THE PRODUCTS OF SOFT QUASI-UNIFORMITIES AND SOFT TOPOLOGIES

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Abstract. In this paper, we investigate the relations among soft topology, soft closure operators and soft quasi-uniformities in complete residuated lattices. We give their examples.

Keywords: Complete residuated lattices; Soft quasi-uniformities; Soft closure operators; Soft topologies.

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1. Introduction

Hájek [6] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [7,11-16,26]. Many researcher introduced the notion of fuzzy uniformities in unit interval [0,1] [3,17], complete distributive lattices [8,32]. Recently, Molodtsov [23] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,5,11-15, 19,23, 30,31]. Pawlak’s rough set [24,25] can be viewed as a special case of soft rough sets [5]. The topological structures of soft sets have been developed by many researchers [4,11-15,27,28].

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Kim [15] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft $F : A \to P(U)$ where $L$ is a complete residuated lattice. Kim [11-15] introduced the soft topological structures, fuzzy quasi-uniformities and soft closure operators in complete residuated lattices.

In this paper, we investigate the relations among soft topology, soft closure operators and soft quasi-uniformities in complete residuated lattices. We give their examples.

2. Preliminaries

**Definition 2.1.** [2,6.7,26] An algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \to)$ is a complete residuated lattice and we denote $L_0 = L - \{0\}$.

**Lemma 2.2.** [2,6.7,26] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x, 0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z, x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \odot y \leq x \land y \leq x \lor y$,

(4) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(7) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(8) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(10) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,
(12) \( x \to y \leq (x \circ z) \to (y \circ z) \) and \( (x \to y) \circ (y \to z) \leq x \to z \).

**Definition 2.3.** [15] Let \( X \) be an initial universe of objects and \( E \) the set of parameters (attributes) in \( X \). A pair \((F,A)\) is called a fuzzy soft set over \( X \), where \( A \subseteq E \) and \( F : A \to \mathcal{L}^X \) is a mapping. We denote \( S(X,A) \) as the family of all fuzzy soft sets under the parameter \( A \).

\[ F \subseteq SU_3 \]

For every \((F,A)\) \((SU_2)\) If \( x \leq F \)

\[ \tau \]

A map \( \tau \subseteq S(X,A) \) is called a soft topology on \( X \) if it satisfies the following conditions.

(1) \((F,A)\) is a fuzzy soft subset of \((G,A)\), denoted by \((F,A) \leq (G,A)\) if \( F(a) \leq G(a) \), for each \( a \in A \).

(2) \((F,A) \wedge (G,A) = (F \wedge G,A)\) if \((F \wedge G)(a) = F(a) \wedge G(a)\) for each \( a \in A \).

(3) \((F,A) \vee (G,A) = (F \vee G,A)\) if \((F \vee G)(a) = F(a) \vee G(a)\) for each \( a \in A \).

(4) \((F,A) \circ (G,A) = (F \circ G,A)\) if \((F \circ G)(a) = F(a) \circ G(a)\) for each \( a \in A \).

(6) \( \alpha \circ (F,A) = (\alpha \circ F,A) \) for each \( \alpha \in L \).

**Definition 2.5.** [12] A map \( \tau \subseteq S(X,A) \) is called a soft cotopology on \( X \) if it satisfies the following conditions.

(ST1) \((0_X,A),(1_X,A) \in \tau \), where \( 0_X(a)(x) = 0, 1_X(a)(x) = 1 \) for all \( a \in A, x \in X \),

(ST2) If \((F,A),(G,A) \in \tau \), then \((F,A) \circ (G,A) \in \tau \),

(T) If \((F_i,A) \in \tau \) for each \( i \in I, \bigvee_{i \in I} (F_i,A) \in \tau \).

A map \( \tau \subseteq S(X,A) \) is called a soft quasi-uniformity on \( X \) if it satisfies (ST1), (ST2) and (CT) If \((F_i,A) \in \tau \) for each \( i \in I, \bigwedge_{i \in I} (F_i,A) \in \tau \).

The triple \((X,A,\tau)\) is called a soft topological (resp. cotopological) space.

Let \((X,A,\tau_1)\) and \((X,A,\tau_2)\) be soft fuzzy topological spaces. Then \( \tau_1 \) is finer than \( \tau_2 \) if \((F,A) \in \tau_1 \), for all \((F,A) \in \tau_2 \).

**Definition 2.6.** [13] A subset \( U \subseteq S(X \times X,A) \) is called a soft quasi-uniformity on \( X \) iff it satisfies the properties.

(SU1) \((1_{X \times X},A) \in U \).

(SU2) If \((V,A) \leq (U,A)\) and \((V,A) \in U \), then \((U,A) \in U \).

(SU3) For every \((U,A),(V,A) \in U \), \((U,A) \circ (V,A) \in U \).
(SU4) If \((U,A) \in \mathbf{U}\) then \((1_{\Delta},A) \leq (U,A)\) where

\[
1_{\Delta}(a)(x,y) = \begin{cases} 
1, & \text{if } x = y \\
\bot, & \text{if } x \neq y,
\end{cases}
\]

(SU5) For every \((U,A) \in \mathbf{U}\), there exists \((V,A) \in \mathbf{U}\) such that \((V,A) \circ (V,A) \leq (U,A)\) where

\[
((V,A) \circ (V,A))(a)(x,y) = (V(a) \circ V(a))(x,y) = \bigvee_{z \in X} (V(a)(z,x) \circ V(a)(x,y)), \quad \forall x,y \in X, a \in A.
\]

The triple \((X,A,\mathbf{U})\) is called a soft quasi-uniform space.

A soft quasi-uniformity \(\mathbf{U}\) on \(X\) is said to be a soft uniformity if

(U) if \((U,A) \in \mathbf{U}\), then \((U^{-1},A) \in \mathbf{U}\) where \(U^{-1}(a)(x,y) = U(a)(y,x)\).

**Definition 2.7.** [8] A mapping \(cl : S(X,A) \to S(X,A)\) is called a soft closure operator if it satisfies the following conditions;

(C1) \(cl(0_X,A) = (0_X,A)\),

(C2) \(cl(F,A) \geq (F,A)\),

(C3) If \((F,A) \leq (G,A)\), then \(cl(F,A) \leq cl(G,A)\),

(C4) \(cl(cl(F,A)) = (F,A)\),

(C5) \(cl((F,A) \circ (G,A)) \leq cl(F,A) \circ cl(G,A)\).

The pair \((X,A,\mathbf{cl})\) is called a soft closure space.

**Theorem 2.8.** [14] Let \((X,A,\mathbf{U})\) be a soft quasi-uniform space. Define \(cl^r_\mathbf{U}, cl^l_\mathbf{U} : S(X,A) \to S(X,A)\) as follows

\[
cl^r_\mathbf{U}(F,A)(y) = \bigwedge_{(U,A) \in \mathbf{U}} \left( \bigvee_{x \in X} (U,A)(y,x) \circ (F,A)(x) \right),
\]

\[
cl^l_\mathbf{U}(F,A)(y) = \bigwedge_{(U,A) \in \mathbf{U}} \left( \bigvee_{x \in X} (U,A)(x,y) \circ (F,A)(x) \right).
\]

Then, for \(cl \in \{cl^r_\mathbf{U}, cl^l_\mathbf{U}\}\), we have following properties.

1. \(cl(0_X,A) = (0_X,A)\) and \(cl(F,A) \leq cl(G,A)\) for \((F,A) \leq (G,A)\).
2. 
(2) \((F,A) \leq cl(F,A)\).
3. \(cl(cl(F,A)) = cl(F,A)\).
4. If \(L\) satisfies \(a \circ \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \circ b_i)\), then \(cl((F,A) \circ (G,A)) \leq cl(F,A) \circ cl(G,A)\).
Remark 2.9. If \((L, \odot)\) is a continuous t-norm, then \(a \odot \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \odot b_i)\).

**Theorem 2.10.** [13] Let \((X, A, U)\) be a soft quasi-uniform space and \(a \odot \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \odot b_i)\) for \(a, b_i \in L\). Define \(\tau^r_U, \tau^l_U \subset S(X, A)\) as follows

\[
\tau^r_U = \{ (F, A) \in S(X, A) \mid cl^r_U(F, A) = (F, A) \},
\]

\[
\tau^l_U = \{ (F, A) \in S(X, A) \mid cl^l_U(F, A) = (F, A) \}.
\]

Then (1) \(\tau^r_U\) is a soft topology on \(X\) such that \(\tau^r_U = \{ cl^r_U(F, A) \mid (F, A) \in S(X, A) \}\).

(2) \(\tau^l_U\) is a soft topology on \(X\) such that \(\tau^l_U = \{ cl^l_U(F, A) \mid (F, A) \in S(X, A) \}\).

**Lemma 2.11.** [13] For every \((F, A), (G, A) \in S(X, A)\), we define \((U_F, A) \in S(X \times X, A)\) by

\[
U_F(a)(x, y) = F(a)(x) \to F(a)(y).
\]

then we have the following statements

(1) \((1_{X \times X}, A) = (U_{0_X}, A) = (U_{1_X}, A)\),

(2) \((1_{\Delta}, A) \leq (U_F, A)\),

(3) \((U_F, A) \odot (U_F, A) = (U_F, A)\),

(4) \((U_F, A) \odot (U_G, A) \leq (U_{F \odot G}, A)\).

**Theorem 2.12.** [13] Let \((X, A, \tau)\) be a soft topological space. Define a function \(U_\tau : S(X \times X, A) \to L\) by

\[
U_\tau = \{ (U, A) \in S(X \times X, A) \mid \bigvee_{i=1}^n (U_{G_i}, A) \leq (U, A), (G_i, A) \in \tau \}
\]

where the first \(\bigvee\) is taken over every finite family \(\{U_{(G_i, A)} \mid i = 1, \ldots, n\}\). Then

(1) \(U_\tau\) is a soft quasi-uniformity on \(X\).

(2) \(\tau \subset \tau^l_U\).

3. The products of soft uniformities and soft topologies
Then we have the following properties.

1. \( U_1 \oplus U_2 \) is the coarsest quasi-uniformity on \( X \) which is finer than \( U_1 \) and \( U_2 \).
2. If \( a \circ \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \circ b_i) \) for \( a, b_i \in L \), then
   \[
   cl^{r}_{U_1}(F,A) \circ cl^{r}_{U_2}(G,A) = cl^{r}_{U_1 \oplus U_2}((F,A) \circ (G,A)).
   \]
3. If \( a \circ \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \circ b_i) \) for \( a, b_i \in \mathbb{L} \), then \( \tau^{r}_{U_1} \oplus \tau^{r}_{U_2} = \tau^{r}_{U_1 \oplus U_2} \) where
   \[
   \tau^{r}_{U_1} \oplus \tau^{r}_{U_2} = \{(F,A) = (F_1,A) \circ (F_2,A) \mid (F_i,A) \in \tau^{r}_{U_i} \text{, } i = 1, 2\}.
   \]
4. If \( a \circ \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \circ b_i) \) for \( a, b_i \in \mathbb{L} \), then \( \tau^{l}_{U_1} \oplus \tau^{l}_{U_2} = \tau^{l}_{U_1 \oplus U_2} \) where
   \[
   \tau^{l}_{U_1} \oplus \tau^{l}_{U_2} = \{(F,A) = (F_1,A) \circ (F_2,A) \mid (F_i,A) \in \tau^{l}_{U_i} \text{, } i = 1, 2\}.
   \]
5. If \( (X,A, \tau_1) \) and \( (X,A, \tau_2) \) are soft fuzzy topological spaces, then \( U_{\tau_1} \oplus \tau_2 \subset U_{\tau_1} \oplus U_{\tau_2} \).

**Proof.** (1) (SU1) \((1_{X \times X}, A) \in U_1 \oplus U_2 \) because \((1_{X \times X}, A) \circ (1_{X \times X}, A) = (1_{X \times X}, A) \) for \((1_{X \times X}, A) \in U_i, i = 1, 2\).

(SU2) If \((V,A) \leq (U,A) \) and \((V,A) \in U_1 \oplus U_2 \), then there exist \((V_i,A) \in U_i, i = 1, 2, \) with \((V_1,A) \circ (V_2,A) \leq (V,A) \leq (U,A). \) Thus \((U,A) \in U_1 \oplus U_2 \).

(SU3) For every \((U,A), (V,A) \in U_1 \oplus U_2 \), there exist \((U_i,A), (V_i,A) \in U_i, i = 1, 2, \) with \((U_1,A) \circ (U_2,A) \leq (U,A) \) and \((V_1,A) \circ (V_2,A) \leq (V,A). \) Thus \((U_1,A) \circ (U_2,A) \circ (V_1,A) \circ (V_2,A) \leq (U,A) \circ (V,A) \). Hence \((U,A) \circ (V,A) \in U_1 \oplus U_2 \).

(SU4) If \((U,A) \in U_1 \oplus U_2 \), then there exist \((U_i,A) \in U_i, i = 1, 2, \) with \((U_1,A) \circ (U_2,A) \leq (U,A). \) Since \((U_i,A) \in U_i, i = 1, 2, \) by (SU4), \((1_\Delta,A) \leq (U_i,A), i = 1, 2, \) Hence \((1_\Delta,A) \leq (U,A). \)

(SU5) For each \((U,A) \in U_1 \oplus U_2 \), there exist \((U_1,A) \in U_1 \) and \((U_2,A) \in U_2 \) such that \((U_1,A) \circ (U_2,A) \leq (U,A). \) For each \((U_i,A) \in U_i, i = 1, 2, \) there exists \((V_i,A) \in U_i \) such that \((V_i,A) \circ (V_i,A) \leq (U_i,A). \)
Thus, there exists \((V_1, A) \circ (V_2, A) \in U_1 \oplus U_2\) such that \[((V_1, A) \circ (V_2, A)) \circ ((V_1, A) \circ (V_2, A)) \leq (U, A)\).

If \((U_1, A) \in U_1\), then \((U_1, A) \circ (1_{X \times X}, A) = (U_1, A)\) such that \((U_1, A) \in U_1, (1_{X \times X}, A) \in U_2\). Hence \((U_1, A) \in U_1 \oplus U_2\); i.e. \(U_1 \subset U_1 \oplus U_2\). Similarly, \(U_2 \subset U_1 \oplus U_2\). If \(U_1 \subset V\) and \(V\) is a soft quasi-uniformity, for \((U, A) \in U_1 \oplus U_2\), there exists \((U, A) \in U_1\) such that \((U_1, A) \circ (U_2, A) \leq (U, A)\). Since \((U_1, A) \in V\), then \((U_1, A) \circ (U_2, A) \in V\). Hence \((U, A) \in V\). So, \(U_1 \oplus U_2 \subset V\).

\((2)\)

\[
\begin{align*}
cl'_{U_1 \oplus U_2}((F, A) \circ (G, A)) (y) \\
= \bigwedge_{y \in X} (U, A) (y, x) \circ (F, A)(x) \circ (G, A)(x) \\
\geq \bigwedge_{y \in X} (U_1, A)(y, x) \circ (U_2, A)(y, x) \circ (F, A)(x) \circ (G, A)(x) \\
= \bigwedge_{y \in X} (U_1, A)(y, x) \circ (U_2, A)(y, x) \circ (F, A)(x) \circ (G, A)(x) \\
= \bigwedge_{y \in X} (U_1, A)(y, x) \circ (F, A)(x) \\
\circ \bigwedge_{y \in X} (U_2, A)(y, x) \circ (G, A)(x) \\
= cl'_{U_1}(F, A)(y) \circ cl'_{U_2}(G, A)(y).
\end{align*}
\]

Suppose there exist \((F, A) \in U_1, (G, A) \in U_2\) and \(y \in X\) such that

\[
cl'_{U_1}(F, A)(y) \circ cl'_{U_2}(G, A)(y) \not\geq cl'_{U_1 \oplus U_2}((F, A) \circ (G, A))(y).
\]

Then there exist \(U_1 \in U_1, U_2 \in U_2\) such that

\[
\bigvee_{x \in X} (U_1(y, x) \circ (F, A)(x)) \circ \bigvee_{z \in X} (U_2(y, z) \circ (G, A)(z)) \not\geq cl'_{U_1 \oplus U_2}((F, A) \circ (G, A))(y).
\]
It follows

\[ \bigvee_{x \in X} ((U_1 \circ U_2)(y,x) \circ ((F,A) \circ (G,A))(x)) \nless c_{U_1 \oplus U_2}^r((F,A) \circ (G,A))(y). \]

It is a contradiction. Hence \( c_{U_1}^r(F,A) \circ c_{U_2}^r(G,A) \geq c_{U_1 \oplus U_2}^r((F,A) \circ (G,A)) \). Thus the result holds.

(3) Let \( (F,A) \in \tau_{U_1} \oplus \tau_{U_2} \)

iff \( (F,A) = (F_1,A) \circ (F_2,A) = c_{U_1}^r(F_1,A) \circ c_{U_2}^r(F_2,A) \)

iff \( (F,A) = (F_1,A) \circ (F_2,A) = c_{U_1 \oplus U_2}^r((F_1,A) \circ (F_2,A)) \)

iff \( (F,A) \in \tau_{U_1 \oplus U_2}^r \).

(4) It is similarly proved as (3).

(5) Let \( (U,A) \in U_{\tau_1 \oplus \tau_2} \). Then there exist \( (F_i,A) \in \tau_i \) such that \( \circ_{j=1}^n(U_{F_{j1}} \circ F_{j2},A) \leq (U,A) \).

Since \( (U_{F_{j1}},A) \circ (U_{F_{j2}},A) \leq (U_{F_{j1}} \circ F_{j2},A) \) from Lemma 2.11(4), we have

\[ \circ_{j=1}^n(U_{F_{j1}},A) \circ (\circ_{j=1}^n(U_{F_{j2}},A)) \leq \circ_{j=1}^n(U_{F_{j1}} \circ F_{j2},A) \leq (U,A). \]

Since \( \circ_{j=1}^n(U_{F_{j1}},A) \in U_{\tau_1}, \circ_{j=1}^n(U_{F_{j2}},A) \in U_{\tau_2} \), we have \( (U,A) \in U_{\tau_1 \oplus U_2} \).

**Theorem 3.2.** Let \( U \) be a soft quasi-uniformities on \( X \). We define

\[ U^{-1} = \{(U,A) \in S(X \times X,A) \mid (U^{-1},A) \in U\}. \]

\[ U \oplus U^{-1} = \{(U,A) \in S(X \times X,A) \mid (U_1,A) \circ (U_2,A) \leq (U,A), \ (U_1,A) \in U, \ (U_2,A) \in U^{-1}\}. \]

Then we have the following properties.

(1) \( U^{-1} \) a soft quasi-uniformities on \( X \).

(2) \( U \oplus U^{-1} \) is the coarsest uniformity on \( X \) which is finer than \( U \) and \( U^{-1} \).

(3) If \( a \circ \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \circ b_i) \) for \( a,b_i \in L \), then

\[ c_{U}^{r}(F,A) = c_{U^{-1}}^{r}(F,A), \ c_{U}^{l}(F,A) = c_{U^{-1}}^{l}(F,A), \]

\[ c_{U}^{r}(F,A) \circ c_{U^{-1}}^{r}(G,A) = c_{U \oplus U^{-1}}^{r}((F,A) \circ (G,A)). \]
(4) If \(a \odot \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \odot b_i)\) for \(a, b_i \in L\), then \(\tau'_U = \tau'_U^{l-1}, \tau_U = \tau_U^{l-1}, \tau'_U \oplus \tau_U^{r-1} = \tau'_U \oplus \tau_U^{l-1}\) where

\[
\tau'_U \oplus \tau_U^{l-1} = \{(F, A) = (F_1, A) \odot (F_2, A) \mid (F_1, A) \in \tau'_U, (F_2, A) \in \tau_U^{l-1}\} = \tau'_U \oplus \tau_U^{l-1}.
\]

**Proof.** (1) (SU5) For \((U, A) \in U^{-1}\) iff \((U^{-1}, A) \in U\), there exists \((V, A) \in U\) such that \((V, A) \odot \\left((V, A) \leq (U^{-1}, A) \iff (V^{-1}, A) \odot (V^{-1}, A) \leq (U, A)\). Other cases are easily proved.

(2) \(U \oplus U^{-1}\) is the coarsest uniformity on \(X\) from Theorem 3.1(1) and

\[
(U, A) \in U \oplus U^{-1}
\]

iff \((U, A) \geq (U_1, A) \odot (U_2, A), (U_1, A) \in U, (U_2, A) \in U^{-1}\)

iff \((U^{-1}, A) \geq (U_1^{-1}, A) \odot (U_2^{-1}, A), (U_1^{-1}, A) \in U^{-1}, (U_2^{-1}, A) \in U\)

iff \((U^{-1}, A) \in U \oplus U^{-1}\)

(3) It follows from Theorem 3.1(2) and the definition of \(cU'_l\).

(4) By (3), we have \(\tau'_U = \tau'_U^{l-1}, \tau_U = \tau_U^{l-1}\) and

\[
\tau'_U \oplus \tau_U^{l-1} = \{(F, A) = (F_1, A) \odot (F_2, A) \mid (F_1, A) \in \tau'_U, (F_2, A) \in \tau_U^{l-1}\}
\]

\[
= \{(F, A) = (F_1, A) \odot (F_2, A) \mid (F_1, A) \in \tau_U^{l-1}, (F_2, A) \in \tau'_U\}
\]

\[
= \tau'_U \oplus \tau_U^{l-1}.
\]

**Example 3.3.** Let \(X = \{h_i \mid i = \{1, \ldots, 4\}\}\) with \(h_i\)-house and \(E_y = \{e, b, w, c, i\}\) with \(e\)-expensive, \(b\)-beautiful, \(w\)-wooden, \(c\)-creative, \(i\)-in the green surroundings.

Let \((L = [0, 1], \odot, \rightarrow)\) be a complete residuated lattice defined by

\[
x \odot y = x \wedge y, \quad x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y; \\ y, & \text{otherwise}. \end{cases}
\]

Let \(X = \{x, y, z\}\) be a set and \(W_i(e), W_i(b) \in S(X \times X, A)\) such that

\[
W_i(e) = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.7 & 1 & 0.8 \\ 0.4 & 0.4 & 1 \end{pmatrix}, \quad W_i(b) = \begin{pmatrix} 1 & 0.6 & 0.7 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{pmatrix}
\]
Also, we have

\[ W_2(e) = \begin{pmatrix} 1 & 0.4 & 0.3 \\ 0.4 & 1 & 0.3 \\ 0.6 & 0.5 & 1 \end{pmatrix}, \quad W_2(b) = \begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.6 & 1 & 0.7 \\ 0.5 & 0.4 & 1 \end{pmatrix} \]

\[ (W_1 \wedge W_2)(e) = \begin{pmatrix} 1 & 0.4 & 0.3 \\ 0.4 & 1 & 0.3 \\ 0.4 & 0.4 & 1 \end{pmatrix}, \quad (W_1 \wedge W_2)(b) = \begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.4 & 1 \end{pmatrix} \]

Define \( U_i = \{(U,A) \in S(X \times X,A) \mid (U,A) \geq (W_i,A)\} \) for \( i = 1,2 \).

1) Since \( W_i(e) \circ W_i(e) = W_i(e) \) and \( W_i(b) \circ W_i(b) = W_i(b), U_i \) is a soft quasi-uniformity on \( X \).

2) From Theorem 2.10(1), we obtain \( \tau_{U_1}^l = \{cl^l_{U_1}(F,A) \mid (F,A) \in L^X\} \) where

\[
cl^l_{U_1}(F,A)(e) = \begin{pmatrix}
F(e)(x) \vee (0.5 \wedge F(e)(y)) \vee (0.5 \wedge F(e)(z)) \\
(0.7 \wedge F(e)(x)) \vee F(e)(y) \vee (0.8 \wedge F(e)(z)) \\
(0.4 \wedge F(e)(x)) \vee (0.4 \wedge F(e)(y)) \vee F(e)(z)
\end{pmatrix}
\]

\[
cl^l_{U_1}(F,A)(b) = \begin{pmatrix}
F(b)(x) \vee (0.6 \wedge F(b)(y)) \vee (0.7 \wedge F(b)(z)) \\
(0.4 \wedge F(b)(x)) \vee F(b)(y) \vee (0.4 \wedge F(b)(z)) \\
(0.5 \wedge F(b)(x)) \vee (0.6 \wedge F(b)(y)) \vee F(b)(z)
\end{pmatrix}
\]

Also, we have \( \tau_{U_2}^l = \{cl^l_{U_2}(F,A) \mid (F,A) \in L^X\} \) where

\[
cl^l_{U_2}(F,A)(e) = \begin{pmatrix}
F(e)(x) \vee (0.4 \wedge F(e)(y)) \vee (0.3 \wedge F(e)(z)) \\
(0.4 \wedge F(e)(x)) \vee F(e)(y) \vee (0.3 \wedge F(e)(z)) \\
(0.6 \wedge F(e)(x)) \vee (0.5 \wedge F(e)(y)) \vee F(e)(z)
\end{pmatrix}
\]

\[
cl^l_{U_2}(F,A)(b) = \begin{pmatrix}
F(b)(x) \vee (0.3 \wedge F(b)(y)) \vee (0.3 \wedge F(b)(z)) \\
(0.6 \wedge F(b)(x)) \vee F(b)(y) \vee (0.7 \wedge F(b)(z)) \\
(0.5 \wedge F(b)(x)) \vee (0.4 \wedge F(b)(y)) \vee F(b)(z)
\end{pmatrix}
\]

3) From Theorem 3.3(3), we obtain \( \tau_{U_1}^r \oplus \tau_{U_2}^r = \tau_{U_1 \oplus U_2}^r = \{cl^r_{U_1 \oplus U_2}(F,A) \mid (F,A) \in L^X\} \) as follows:
\[
cl^r_{U_1 \oplus U_2}(F, A)(e) = \begin{pmatrix}
F(e)(x) \vee (0.4 \wedge F(e)(y)) \vee (0.3 \wedge F(e)(z)) \\
(0.4 \wedge F(e)(x)) \vee F(e)(y) \vee (0.3 \wedge F(e)(z)) \\
(0.4 \wedge F(e)(x)) \vee (0.4 \wedge F(e)(y)) \vee F(e)(z)
\end{pmatrix}
\]

\[
cl^r_{U_1 \oplus U_2}(F, A)(b) = \begin{pmatrix}
F(b)(x) \vee (0.3 \wedge F(b)(y)) \vee (0.3 \wedge F(b)(z)) \\
(0.4 \wedge F(b)(x)) \vee F(b)(y) \vee (0.4 \wedge F(b)(z)) \\
(0.5 \wedge F(b)(x)) \vee (0.4 \wedge F(b)(y)) \vee F(b)(z)
\end{pmatrix}
\]

Similarly, we obtain \( \tau^l_{U_1} \oplus \tau^l_{U_2} = \tau^l_{U_1 \oplus U_2} = \{cl^l_{U_1 \oplus U_2}(F, A) \mid (F, A) \in L^X \} \) as follows:

\[
cl^l_{U_1 \oplus U_2}(F, A)(e) = \begin{pmatrix}
F(e)(x) \vee (0.4 \wedge F(e)(y)) \vee (0.4 \wedge F(e)(z)) \\
(0.4 \wedge F(e)(x)) \vee F(e)(y) \vee (0.4 \wedge F(e)(z)) \\
(0.3 \wedge F(e)(x)) \vee (0.3 \wedge F(e)(y)) \vee F(e)(z)
\end{pmatrix}
\]

\[
cl^l_{U_1 \oplus U_2}(F, A)(b) = \begin{pmatrix}
F(b)(x) \vee (0.4 \wedge F(b)(y)) \vee (0.5 \wedge F(b)(z)) \\
(0.3 \wedge F(b)(x)) \vee F(b)(y) \vee (0.4 \wedge F(b)(z)) \\
(0.3 \wedge F(b)(x)) \vee (0.4 \wedge F(b)(y)) \vee F(b)(z)
\end{pmatrix}
\]

(4) We obtain a soft quasi-uniformity \( U^{-1}_1 = \{(U, A) \in S(X \times X, A) \mid (U, A) \geq (W^{-1}_1, A)\} \)
where

\[
W^{-1}_1(e) = \begin{pmatrix}
1 & 0.7 & 0.4 \\
0.5 & 1 & 0.4 \\
0.5 & 0.8 & 1
\end{pmatrix}
\]

\[
W^{-1}_1(b) = \begin{pmatrix}
1 & 0.4 & 0.5 \\
0.6 & 1 & 0.6 \\
0.7 & 0.4 & 1
\end{pmatrix}
\]

From Theorem 3.2 (2), we obtain a soft uniformity \( U_1 \oplus U^{-1}_1 = \{(U, A) \in S(X \times X, A) \mid (U, A) \geq (W \wedge W^{-1}_1, A)\} \)
where

\[
W \wedge W^{-1}_1(e) = \begin{pmatrix}
1 & 0.5 & 0.4 \\
0.5 & 1 & 0.4 \\
0.4 & 0.4 & 1
\end{pmatrix}
\]

\[
W \wedge W^{-1}_1(b) = \begin{pmatrix}
1 & 0.4 & 0.5 \\
0.4 & 1 & 0.4 \\
0.5 & 0.4 & 1
\end{pmatrix}
\]

(5) Let \( \tau_1 = \{(0_X, A), (1_X, A), (F_1, A)\} \) and \( \tau_2 = \{(0_X, A), (1_X, A), (F_2, A)\} \)
where

\[
F_1(e) = (0.4, 0.5, 0.6), \quad F_1(b) = (0.7, 0.4, 0.9),
\]

\[
F_2(e) = (0.5, 0.1, 0.3), \quad F_2(b) = (0.6, 0.7, 0.4).
\]
Define $U_{\tau_i} = \{(U,A) \in S(X \times X, A) \mid (U,A) \geq (U_{F_i}, A)\}$ for $i = 1, 2$. Since $(U_{F_i}, A) \circ (U_{F_i}, A) = (U_{F_i}, A)$, $U_i$ is a soft quasi-uniformity for $i = 1, 2$ where

\[
U_{\tau_1} = U_{\tau_2} = \{(U,A) \in S(X \times X, A) \mid (U,A) \geq (U_{F_1}, A)\}
\]

Then $U_{\tau_1} \oplus U_{\tau_2} \subset U_{\tau_1} \oplus U_{\tau_2}$. 

Conflict of Interests

The authors declare that there is no conflict of interests.

References


[16] W. Kotzé, uniform spaces, Chapter 8, 553-580 in [7].


