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THE ALEKSANDROV PROBLEM IN QUASI CONVEX N-NORMED LINEAR SPACES

XINKUN WANG*, MEIMEI SONG

College of Science, Tianjin University of Technology, Tianjin 300384, China

Abstract. We prove that the Aleksandrov problem holds without the condition "n-Lipschitz mapping" in quasi convex n-normed linear spaces and also we show that the Mazur-Ulam theorem holds in quasi convex n-normed linear space.

Keywords: Aleksandrov problem; Mazur-Ulam theorem; nDOPP; n-isometry.

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1. Introduction

Let E and F be metric spaces. A mapping $f : E \to F$ is called an isometry if f satisfies

$$d_F(f(x), f(y)) = d_E(x,y)$$

for all $x, y \in E$, where $d_E(,)$ and $d_F(,)$ denote the metric in the space E and F, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; ie, for all $x, y \in E$ with $d_E(x,y) = r$, we have $d_F(f(x), f(y)) = r$. Then r is called a conservative distance for the mapping f. The classical Mazur-Ulam theorem states that every surjective isometry between normed spaces is

*Corresponding author

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a linear mapping up to translation. In 1970, Aleksandrov [1] posed the following question : ”Whether or not a mapping with distance one preserving property is an isometry? ” It is called the *Aleksandrov problem*. The Aleksandrov problem has been investigated in several papers [4]-[5]. Misiak [8]-[9] defined n-normed spaces and investigeted the properties of these spaces. The concept of an n-normed spaces is a generlization of the concept of a normed spaces and a 2-normed space.

Chu et al. [3] defined the concept of n-isometry which is suitable for representing the notion of n-distance preserving mappings in linear n-normed spaces and studied the Aleksandrov problem in linear n-normed spaces. and proved also that the Rassias and Šemrl theorem holds under some conditions in linear 2-normed spaces as follows:

Theorem 1.1.[3] Let f be a n-Lipschitz mapping with the n-Lipschitz constant $K \leq 1$. Assume that if x, y and z are m-colinear, then $f(x), f(y)$ and $f(z)$ are m-colinear, $m = 2, n$, and that f satisfies (nDOPP). Then f is a n-isometry.

Zheng and Ren [14] defined the quasi convex linear space and studied the Aleksandrov problem.

In this paper, We prove that the Aleksandrov problem holds without the condition ”n-Lipschitz mapping” in quasi convex n-normed linear spaces and also we show that the Mazur-Ulam theorem holds in quasi convex n-normed linear space.

2. Preliminaries

In the remainder of this introduction, we will recall some definitions and give some Lemmas about them in quasi convex n-normed linear space.

Definition 2.1. Let E be a real linear space that has dimension greater than one and $\| \cdot, \dots, \cdot \|$ be a function from E^n into R. Then ($E, \| \cdot, \dots, \cdot \|$) is called a quasi convex n-normed linear space if

(a) $\| x_1, \dots, x_n \| = 0 \Leftrightarrow x_1, \dots, x_n$ are linearly dependent.

(b) $\| x_1, \dots, x_n \| = \| x_{j_1}, \dots, x_{j_n} \|$ for every permutation $(j_1, \dots, j_n)$ of $(1, \dots, n)$.

(c) $\| \alpha x_1, \dots, x_n \| = | \alpha | \| x_1, \dots, x_n \|$.

(d) $\| tx+(1-t)y, x_2, \ldots, x_n \| \leq max\{\| x, x_2, \ldots, x_n \|, \| y, x_2, \ldots, x_n \|\}$.

for any $\alpha \in R, t \in [0,1]$ and $x, y, x_1, \ldots, x_n \in E$. The function $\| \cdot, \ldots, \cdot \|$ is called the quasi convex n-norm on E.

Definition 2.2.[3] A mapping $f : E \to F$ satisfies the distance one preserving property (briefly nDOPP), if for all $x_i \in E$, $i = 0,1,2,\ldots,n$, $\| x_1 - x_0, \ldots, x_n - x_0 \| = 1$ implies $\| f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0) \| = 1$.

Definition 2.3.[3] A mapping $f : E \to F$ is said to be an n-isometry if for all $x_1, \ldots, x_n, x_0 \in E$, it satifies

$$\| x_1 - x_0, \ldots, x_n - x_0 \| = \| f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0) \| .$$

Definition 2.4.[3] The points $x_0, x_1, \ldots, x_n$ of E are said to be n-collinear, if every $i, \{x_i - x_j \mid 0 \leq i \neq j \leq n\}$ is linearly dependent.

Definition 2.5.[4] We say that a mapping $f : E \to F$ preserves 2-collinearity, if $x, y, z \in E$ are collinear, then $f(x), f(y), f(z)$ are collinear.

Definition 2.6.[14] A mapping $f : E \to F$ on two real linear spaces E and F is called an affine mapping, if for all $x, y \in E$ and $\lambda \in [0,1]$ satifies

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y).$$

Definition 2.7.[3] We call f is a n-*Lipschitz* mapping if there is a $k \geq 0$ such that

$$\| f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0) \| \leq k \| x_1 - x_0, \ldots, x_n - x_0 \|$$

for all $x, y, p, q \in E$. In this case, the constant k is called the n-*Lipschitz* constant.

Lemma 2.8. Let E be a quasi convex n-nomed linear space with $dimE \geq n$, for $y_i, x_i \in E, t_i > 0, \sum_{i=1}^{n} t_i = 1 (i = 1,2,\cdots,n)$, we have

$$\| \sum_{i=1}^{n} t_i y_i, x_2, \ldots, x_n \| \leq max\{\| y_1, x_2, \ldots, x_n \|, \| y_2, x_2, \ldots, x_n \|, \\ \cdots, \| y_{n-1}, x_2, \ldots, x_n \|, \| y_n, x_2, \ldots, x_n \|\}$$

The next result follows easily from [6][Lemma 8].

Lemma 2.9. Let E be a n-normed linear space with $dimE > n$, suppose $0 <\| x_1 - y_1, \ldots, x_n - y_n \| \leq 2r$, for any $r > 0$, and $x_i, y_i \in E$, $i = 1, 2, \ldots, n$, then there exists $z \in E$, such that $\| z - y_1, \ldots, x_n - y_n \| = r$ and$\| x_1 - z, \ldots, x_n - y_n \| = r$.

3. Main results

In this section, let E and F be quasi convex n-normed linear spaces with dimension greater than n.

Lemma 3.1. *Let E and F be two quasi convex n-normed linear spaces , if $f : E \to F$ satisfies (nDOPP) and preserves 2-collinearity, then f is injective and for all $x, y \in E$, we have*

$$f(\frac{y+x}{2}) = \frac{f(y)+f(x)}{2}$$

Proof. *we prove that f is injective. Since $dimE > n$, for any $x, y \in E$ with $x \neq y$, there exists $x_i \in E$, $i = 1, 2, \ldots, n-1$ such that $\| x - y, x_1 - y, \cdots, x_{n-1} - y \| = 1$.*
Since f satisfies (nDOPP), thus

$$\| f(x) - f(y), f(x_1) - f(y), \cdots, f(x_{n-1}) - f(y) \| = 1.$$

Hence $f(x) \neq f(y)$, So we prove f is injective.
On the other hand, let $z = \frac{y+x}{2}$ for distinct $y, x \in E$, then $z - y = x - z$. Since f preserves 2-collinearity, there exists a real numble $t \neq 0$ such that $f(z) - f(y) = t(f(z) - f(x))$.
Since $dimE > n$, there exists $x_i \in E$, $i = 1, 2, \ldots, n-1$ with

$$\| z - y, 2x_1 - 2y, \cdots, x_{n-1} - y \| = 1.$$

Then

$$\| f(z) - f(y), f(2x_1) - f(2y), \cdots, f(x_{n-1}) - f(y) \| = 1. \tag{1}$$

Because f is injective, and it follows from the above equation(1) we conclude that $t = -1$. Thus $f(z) - f(y) = f(x) - f(z)$ and

$$f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}.$$

This completes the proof.

Theorem 3.2. *Let E and F be two quasi convex n-normed linear spaces , if $f : E \to F$ is a n-isometry, then f is affine.*

Proof. *Assume that x,y and z are 2-colinear, then f preserves collinearity by the condition that $\|x-z,y-z\| = 0$ implies $\|f(x)-f(z), f(y)-f(z)\| = 0$. Let $g(x) = f(x)-f(0)$. It suffices to prove that the mapping g is linear. Since g satisfies (DOPP) and $g(0) = 0$. From Lemma 2.1, the mapping g is $\mathbb{Q}$-linear. Let $\xi \in R^+$ with $\xi \neq 1$ and $x \in E$. Since $0,x,\xi x$ are collinear, g preserves collinearity and also $g(0) = 0$, so there exists a real number η such that*

$$g(\xi x) = \eta g(x).$$

For any $x \in E$ with $x \neq 0$, there exists $x_i \in E$, $i = 1,2,\ldots,n-1$ such that $\|x,x_1,\cdots,x_{n-1}\| = 1$. Hence we obtain

$$\begin{aligned} \xi = \|\xi x, x_1, \cdots, x_{n-1}\| &= \|g(\xi x), g(x_1), \cdots, g(x_{x_{n-1}})\| = \|\eta g(x), g(x_1), \cdots, g(x_{x_{n-1}})\| \\ &= |\eta| \|g(x), g(x_1), \cdots, g(x_1)\| = |\eta|. \end{aligned}$$

Thus $\eta = \pm\xi$. While $\eta = -\xi$, that is to say $g(\xi x) = -\xi g(x)$, it deduces that

$$\begin{aligned} |\ 1-\xi\ | &= \|x-\xi x, x_1, \cdots, x_{n-1}\| \\ &= \|g(x)-g(\xi x), g(x_1), \cdots, g(x_{x_{n-1}})\| \\ &= \|g(x)+\xi g(x), g(x_1), \cdots, g(x_{x_{n-1}})\| \\ &= (1+\xi)\|g(x), g(x_1), \cdots, g(x_{x_{n-1}})\| \\ &= 1+\xi. \end{aligned}$$

So $\xi = 0$, while it conflict with $\xi \in R^+$. Hence we get $\xi = \eta$, that is to say $g(\xi x) = \xi g(x)$. This completes the proof.

Theorem 3.3. *Let E and F be two quasi convex n-normed linear spaces. If $f : E \to F$ satisfies (nDOPP) and preserves 2-collinearity , then f is an affine n-isometry.*

Proof.*(1) we prove f preserves distance $\frac{m}{k}$. Let $\| x_1-x_0, x_2-x_0, \cdots, x_n-x_0 \| = \frac{1}{k}$ with $x_i \in E$, $i = 0,1,2,\ldots,n$, we define*

$$\omega_i = x_1 + i(x_0 - x_1)$$

Then

$$\omega_i = \frac{\omega_{i-1} + \omega_{i+1}}{2}, \quad \forall i = 1, \cdots, k-1.$$

According to Lemma 2.1, we have

$$f(\omega_i) = \frac{f(\omega_{i-1}) + f(\omega_{i+1})}{2}, \quad \forall i = 1, \cdots, k-1.$$

That is

$$f(\omega_{i+1}) - f(\omega_i) = f(\omega_i) - f(\omega_{i-1}), \quad \forall i = 1, \cdots, k-1.$$

Hence

$$\begin{aligned} f(\omega_k) - f(x) &= f(\omega_k) - f(\omega_{k-1}) + f(\omega_{k-1}) - f(\omega_{k-2}) + \cdots + f(\omega_1) - f(\omega_0) \\ &= k(f(\omega_1) - f(\omega_0)) = k(f(y) - f(x)). \end{aligned}$$

Since $\|\omega_k - x_1, x_2 - x_0, \cdots, x_n - x_0\| = 1$*,*

$$\begin{aligned} & k\|f(x_1) - f(x_0), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0)\| \\ = \; & \|f(\omega_k) - f(x_1), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0)\| = 1. \end{aligned}$$

Therefore $\|f(x_1) - f(x_0), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0)\| = \frac{1}{k}$.

Next, we shall show that f *preserves distance* $\frac{m}{k}$ *for integers* m, k*. Let* $\|x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0\| = \frac{m}{k}$ *with* $x_i \in E$*,* $i = 0, 1, 2, \ldots, n$*. We define*

$$z_i := x + \frac{i}{m}(x_1 - x_0), \quad \forall i = 0, 1, \cdots, k.$$

Then

$$z_i = \frac{z_{i-1} + z_{i+1}}{2}, \quad \forall i = 1, \cdots, k-1.$$

By the same method as above,

$$f(x_1) - f(x_0) = f(z_m) - f(z_0) = m(f(z_1) - f(z_0)).$$

Note that $\|z_1 - z_0, x_2 - x_0, \cdots, x_n - x_0\| = \frac{1}{k}$ *and* f *preserves distance* $\frac{1}{k}$,

$$\begin{aligned} & \|f(x_1) - f(x_0), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0)\| \\ = & \|m(f(z_1) - f(z_0)), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0)\| \\ = & \frac{m}{k}. \end{aligned}$$

(2) we prove that $\| f(x_1) - f(x_0), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0) \| \leq \| x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0 \|$ *for any* $x_i \in E$, $i = 0, 1, 2, \ldots, n$. *when* $\| x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0 \| = 0$, *the theorem is successful obviously.*

Suppose $x_i \in E$, $i = 0, 1, 2, \ldots, n$ *with* $\| x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0 \| > 0$ *and* $k, m \in N$, *such that*

$$\frac{m-1}{k} \leq \| x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0 \| \leq \frac{m}{k}$$

Set

$$\omega_i = x_1 + \frac{i}{k} \frac{x_0 - x_1}{\| x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0 \|}$$

and also define $\omega_m = x_0$. *Then*

$$\|\omega_i - \omega_{i-1}, x_2 - x_0, \cdots, x_n - x_0\| = \frac{1}{k}, \quad i = 1, \cdots, m-2.$$

Moreover,

$$\begin{aligned} 0 < & \|\omega_m - \omega_{m-2}, x_2 - x_0, \cdots, x_n - x_0\| \\ = & \|\frac{m-2}{k} \frac{x_0 - x_1}{\|x_0 - x_1, x_2 - x_0, \cdots, x_n - x_0\|} + (x_0 - x_1), x_2 - x_0, \cdots, x_n - x_0\| \\ = & \|x_0 - x_1, x_2 - x_0, \cdots, x_n - x_0\| - \frac{m-2}{k} \\ \leq & \frac{m}{k} - \frac{m-2}{k} = \frac{2}{k}. \end{aligned}$$

From Lemma 2.9, we can choose $\omega_{m-1} \in E$, *such that*

$$\|\omega_{m-1} - \omega_{m-2}, x_2 - x_0, \cdots, x_n - x_0\| = \|\omega_{m-1} - \omega_m, x_2 - x_0, \cdots, x_n - x_0\| = \frac{1}{k}$$

Therefore, for $i = 0, 1, \cdots, m$, *we have*

$$\|f(\omega_i) - f(\omega_{i-1}), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0)\| = \frac{1}{k}.$$

From Lemma 2.8, we can obtain

$$\begin{aligned}
& \|f(x_1)-f(x_0), f(x_2)-f(x_0), \cdots, f(x_n)-f(x_0)\| \\
= & \|f(\omega_0)-f(\omega_m), f(x_2)-f(x_0), \cdots, f(x_n)-f(x_0)\| \\
= & \|\sum_{i=0}^{m-1}(f(\omega_i)-f(\omega_{i+1})), f(x_2)-f(x_0), \cdots, f(x_n)-f(x_0)\| \\
= & m\|\sum_{i=0}^{m-1}\frac{1}{m}(f(\omega_i)-f(\omega_{i+1})), f(x_2)-f(x_0), \cdots, f(x_n)-f(x_0)\| \\
\leq & m max\{\|f(\omega_i)-f(\omega_{i+1}), f(x_2)-f(x_0), \cdots, f(x_n)-f(x_0)\| : i=0,1,\cdots,m-1\} \\
\leq & \frac{m}{k}.
\end{aligned}$$

Hence $\|f(x_1)-f(x_0), f(x_2)-f(x_0), \cdots, f(x_n)-f(x_0)\| \leq \|x_1-x_0, x_2-x_0, \cdots, x_n-x_0\|$.

(3) we will show that f is a generalized n-isometry. Otherwise, there exists $x_i \in E$, $i = 0,1,2,\ldots,n$ and $m \in \mathbb{N}$ such that $0 < \|x_1-x_0, x_2-x_0, \cdots, x_n-x_0\| < m$ *and*

$$\|f(x_1)-f(x_0), f(x_2)-f(x_0), \cdots, f(x_n)-f(x_0)\| < \|x_1-x_0, x_2-x_0, \cdots, x_n-x_0\|.$$

Set $z := x_1 + \frac{m(x_0-x_1)}{\|x_1-x_0, x_2-x_0, \cdots, x_n-x_0\|}$. *Then we obtain that*

$$\begin{aligned}
\|z-x_1, x_2-x_0, \cdots, x_n-x_0\| &= m \\
\|z-x_0, x_2-x_0, \cdots, x_n-x_0\| &= m-\|x_1-x_0, x_2-x_0, \cdots, x_n-x_0\|.
\end{aligned}$$

Since f preserves 2-collinearity, there exists a real number t such that

$$f(z)-f(x_1) = t(f(x_0)-f(x_1)).$$

Then $f(z)-f(x_0)=(t-1)(f(x_0)-f(x_1))$. By (1), f preserves distance m. So we have

$$\begin{aligned} m &= \|f(z)-f(x_1),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ &= |t|\|f(x_1)-f(x_0),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ &\leq |t-1|\|f(x_1)-f(x_0),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ &+ \|f(x_2)-f(y),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ &= \|f(z)-f(x_0),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ &+ \|f(x_1)-f(x_0),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ &< m-\|x_1-x_0,x_2-x_0,\cdots,x_n-x_0\|+\|x_1-x_0,x_2-x_0,\cdots,x_n-x_0\|=m, \end{aligned}$$

which is a contraction. By Theorem 3.2, the proof of the theorem is finished.

Theorem 3.4. *Let X and Y be two quasi convex n-normed linear spaces. If $f: X\to Y$ is an affine such that it preserves all areas $m<1$. Then f is an n-isometry.*

Proof. *Since $dimX\geq n$, there exist $x_0,x_1,x_2,\cdots,x_n\in X$ such that $\|x_1-x_0,x_2-x_0,\cdots,x_n-x_0\|\neq 0$, also $\lambda x_1+(1-\lambda)x_0\in X$, for all $\lambda\in[0,1]$. we can choose $\lambda_i\in[0,1]$ such that $0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n=1$ and $\|p_k-p_{k-1},x_2-x_0,\cdots,x_n-x_0\|<1$, while $p_k=\lambda_k x_1+(1-\lambda_k)x_0$. Since f preserves all areas $m<1$, so we have*

$$\|f(p_k)-f(p_{k-1}),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\|=\|p_k-p_{k-1},x_2-x_0,\cdots,x_n-x_0\|.$$

By the condition f is an affine, we can get $f(p_k)=\lambda_k f(x_1)+(1-\lambda_k)f(x_0)$. According to Remark, we obtain

$$\begin{aligned} & \|f(x_1)-f(x_0),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ = & \|\sum_{i=1}^{n}(f(p_k)-f(p_{k-1})),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ = & \sum_{i=1}^{n}\|f(p_k)-f(p_{k-1}),f(x_2)-f(x_0),\cdots,f(x_n)-f(x_0)\| \\ = & \sum_{i=1}^{n}\|p_k-p_{k-1},x_2-x_0,\cdots,x_n-x_0\|=\|\sum_{i=1}^{n}(p_k-p_{k-1}),x_2-x_0,\cdots,x_n-x_0\| \\ = & \|x_1-x_0,x_2-x_0,\cdots,x_n-x_0\| \end{aligned}$$

This proves that f is n-isometry.

Conflict of Interests

The authors declare that there is no conflict of interests.

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