^j] + \frac{Q_l D_0 h_l \sigma_l}{6(1+\sigma_l)}[\overline{U}_{ttl+1}^j - \overline{U}_{ttl-1}^j] + \frac{2R_l D_2 h_l}{6}[\overline{U}_{ttl}^j + 2\overline{U}_{ttl-1}^j]
\end{aligned}
$$

Note that the numerical method (31) is of $O(k^2 + k^2 h_l + h_l^3)$ accuracy and free from the terms $1/(x_{l\pm1})$, hence very easily solved for $l = 1(1)N$ in the solution region $0 < x < 1$, $t > 0$. This technique shows that the proposed numerical method is applicable to wave equation with singular coefficients and we do not require the presence of any fictitious points outside the solution region to handle the numerical scheme near the boundary.

## 4. Numerical illustrations

Substituting the approximations (4a), (4d), (5a) and (5d) directly into the differential equations (1), we obtain a lower order variable mesh method

$$(32)\qquad \overline{U}_{ttl}^j = \overline{U}_{xxl}^j + G(x_l, t_j, U_l^j, \overline{U}_{xl}^j, \overline{U}_{tl}^j) + O(k^2 + h_l), \quad l = 1(1)N, j = 1.2.\ldots$$

In this section, we have solved some benchmark problems using the method described by equation (25) and compared our results with the results obtained by using the method (32) for the solution of 1-D non-linear wave equations. The exact solutions are provided in each case. The right hand side homogeneous functions, initial and boundary conditions may be obtained using the exact solution as a test procedure. The linear difference equations have been solved using a tri-diagonal solver, whereas non-linear difference equations have been solved using the Newton-Raphson method. While using the Newton-Raphson method, the iterations were stopped when absolute error tolerance $\leq 10^{-12}$ was achieved. All computations were carried out using double precision arithmetic.

Note that, the proposed method (25) for second order hyperbolic equations is a three-level scheme. The value of $u$ at $t = 0$ is known from the initial condition. To start any computation, it is necessary to know the numerical value of $u$ of required accuracy at $t = k$. In this section, we discuss an explicit scheme of $O(k^2)$ for $u$ at first time level, i.e., at $t = k$ in order to solve the differential equation (1) using the method (25), which is applicable to problems in Cartesian and polar coordinates.

Since the values of $u$ and $u_t$ are known explicitly at $t = 0$, this implies all their successive tangential derivatives are known at $t = 0$, i.e. the values of $u, u_x, u_{xx}, \cdots, u_t, u_{tx}, \cdots$, etc. are known at $t = 0$.

An approximation for $u$ of $O(k^2)$ at $t = k$ may be written as

$$u_l^1 = u_l^0 + k u_{tl}^0 + \frac{k^2}{2}(u_{tt})_l^0 + O(k^3) \tag{33}$$

From equation (1), we have

$$(u_{tt})_l^0 = [u_{tt} + G(x, t, u, u_x, u_t)]_l^0 \tag{34}$$

Thus using the initial values and their successive tangential derivative values, from (34) we can obtain the value of $(u_{tt})_l^0$, and then ultimately, from (33) we can compute the value of $u$ at first time level, i.e. at $t = k$. Replacing the variable $x$ by $r$ in (33), we can also obtain an approximation of $O(k^2)$ for $u$ at $t = k$.

Since

$$1 = x_{N+1} - x_0 = (x_{N+1} - x_N) + (x_N - x_{N-1}) + \ldots + (x_1 - x_0)$$
$$= h_{N+1} + h_N + \ldots + h_1 = h_1(1 + \sigma_1 + \sigma_1\sigma_2 + \sigma_1\sigma_2\sigma_3 + \ldots + \sigma_1\sigma_2\cdots\sigma_N)$$

Thus

$$h_1 = \frac{1}{1 + \sigma_1 + \sigma_1\sigma_2 + \ldots + \sigma_1\sigma_2 \ldots \sigma_N} \tag{35}$$

This determines the starting value of the first step length in $x$-direction and the subsequent step lengths in $x$-direction are calculated by

$$h_2 = \sigma_1 h_1, h_3 = \sigma_2 h_2, \cdots \text{etc.}$$

For simplicity, we may consider $\sigma_l = \sigma$ (a constant), then $h_1$ reduces to

$$h_1 = \frac{(1-\sigma)}{1-\sigma^{N+1}}, \quad \sigma \neq 1 \tag{36}$$

Therefore, by prescribing the value of $N$ and $\sigma$, we can calculate $h_1$ from the above relation and the remaining mesh points in $x$-direction is determined by $h_{l+1} = \sigma h_l$, $l = 1(1)N$. We have chosen the value of $\sigma = 1.02$. Throughout our computation we use the time step $k = 1.6/(N+1)^2$

**Example 1** (Wave equation in polar coordinates)

$$u_{tt} = u_{rr} + \frac{\alpha}{r} u_r + f(r,t), \quad 0 < r < 1, \quad t > 0 \tag{37}$$

The exact solution is $u = \cosh r \sin t$. The maximum absolute errors are tabulated in Table 1 at $t = 1$ for $\alpha = 0, 1$ and 2.

**Example 2** (Van der Pol type nonlinear wave equation)

$$u_{tt} = u_{xx} + \gamma(u^2 - 1)u_t + f(x,t), \quad 0 < x < 1, \quad t > 0 \tag{38}$$

with exact solution $u = e^{-\gamma t} \sin(\pi x)$. The maximum absolute errors are tabulated in Table 2 at $t = 2$ for $\gamma = 1, 2$ and 3.

**Example 3** (Dissipative nonlinear wave equation)

$$u_{tt} = u_{xx} - 2uu_t + f(x,t), \quad 0 < x < 1, \quad t > 0 \tag{39}$$

with exact solution $u = \sin(\pi x) \sin t$. The maximum absolute errors are tabulated in Table 3 at $t = 1$ and 2.

**Example 4** (Non-linear wave equation)

$$u_{tt} = u_{xx} + \gamma u(u_x + u_t) + f(x,t), \quad 0 < x < 1, t > 0 \tag{40}$$

with exact solution $u = e^{-t}\cosh x$. The maximum absolute errors are tabulated in Table 4 for $\gamma = 2, 5$, and 10 at $t = 1$.

## 5. Concluding remarks

Available numerical methods on a variable mesh for the numerical solution of second order non-linear wave equations are of $O(k^2 + h_l)$ accurate. In this article, using the same variable mesh and same number of grid points and three evaluations of the function $G$ (as compared to five and nine evaluations of the function $G$ discussed in [13, 20, 21] for constant mesh), we have derived a new stable cubic spline discretization of $O(k^2+k^2h_l+h_l^3)$ accuracy for the solution of non-linear wave equation (1). The proposed method produces stable results for nonlinear equations, which is exhibited from the computed results. The proposed numerical method (25) is applicable to wave equation in polar coordinates which produces stable results, whereas the corresponding lower order method (32) is unstable.

## Acknowledgements

This research was supported by 'The University of Delhi' under research grant No. Dean (R)/ R & D/2011/423.

## References

[1] A. Khan and T. Aziz, Parametric cubic spline approach to the solution of a system of second order boundary value problems, *J. Optim. Theory Appl.*, 118 (2003) 45-54.

[2] A. Mohebbi and M. Dehghan, High order compact solution of the one-space dimensional linear hyperbolic equation, *Numer. Meth. Partial Diff. Eq.*, 24(2008) 1222-1235.

[3] C.T. Kelly, Iterative Methods for Linear and Non-linear Equations, *SIAM Publications*, Philadelphia, 1995.

[4] D.J. Evans and R.K. Mohanty, On the application of the SMAGE parallel algorithms on a non-uniform mesh for the solution of non-linear two point boundary value problems with singularity, *Int. J. Comput. Math.*, 82 (2005) 341-353.

[5] D.J. Fyfe, The use of cubic splines in the solution of two point boundary value problems, *Comput. J.*, 12 (1969) 188-192.

[6] E.H. Twizell, An explicit difference method for the wave equation with extended stability range, *BIT*, 19 (1979) 378-383.

[7] G.F. Raggett, P.D. Wilson, A fully implicit finite difference approximation to the one-dimensional wave equation using a cubic spline technique, *J. Inst. Maths. Applics.*, 14 (1974) 75-77.

[8] Hengfei Ding, Yuxin Zhang, Parametric spline methods for the solution of hyperbolic equations, *Appl. Math. Comput.*, 204 (2008) 938-941.

[9] J.A. Fleck, Jr., A cubic spline method for solving the wave equation of nonlinear optics, *J. Comput. Phys.*, 16 (1974) 324-341.

[10] J. Rashidinia, R. Jalilian, V. Kazemi, Spline methods for the solutions of hyperbolic equations, *Appl. Math. Comput.*, 190 (2007) 882-886.

[11] L.A. Hageman and D.M. Young, Applied Iterative Methods, *Dover Publication*, New York, 2004.

[12] M.K. Jain, T. Aziz, Cubic spline solution of two point boundary value problems with significant first derivatives, *Comput. Meth. Appl. Mech. Engg.*, 39 (1983) 83-91.

[13] M.K. Jain, R.K. Jain and R.K. Mohanty, Higher order difference methods for the system of one-dimensional second order hyperbolic equations with nonlinear first derivative terms, *J. Mat. Phy. Sci.*, 26 (1992) 401-411.

[14] M.K. Jain, S.R.K. Iyenger and G.S. Subramanyam, Variable mesh methods for the numerical solution of two point singular perturbation problems, *Comput. Meth. Appl. Mech. Engg.*, 42 (1984) 273-286.

[15] M.K. Kadalbajoo, R.K. Bawa, Cubic spline method for a class of non-linear singularly perturbed boundary value problems, *J. Optim. Theory Appl.*, 76 (1993) 415-428.

[16] M.K. Kadalbajoo, V. K. Aggarwal, Cubic spline for solving singular two-point boundary value problems, *Appl. Math. Comput.*, 156 (2004) 249-259.

[17] Manoj Kumar and P.K. Srivastava, Computational techniques for solving differential equations by cubic, quintic and sextic splines, *Int. J. Comput. Meth. Eng. Sci. Mech.*, 10 (2009) 108-115.

[18] N. Papamichael, J.R. Whiteman, A cubic spline technique for the one-dimensional heat conduction equation, *J. Inst. Maths. Applics.*,11 (1973)111-113.

[19] R.K. Mohanty, M.K. Jain, High accuracy cubic spline alternating group explicit methods for 1D quasi-linear parabolic equations, *Int. J. Comput. Math.*, 86 (2009) 1556-1571.

[20] R. K. Mohanty, M. K. Jain, K. George, On the use of high order difference methods for the system of one space second order non-linear hyperbolic equations with variable coefficients, *J. Comp. Appl. Math.*, 72(1996) 421-431.

[21] R. K. Mohanty, U. Arora, A new discretization method of order four for the numerical solution of one space dimensional second order quasi-linear hyperbolic equation, *Int. J. Math. Edu. Sci. Technol.*, 33 (2002) 829-838.

[22] R.K. Mohanty and Suruchi Singh, High accuracy Numerov type discretization for the solution of one space dimensional nonlinear wave equations with variable coefficients, *J. Adv. Res. Sci. Comput.*, 03 (2011) 53-66.

[23] R.K. Mohanty and Suruchi Singh, A new high order approximation for the solution of two-space dimensional quasi-linear hyperbolic equations, *Adv. Math.Phys.*, Vol. 2011, ID: 420608 (2011).

[24] R.K. Mohanty and Venu Gopal, High accuracy cubic spline finite difference approximation for the solution of one-space dimensional non-linear wave equations, *Appl. Math. Comput.*, 218 (2011) 4234-4244.

[25] R.K. Mohanty, An unconditionally stable difference scheme for the one space dimensional linear hyperbolic equation, *Appl. Math. Lett.*, 17 (2004) 101-105.

[26] R.K. Mohanty, Stability interval for explicit difference schemes for multi-dimensional second order hyperbolic equations with significant first order space derivative terms, *Appl. Math. Comput.*, 190 (2007)1683-1690.

[27] R.K. Mohanty, New unconditionally stable difference schemes for the solution of multi-dimensional telegraphic equations, *Int. J. Comput. Math.*, 86 (2009) 2061-2071.

[28] R.K. Mohanty, A family of variable mesh methods for the estimates of (du/dr) and the solution of non-linear two point boundary value problems with singularity, *J. Comp. Appl. Math.*, 182 (2005) 173-187.

[29] R.K. Mohanty and Noopur Khosla, A third order accurate variable mesh TAGE iterative method for the numerical solution of two point non-linear singular boundary value problems, *Int. J. Comput. Math.*, 82 (2005) 1261-1273.

[30] R.K. Mohanty, An implicit high accuracy variable mesh scheme for 1D nonlinear singular parabolic partial differential equations, *Appl. Math. Comp.*, 186 (2007) 219-229.

[31] R.S. Varga, Matrix Iterative Analysis, *Springer Verlag*, New York, 2000.

[32] W.G. Bickley, Piecewise cubic interpolation and two point boundary value problems, *Comput. J.*, 11 (1968) 206-208.

[33] Y.Saad, Iterative methods for sparse linear systems, *SIAM Publications*, Philadelphia, 2004.

Table 1. The maximum absolute errors

| $N+1$ | Method (31) | | | Method (32) |
|---|---|---|---|---|
| | $\alpha = 0$ | $\alpha = 1$ | $\alpha = 2$ | $\alpha = 0$,1 and 2 |
| 08 | 0.1045(-03) | 0.2574(-04) | 0.7176(-04) | |
| 16 | 0.6586(-05) | 0.1606(-05) | 0.4543(-05) | Unstable |
| 32 | 0.4124(-06) | 0.1003(-06) | 0.2849(-06) | |
| 64 | 0.2579(-07) | 0.6274(-08) | 0.1782(-07) | |

Table 2. The maximum absolute errors

| $N+1$ | Method (25) | | | Method (32) | | |
|---|---|---|---|---|---|---|
| | $\gamma = 1$ | $\gamma = 2$ | $\gamma = 3$ | $\gamma = 1$ | $\gamma = 2$ | $\gamma = 3$ |
| 04 | 0.3700(-03) | 0.2380(-03) | 0.2443(-03) | 0.1310(-01) | 0.5800(-02) | 0.2400(-02) |
| 08 | 0.2676(-04) | 0.1610(-04) | 0.1473(-04) | 0.3300(-02) | 0.1600(-02) | 0.6426(-02) |
| 16 | 0.2138(-05) | 0.1210(-05) | 0.9764(-06) | 0.8805(-03) | 0.4164(-03) | 0.1728(-03) |
| 32 | 0.2363(-06) | 0.1236(-06) | 0.7923(-07) | 0.2573(-04) | 0.1217(-03) | 0.5039(-04) |

Table 3. The maximum absolute errors

| $N+1$ | Method (25) | | Method (32) | |
|---|---|---|---|---|
| | $t = 1$ | $t = 2$ | $t = 1$ | $t = 2$ |
| 08 | 0.8583(-04) | 0.1064(-04) | 0.9300(-02) | 0.1410(-01) |
| 16 | 0.6570(-05) | 0.8302(-05) | 0.2500(-02) | 0.3700(-02) |
| 32 | 0.7055(-06) | 0.9182(-06) | 0.7250(-03) | 0.1100(-02) |
| 64 | 0.1108(-06) | 0.1810(-06) | 0.2781(-03) | 0.3941(-03) |

Table 4. The maximum absolute errors

| $N+1$ | Method (25) | | | Method (32) | | |
|---|---|---|---|---|---|---|
| | $\gamma = 2$ | $\gamma = 5$ | $\gamma = 10$ | $\gamma = 2$ | $\gamma = 5$ | $\gamma = 10$ |
| 08 | 0.5008(-04) | 0.2193(-03) | 0.6101(-02) | 0.1100(-02) | 0.7500(-02) | 0.2670(-00) |
| 16 | 0.3157(-05) | 0.1297(-04) | 0.1278(-03) | 0.3195(-03) | 0.2200(-02) | 0.6850(-01) |
| 32 | 0.2001(-06) | 0.9454(-06) | 0.1118(-04) | 0.1077(-03) | 0.6869(-03) | 0.2120(-01) |
| 64 | 0.2843(-07) | 0.5431(-07) | 0.1692(-05) | 0.4569(-04) | 0.2748(-03) | 0.8400(-02) |