CR-SUBMANIFOLDS OF A NEARLY HYPERBOLIC KENMOTSU MANIFOLD ADMITTING A QUARTER-SYMMETRIC SEMI-METRIC CONNECTION

NIKHAT ZULEKHA¹*, SHADAB AHMAD KHAN¹, MOBIN AHMAD²

¹Department of Mathematics, Integral University, Kursi Road, Lucknow-226026, India
²Department of Mathematics, Jazan University, Jazan-2069, Saudi Arabia

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Abstract. We consider a nearly hyperbolic Kenmotsu manifold with a quater symmetric semi metric connection and study Cr-Submanifolds of a nearly hyperbolic Kenmotsu manifold with quater symmetric semi metric connection. We also study parallel distributions on nearly hyperbolic Kenmotsu manifold with a quater symmetric semi metric connection and find the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold with a quater symmetric semi metric connection.

Keywords: Cr-Submanifolds; Nearly hyperbolic Kenmotsu manifold; Quater symmetric semi metric connection; Integrability conditions and parallel distribution.

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1. Introduction

The notion of CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was introduced and studied by A. Bejancu in ([1],[2]). Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J. Hsu in [5] and M. Kobayashi in [18]. CR-submanifolds of Kenmotsu

*Corresponding author

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manifold was studied by A. Bejancu and N. Papaghiuc in [4]. Later, several geometers (see, [9],[12],[13],[15],[16]) enriched the study of CR-submanifolds of almost contact manifolds. The almost hyperbolic \((f, \xi, \eta, g)\)-structure was defined and studied by Upadhyay and Dube in [17]. Dube and Bhatt studied CR-submanifolds of trans-hyperbolic Sasakian manifold in [10]. On the other hand, S. Golab introduced the idea of semi-symmetric and quarter symmetric connections in [8]. CR-submanifolds of LP-Sasakian manifold with quarter symmetric non-metric connection were studied by the first author S. K. Lovejoy Das in [11]. CR-submanifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connections were studied by the first author, M. D. Siddiqi and S. Rizvi in [14]. M. Ahmad and Kasif Ali, studied CR-submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a quarter symmetric non-metric connection in [19]. In this paper, we study some properties of CR-submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a quarter symmetric semi-metric connection.

### 2. Preliminaries

Let \(\overline{M}\) be an \(n\)-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure \((\phi, \xi, \eta, g)\), where a tensor \(\phi\) of type (1,1), a vector field \(\xi\), called structure vector field, \(\eta\) that dual 1-form of \(\xi\) and \(g\) is Riemannian metric satisfying the following

\[
\phi^2 X = X + \eta(X)\xi, \quad g(X, \xi) = \eta(X) \quad (2.1)
\]

\[
\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0 \quad (2.2)
\]

\[
g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \quad (2.3)
\]

for any \(X, Y\) tangent to \(\overline{M}\) [17]. In this case

If addition to the above condition, we have

\[
g(\phi X, Y) = -g(\phi Y, X) \quad (2.4)
\]
An almost hyperbolic contact metric structure \((\phi, \xi, \eta, g)\) on \(\overline{M}\) is called hyperbolic Kenmotsu manifold \([7]\) if and only if
\[
(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X
\]  
(2.5)
for all \(X, Y\) tangent to \(\overline{M}\).

On a hyperbolic Kenmotsu manifold \(\overline{M}\), we have
\[
\nabla_X \xi = X + \eta(X)\xi
\]  
(2.6)
For a Riemannian metric \(g\) and Riemannian connection \(\nabla\).

Further, an almost hyperbolic contact metric manifold \(\overline{M}\) on \((\phi, \xi, \eta, g)\) is called a nearly hyperbolic Kenmotsu manifold \([7]\), if
\[
(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X
\]  
(2.7)
where \(\nabla\) is Riemannian connection on \(\overline{M}\).

Now, Let \(M\) be a submanifold immersed in \(\overline{M}\). The Riemannian metric symbol \(g\) induced on \(M\). Let \(TM\) and \(T^\perp M\) be the Lie algebra of vector field tangential to \(M\) and normal to \(M\) respectively and \(\nabla^*\) be induced Levi-Civita connection on \(M\) then the Gauss formula and Weingarten formula are given respectively
\[
\nabla_X Y = \nabla^*_X Y + h(X, Y)
\]  
(2.8)
\[
\nabla_X N = -A_N X + \nabla^*_X N
\]  
(2.9)
for any \(X, Y \in TM\) and \(N \in T^\perp M\), where \(\nabla^\perp\) is a connection on the normal bundle \(T^\perp M\), \(h\) is the second fundamental form and \(A_N\) is the Weingarten map associated with \(N\) as
\[
g(h(X, Y), N) = g(A_N X, Y)
\]  
(2.10)
any vector \(X\) tangent to \(M\) is given as
\[
X = PX + QX,
\]  
(2.11)
where \( PX \in D \) and \( QX \in D^\perp \). For any \( N \) normal to \( M \), we have

\[
\phi N = BN + CN, \tag{2.12}
\]

where \( BN \) (resp. \( CN \)) is the tangential component (resp. normal component) of \( \phi N \).

Now, we define a quarter-symmetric semi-metric connection

\[
\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y + g(\phi X, Y)\xi \tag{2.13}
\]

such that

\[
(\widetilde{\nabla}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y)
\]

From (2.13) and using (2.1) and (2.3), we have

\[
(\widetilde{\nabla}_X \phi)Y + \phi(\widetilde{\nabla}_X Y) = (\nabla_X \phi)Y + \phi(\nabla_X Y) - \eta(X)Y - 2\eta(X)\eta(Y)\xi - g(X, Y)\xi
\]

Interchanging \( X \) and \( Y \), we have

\[
(\widetilde{\nabla}_Y \phi)X + \phi(\widetilde{\nabla}_Y X) = (\nabla_Y \phi)X + \phi(\nabla_Y X) - \eta(Y)X - 2\eta(Y)\eta(X)\xi - g(X, Y)\xi
\]

Adding above two equations, we get

\[
(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X + \phi(\widetilde{\nabla}_X Y - \nabla_X Y) + \phi(\widetilde{\nabla}_Y X - \nabla_Y X) = (\nabla_X \phi)Y + (\nabla_Y \phi)X - \eta(X)Y - \eta(Y)X - 4\eta(Y)\eta(X)\xi - 2g(X, Y)\xi
\]

Using equation (2.7) and (2.13) in above, we have

\[
(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi \tag{2.14}
\]

\[
\widetilde{\nabla}_X \xi = X + \eta(X)\xi \tag{2.15}
\]

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure \((\phi, \xi, \eta, g)\) is called nearly hyperbolic Kenmotsu manifold with quarter-symmetric semi-metric connection if it is satisfied (2.14) and (2.15).
The Gauss formula and Weingarten formula for a nearly hyperbolic Kenmotsu manifold admitting quarter symmetric semi metric connection is

\[
\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \\
\overline{\nabla}_X N = -A_N X + \nabla_X N - \eta(X) \phi N + g(\phi X, N) \xi
\]

(2.16)  
(2.17)

**Definition 2.1.** An m-dimensional sub-manifold \( M \) of an n-dimensional nearly hyperbolic Kenmotsu manifold \( \overline{M} \) is called a CR- submanifold if there exist a differentiable distribution \( D : x \to D_x \) on \( M \) satisfying the following conditions:

(i) The distribution \( D \) is invariant under \( \phi \) that is \( \phi D_x = D_x \), for each \( x \in M \),

(ii) The complementary orthogonal distribution \( D^\perp \) of \( D \) is anti-invariant under \( \phi \), that is \( \phi D_x^\perp \subset T_x^\perp M \) for each \( x \in M \).

If \( \text{dim } D_x^\perp = 0 \) (resp., \( \text{dim } D_x^\perp = 0 \)), then the CR-Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution \( D \) (resp., \( D^\perp \)) is called the horizontal (resp., vertical) distribution. Also, the pair \( (D, D^\perp) \) is called \( \xi \)- horizontal (resp., vertical) if \( \xi \in \overline{D} \) (resp., \( \xi \in D_x^\perp \)).

### 3. Some Basic Lemmas

**Lemma 3.1.** If \( M \) be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold \( \overline{M} \) with quarter symmetric semi metric connection. Then

\[
- \eta(X) \phi PY - \eta(Y) \phi PX - 2 \eta(X) \eta(Y) P\xi - 2g(X, Y)P\xi + \phi P(\nabla_X Y) \\
+ \phi P(\nabla_Y X) = P \nabla_X (\phi PY) + P \nabla_Y (\phi PX) - PA_{\phi QY} X - PA_{\phi QX} Y \\
-g(X, QY)P\xi - g(Y, QX)P\xi - 2 \eta(X) \eta(QY)P\xi - 2 \eta(Y) \eta(QX)P\xi \\
-2 \eta(X) \eta(Y) Q\xi - 2g(X, Y)Q\xi + 2Bh(X, Y) = Q \nabla_X (\phi PY) \\
+ Q \nabla_Y (\phi PX) - QA_{\phi QY} X - QA_{\phi QX} Y - \eta(X) QY - \eta(Y) QX \\
-g(X, QY)Q\xi - g(Y, QX)Q\xi - 2 \eta(X) \eta(QY)Q\xi - 2 \eta(Y) \eta(QX)Q\xi \\
- \eta(X) \phi QY - \eta(Y) \phi QX + \phi Q(\nabla_X Y) + \phi Q(\nabla_Y X) + 2Ch(X, Y) = \\
h(X, \phi PY) + h(Y, \phi PX) + \nabla_X (\phi QY) + \nabla_Y (\phi QX)
\]

(3.1)  
(3.2)  
(3.3)
for any \( X, Y \in TM \).

**Proof.** From (2.11), we have

\[
\phi Y = \phi PY + \phi QY.
\]

Differentiating covariantly and using equation (2.16) and (2.17), we have

\[
(\bar{\nabla}_X \phi)Y + \phi(\nabla_X Y) + \phi h(X, Y) = \nabla_X (\phi PY) + h(X, \phi PY)
\]

\[
-A_\phi QY X + \nabla_X^\perp (\phi QY) - \eta(X) QY - g(X, QY) \xi - 2\eta(X) \eta(QY) \xi
\]

Interchanging \( X \) and \( Y \) in above equation, we have

\[
(\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(Y, X) = \nabla_Y (\phi PX) + h(Y, \phi PX)
\]

\[
-A_\phi QX Y + \nabla_Y^\perp (\phi QX) - \eta(Y) QX - g(Y, QX) \xi - 2\eta(Y) \eta(QX) \xi
\]

Adding above two equations, we obtain

\[
(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X + \phi(\nabla_X Y) + \phi(\nabla_Y X) + 2\phi h(X, Y) =
\]

\[
\nabla_X (\phi PY) + \nabla_Y (\phi PX) + h(X, \phi PY) + h(Y, \phi PX) - A_\phi QY X
\]

\[
-A_\phi QX Y + \nabla_X^\perp (\phi QY) + \nabla_Y^\perp (\phi QX) - \eta(X) QY - \eta(Y) QX
\]

\[
-g(X, QY) \xi - g(Y, QX) \xi - 2\eta(X) \eta(QY) \xi - 2\eta(Y) \eta(QX) \xi
\]

Adding (2.14) in above equation and using equations (2.11) and (2.12), we have

\[
-\eta(X) \phi PY - \eta(X) \phi QY - \eta(Y) \phi PX - \eta(Y) \phi QX - 2\eta(X) \eta(Y) P \xi
\]

\[
-2\eta(X) \eta(Y) Q \xi - 2g(X, Y) P \xi - 2g(X, Y) Q \xi + \phi P(\nabla_X Y) + \phi Q(\nabla_Y X)
\]

\[
+\phi P(\nabla_Y X) + \phi Q(\nabla_Y X) + 2Bh(X, Y) + 2Ch(Y, X) = P \nabla_X (\phi PY)
\]

\[
+Q \nabla_X (\phi PY) + P \nabla_Y (\phi PX) + Q \nabla_Y (\phi PX) + h(X, \phi PY) + h(Y, \phi PX) - PA_\phi QY X
\]

\[
-QA_\phi QY X - PA_\phi QX Y - QA_\phi QX Y + \nabla_X (\phi QY) + \nabla_Y (\phi QX) - \eta(X) QY - \eta(Y) QX
\]

\[
-g(X, QY) P \xi - g(X, QY) Q \xi - g(Y, QX) P \xi - g(Y, QX) Q \xi - 2\eta(X) \eta(QY) P \xi
\]

\[
-2\eta(X) \eta(QY) Q \xi - 2\eta(Y) \eta(QX) P \xi - 2\eta(Y) \eta(QX) Q \xi
\]

Compairing tangential, vertical and normal components in (3.4), we get desired results. Hence the lemma is proved.
Lemma 3.2. If $M$ be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then

\[ 2(\overline{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y] \quad (3.5) \]

\[ -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi \]

\[ 2(\overline{\nabla}_Y\phi)X = \nabla_Y\phi X - \nabla_X\phi Y + h(Y, \phi X) - h(X, \phi Y) + \phi [X, Y] \quad (3.6) \]

\[ -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi, \]

for any $X, Y \in D$.

Proof. from Gauss formula (2.16), we have

\[ \overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = \nabla_X\phi Y - \nabla_Y\phi X + h(X, \phi Y) - h(Y, \phi X) \quad (3.7) \]

Also by covariant differentiation, we have

\[ \overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi [X, Y] \quad (3.8) \]

From (3.7) and (3.8), we have

\[ (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = \nabla_X\phi Y - \nabla_Y\phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y] \quad (3.9) \]

Adding (3.9) and (2.14), we have

\[ 2(\overline{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y] \]

\[ -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi \]

Subtracting (3.9) and (2.14), we have

\[ 2(\overline{\nabla}_Y\phi)X = \nabla_Y\phi X - \nabla_X\phi Y + h(Y, \phi X) - h(X, \phi Y) + \phi [X, Y] \]

\[ -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi \]

for any $X, Y \in D$.

Hence lemma is proved.
Corollary 3.1. If $M$ be a $\xi$ - vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then
\[
2(\overline{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y] - 2g(X, Y)\xi \\
2(\overline{\nabla}_Y \phi)X = \nabla_Y \phi X - \nabla_X \phi Y + h(Y, \phi X) - h(X, \phi Y) - \phi [X, Y] - 2g(X, Y)\xi,
\]
for any $X, Y \in D$.

Lemma 3.3. If $M$ be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then
\[
2(\overline{\nabla}_X \phi)Y = A_{\phi X} Y - A_{\phi Y} X + \nabla^\bot_X \phi Y - \nabla^\bot_Y \phi X + \eta(Y) X - \eta(X) Y - \phi [X, Y] \\
- \eta(X) \phi Y - \eta(Y) \phi X - 2\eta(X) \eta(Y) \xi - 2g(X, Y)\xi \\
2(\overline{\nabla}_Y \phi)X = A_{\phi Y} X - A_{\phi X} Y + \nabla^\bot_Y \phi X - \nabla^\bot_X \phi Y + \eta(X) Y - \eta(Y) X + \phi [X, Y] \\
- \eta(X) \phi Y - \eta(Y) \phi X - 2\eta(X) \eta(Y) \xi - 2g(X, Y)\xi,
\]
for any $X, Y \in D^\bot$.

Proof. For any $X, Y \in D^\bot$, from Weingarten formula (2.17), we have
\[
\overline{\nabla}_X \phi Y = -A_{\phi Y} X + \nabla^\bot_X \phi Y - \eta(X) Y - 2\eta(X) \eta(Y) \xi - g(X, Y)\xi
\]
Interchanging $X$ and $Y$ in above, we have
\[
\overline{\nabla}_Y \phi X = -A_{\phi X} Y + \nabla^\bot_Y \phi X - \eta(Y) X - 2\eta(Y) \eta(X) \xi - g(X, Y)\xi
\]
From above two equations, we have
\[
\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = A_{\phi X} Y - A_{\phi Y} X + \nabla^\bot_X \phi Y - \nabla^\bot_Y \phi X + \eta(Y) X - \eta(X) Y
\]
Compairing equations (3.12) and (3.8), we have
\[
(\overline{\nabla}_X \phi) Y - (\overline{\nabla}_Y \phi) X = A_{\phi X} Y - A_{\phi Y} X + \nabla^\bot_X \phi Y - \nabla^\bot_Y \phi X + \eta(Y) X \\
- \eta(X) Y - \phi [X, Y]
\]
Adding (3.13) and (2.14), we get
\[
2(\overline{\nabla}_X \phi) Y = A_{\phi X} Y - A_{\phi Y} X + \nabla^\bot_X \phi Y - \nabla^\bot_Y \phi X + \eta(Y) X - \eta(X) Y - \phi [X, Y]
\]
\[ -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi \]

Subtracting (3.13) from (2.14), we get

\[ 2(\nabla_Y \phi)X = A_{\phi Y}X - A_{\phi X}Y + \nabla^\perp_X \phi Y - \nabla^\perp_Y \phi X - \eta(X)Y - \eta(Y)X + \phi[X,Y] \]

\[ -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi \]

for all \(X, Y \in D^\perp\). Hence the Lemma is proved.

**Corollary 3.2.** If \(M\) be a \(\xi\)-horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold \(\overline{M}\) with quarter symmetric semi metric connection. Then

\[ 2(\nabla_X \phi)Y = A_{\phi Y}X - A_{\phi X}Y + \nabla^\perp_X \phi Y - \nabla^\perp_Y \phi X - \phi[X,Y] - 2g(X,Y)\xi \]

\[ 2(\nabla_Y \phi)X = A_{\phi Y}X - A_{\phi X}Y + \nabla^\perp_Y \phi X - \nabla^\perp_X \phi Y + \phi[X,Y] - 2g(X,Y)\xi, \]

for all \(X, Y \in D^\perp\).

**Lemma 3.4.** If \(M\) be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold \(\overline{M}\) with quarter symmetric semi metric connection. Then

\[ 2(\nabla_X \phi)Y = -A_{\phi Y}X + \nabla^\perp_X \phi Y - \nabla^\perp_Y \phi X - h(Y, \phi X) - \eta(X)Y - \phi[X,Y] \] \hspace{1cm} (3.16)

\[ -\eta(Y)\phi X - \eta(X)\phi Y - 4\eta(X)\eta(Y)\xi - 3g(X,Y)\xi \]

\[ 2(\nabla_Y \phi)X = A_{\phi Y}X - \nabla^\perp_X \phi Y + \nabla^\perp_Y \phi X + h(Y, \phi X) + \eta(X)Y + \phi[X,Y] \] \hspace{1cm} (3.17)

\[ -\eta(X)\phi Y - \eta(Y)\phi X - g(X,Y)\xi, \]

for any \(X \in D\) and \(Y \in D^\perp\).

**Proof.** Let \(X \in D, Y \in D^\perp\), from Gauss formula (2.16), we have

\[ \nabla_Y \phi X = \nabla_Y \phi X + h(Y, \phi X) \]

From Weingarten formula (2.17), we have

\[ \nabla_X \phi Y = -A_{\phi Y}X + \nabla^\perp_X \phi Y - \eta(X)Y - 2\eta(X)\eta(Y)\xi - g(X,Y)\xi \]

Now, from Gauss and Weingarten formula, we have

\[ \nabla_X \phi Y - \nabla_Y \phi X = -A_{\phi Y}X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \eta(X)Y \] \hspace{1cm} (3.18)
Comparing equations (3.18) and (3.8), we have

\[ (\nabla_X \phi)Y - (\nabla_Y \phi)X = -A_{\phi Y}X + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \eta(X)Y \]

\[ -\phi[X,Y] - 2\eta(X)\eta(Y)\xi - g(X,Y)\xi \]

Adding (3.19) and (2.14), we have

\[ 2(\nabla_X \phi)Y = -A_{\phi Y}X + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \eta(X)Y - \phi[X,Y] \]

\[ -\eta(X)\phi Y - \eta(Y)\phi X - 4\eta(X)\eta(Y)\xi - 3g(X,Y)\xi \]

Subtracting (3.19) from (2.14), we find

\[ 2(\nabla_Y \phi)X = A_{\phi Y}X - \nabla_Y \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \eta(Y)Y + \phi[X,Y] - \eta(X)\phi Y \]

\[ -\eta(Y)\phi X - g(X,Y)\xi \]

for any \( X \in D \) and \( Y \in D^\perp \). Hence the Lemma is proved.

**Corollary 3.3.** If \( M \) be a \( \xi \)-horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold \( \overline{M} \) with quarter symmetric semi metric connection. Then

\[ 2(\nabla_X \phi)Y = -A_{\phi Y}X + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \eta(X)Y - \phi[X,Y] \]

\[ -\eta(X)\phi Y - 3g(X,Y)\xi \]

\[ 2(\nabla_Y \phi)X = A_{\phi Y}X - \nabla_Y \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \eta(Y)Y + \phi[X,Y] - \eta(X)\phi Y - g(X,Y)\xi \]

for any \( X \in D \) and \( Y \in D^\perp \).

**Corollary 3.4.** If \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold \( \overline{M} \) with quarter symmetric semi metric connection. Then

\[ 2(\nabla_X \phi)Y = -A_{\phi Y}X + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X,Y] - \eta(Y)\phi X - 3g(X,Y)\xi \]

\[ 2(\nabla_Y \phi)X = A_{\phi Y}X - \nabla_Y \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X,Y] - \eta(Y)\phi X - g(X,Y)\xi \]

for any \( X \in D \) and \( Y \in D^\perp \).
3. Parallel Distribution

**Definition 4.1.** The horizontal (resp., vertical) distribution \( D(\text{resp.}, D^\perp) \) is said to be parallel [3] with respect to the connection on \( M \) if \( \nabla_X Y \in D(\text{resp.}, \nabla_Z W \in D^\perp) \) for any vector field \( X, Y \in D(\text{resp.}, W, Z \in D^\perp) \)

**Theorem 4.1.** If \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold \( \overline{M} \) with quarter symmetric semi metric connection. Then

\[
h(X, \phi Y) = h(Y, \phi X), \tag{4.1}
\]

for any \( X, Y \in D \).

**Proof.** Using parallelism of horizontal distribution \( D \), we have

\[
\nabla_X (\phi Y) \in D \quad \text{and} \quad \nabla_Y (\phi X) \in D
\]

for any \( X, Y \in D \).

From (3.2), we have

\[
Bh(X, Y) = g(X, Y) \xi \tag{4.2}
\]

From (2.12) and (4.2), we have

\[
Ch(X, Y) = \phi h(X, Y) - g(X, Y) \xi \tag{4.3}
\]

Now, from (3.3), we have

\[
h(X, \phi Y) + h(Y, \phi X) = 2Ch(X, Y)
\]

Using (4.3) in above, we have

\[
h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X, Y) - 2g(X, Y) \xi \tag{4.4}
\]

Replacing \( Y \) by \( \phi Y \) in (4.4) and using (2.1), we have

\[
h(X, Y) + h(\phi Y, \phi X) = 2\phi h(X, \phi Y) - 2g(X, \phi Y) \xi \tag{4.5}
\]

Similarly, replacing \( X \) by \( \phi X \) in (4.4) and using (2.1), we have

\[
h(\phi X, \phi Y) + h(Y, X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y) \xi \tag{4.6}
\]
Comparing (4.5) and (4.6), we have
\[ \phi h(X, \phi Y) - g(X, \phi Y)\xi = \phi h(\phi X, Y) - g(\phi X, Y)\xi \]
\[ \phi^2 h(X, \phi Y) - g(X, \phi Y)\phi \xi = \phi^2 h(\phi X, Y) - g(\phi X, Y)\phi \xi \]

Using (2.2), we have
\[ h(X, \phi Y) = h(\phi X, Y) \]
for any \( X, Y \in D \). Hence the theorem is proved.

**Theorem 4.2.** If \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold \( \tilde{M} \) with quarter symmetric semi metric connection. If the distribution \( D^\perp \) is parallel with respect to the connection on \( M \), then
\[ A_{\phi X}Y + A_{\phi Y}X \in D^\perp, \quad (4.7) \]
for any \( X, Y \in D^\perp \).

**Proof.** Let \( X, Y \in D^\perp \), then from Weingarten formula (2.17), we have
\[ (\overline{\nabla}_X \phi)Y + \phi(\overline{\nabla}_Y \phi) = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \eta(X)Y - 2\eta(X)\eta(Y)\xi - g(X, Y)\xi \]  
(4.8)

Using Gauss equation (2.16) in (4.8), we have
\[ (\overline{\nabla}_X \phi)Y = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \phi(\overline{\nabla}_X \phi Y) - \phi h(X, Y) - \eta(X)Y - 2\eta(X)\eta(Y)\xi \]
(4.9)

\[-g(X, Y)\xi\]

Interchanging \( X \) and \( Y \), we have
\[ (\overline{\nabla}_Y \phi)X = -A_{\phi X}Y + \nabla_Y^\perp \phi X - \phi(\overline{\nabla}_Y \phi X) - \phi h(X, Y) - \eta(Y)X - 2\eta(X)\eta(Y)\xi \]
(4.10)

\[-g(X, Y)\xi\]

Adding (4.9) and (4.10), we get
\[ (\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = -A_{\phi Y}X - A_{\phi X}Y + \nabla_X^\perp \phi Y + \nabla_Y^\perp \phi X - \phi(\overline{\nabla}_X \phi Y) \]
\[ -\phi(\overline{\nabla}_Y \phi X) - 2\phi h(X, Y) - \eta(X)Y - \eta(Y)X \]
\[ -4\eta(X)\eta(Y)\xi - 2g(X, Y)\xi \]  
(4.11)
Using (2.14) in (4.11), we have
\[-\eta(X)\phi Y - \eta(Y)\phi X = -A_{\phi Y}X - A_{\phi X}Y + \nabla_{\phi Y}^\perp \phi Y + \nabla_{\phi X}^\perp \phi X - \phi(\nabla_X Y) \tag{4.12}\]
\[-\phi(\nabla_Y X) - 2\phi h(X,Y) - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi\]

Taking inner product with \(Z \in D\) in (4.12), we have
\[-\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) = -g(A_{\phi Y}X, Z) - g(A_{\phi X}Y, Z) + g(\nabla_{\phi Y}^\perp \phi Y, Z)\]
\[+ g(\nabla_{\phi X}^\perp \phi X, Z) - g(\phi(\nabla_X Y), Z) - g(\phi(\nabla_Y X), Z) - 2g(\phi h(X,Y), Z)\]
\[-\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - 2\eta(X)\eta(Y)g(\xi, Z)\]

If \(D^\perp\) is parallel then \(\nabla_X Y \in D^\perp\) and \(\nabla_Y X \in D^\perp\), so that from above
\[g(A_{\phi Y}X + A_{\phi X}Y, Z) = 0 \tag{4.13}\]

Consequently, we have
\[A_{\phi Y}X + A_{\phi X}Y \in D^\perp \tag{4.14}\]

for any \(X, Y \in D^\perp\). Hence theorem is proved.

**Definition 4.2.** A CR-submanifold is said to be mixed-totally geodesic if \(h(X,Y) = 0\) for all \(X \in D\) and \(Y \in D^\perp\).

**Definition 4.3.** A Normal vector field \(N \neq 0\) is called \(D - parallel\) normal section if \(\nabla_X N = 0\) for all \(X \in D\).

**Theorem 4.3.** Let \(M\) be a mixed totally geodesic \(\xi\)-vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold \(\overline{M}\) with quarter symmetric semi metric connection. Then the normal section \(N \in \phi D^\perp\) is \(D - parallel\) if and only if \(\nabla_X \phi N \in D\), for all \(X \in D\).

**Proof.** Let \(N \in \phi D^\perp\), for all \(X \in D\) and \(Y \in D^\perp\) then from (3.2), we have
\[-2\eta(X)\eta(Y)Q \xi - 2g(X,Y)Q \xi + 2B h(X,Y) = Q \nabla_X (\phi PY) + Q \nabla_Y (\phi PX)\]
\[-QA_{\phi Y}X - QA_{\phi X}Y - \eta(X)QY - \eta(Y)QX - g(X,QY)Q \xi - g(Y,QX)Q \xi\]
\[-2\eta(X)\eta(QY)Q \xi - 2\eta(Y)\eta(QX)Q \xi\]
As $M$ is a $\xi$-vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection, so we have from above

$$2Bh(X,Y) = Q\nabla_Y(\phi X) - QA_{\phi Y}X$$

(4.15)

Using definition of mixed geodesic CR-submanifold, we have

$$Q\nabla_Y(\phi X) - QA_{\phi Y}X = 0$$

(4.16)

$$Q\nabla_Y(\phi X) = QA_{\phi Y}X$$

(4.17)

As $Q\nabla_Y(\phi X) = 0$, for $X \in D$.

In particular, we have

$$Q\nabla_Y X = 0$$

(4.18)

From (3.3), we have

$$-\eta(X)\phi QY - \eta(Y)\phi QX + \phi Q(\nabla_X Y) + \phi Q(\nabla_Y X) + 2Ch(X,Y) =$$

$$h(X,\phi PY) + h(Y,\phi PX) + \nabla^\perp_X(\phi QY) + \nabla^\perp_Y(\phi QX)$$

Using (4.18) in above, we have

$$\phi Q\nabla_X Y = \nabla^\perp_X(\phi Y)$$

That is

$$\phi Q\nabla_X(\phi N) = \nabla^\perp_X(\phi^2 N)$$

$$\phi Q\nabla_X(\phi N) = \nabla^\perp_X(N + \eta(N)\xi)$$

$$\phi Q\nabla_X(\phi N) = \nabla^\perp_X(N)$$

$$\phi Q\nabla_X(\phi N) = \nabla^\perp_X N$$

(4.19)

Then by definition of parallelism of $N$, we have

$$\phi Q\nabla_X(\phi N) = 0$$

Consequently, we have

$$\nabla_X(\phi N) \in D$$

(4.20.)

for all $X \in D$. 

Converse part is easy consequence of (4.20).

Conflict of Interests
The authors declare that there is no conflict of interests.

REFERENCES

