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BAYES ESTIMATION FOR THE PARAMETERS OF THE WEIBULL-GEOMETRIC DISTRIBUTION BASED ON PROGRESSIVE FIRST FAILURE CENSORED DATA

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Abstract. In this paper, we deal with the problem of estimating the parameters of the Weibull-Geometric distribution based on progressive first-failure censoring scheme. The maximum likelihood and Bayes methods of estimation are used for this purpose. The Monte Carlo Integration (MCI) technique is used for computing the Bayes estimates. The Bayes estimates of the parameters are compared with their corresponding maximum likelihood estimates via Monte Carlo simulation study.

Keywords: Weibull-Geometric distribution; progressive first-failure censoring scheme; Monte Carlo Integration.

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1. Introduction

In experiments where failure information is available only on a part of the sample, the data are said to be censored data. Many types of censoring schemes are well known. One of the most common censored test is type II censoring. It is noted that one can use type II censoring for saving time and money. However, when the lifetimes of products are very high, the experimental time of a type II censoring life test can be still too long. A generalization of type II censoring

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is progressive type II censoring, which is useful when the loss of live test units at points other than the termination point is unavoidable. Recently, the estimation of Parameters from different lifetime distribution based on progressive type II censored samples is studied by several authors including Gupta et al. [9], Childs and Balakrishnan [8], Tse et al. [18], Ali Mousa and Jaheen [3], Ng et al. [13], Wu and Chang [19], Balakrishnan et al. [6], Wu [20], Soliman [16], and Sarhan and Abuammoh [15].

Johnson [12] in (1964) described a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units, and then run all the test units simultaneously until occurrence the first failure in each group. Such a censoring scheme is called a first-failure censoring scheme. If an experimenter desires to remove some sets of test units before observing the first failures in these sets this life test plan is called a progressive first-failure censoring scheme which recently has been introduced by Wu and Kus [21] in (2009).

2. Progressive first-failure censoring scheme

First-failure censoring is combined with progressive censoring and can be described as: Suppose that n independent groups with k items within each group are put on a life test, R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure (say $X_{1:m:n:k}$) has occurred, R_2 groups and the group in which the second first failure is observed are randomly removed from the test when the second failure (say $X_{2:m:n:k}$) has occurred, and finally R_m $(m \leq n)$ groups and the group in which the m-th first failure is observed are randomly removed from the test as soon as the m-th failure (say $X_{m:m:n:k}$) has occurred.The $X_{1:m:n:k} < X_{2:m:n:k} < ... < X_{m:m:n:k}$ are called progressively first-failure censored order statistics with the progressive censoring scheme $R = (R_1, R_2, ..., R_m)$. It is clear that m is number of the first-failure observed $(1 < m \leq n)$ and $\sum_{i=1}^{m} R_i + m = n$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with distribution function F(x) and probability density function f(x), the joint probability density function for $X_{1:m:n:k}, X_{2:m:n:k}, \ldots, X_{m:m:n:k}$ is given by

$$f_{1,2,\ldots,m}(x_{1:m:n:k},x_{2:m:n:k},\ldots,x_{m:m:n:k}) = Ck^m\prod_{i=1}^{m} f(x_{i:m:n:k})(1-F(x_{i:m:n:k}))^{k(R_i+1)-1}, \tag{2.1}$$

$$0 < x_{1:m:n:k} < x_{2:m:n:k} < \ldots < x_{m:m:n:k} < \infty,$$

where

$$C = n(n-R_1-1)(n-R_1-R_2-2)\ldots(n-R_1-R_2-\ldots-R_{m-1}-m+1).$$

Special cases

It is clear from (2.1) that the progressive first-failure censored scheme containing the following censoring schemes as special cases:

(1) The first-failure censored scheme when $R=(0,0,\ldots,0)$.

(2) The progressive type-II censored order statistics if $k=1$.

(3) Usually type II censored order statistics when $k=1$ and $R=(0,0,\ldots,n-m)$.

(4) The complete sample case when $k=1$ and $R=(0,0,\ldots,0)$.

Also, It should be noted that $X_{1:m:n:k}, X_{2:m:n:k}, \ldots, X_{m:m:n:k}$ can be viewed as a progressive type-II censored sample from a population with distribution function $1-(1-F(x))^k$. For this reason, results for progressive type-II censored order statistics can be extended to progressively first-failure censored order statistics easily. Also, the progressive first-failure censored scheme has advantages in terms of reducing the test time, in which more items are used, but only m of $n\times k$ items are failures.

3. The Weibull-Geometric model

The Weibull-Geometric (W-G) distribution was first introduced by Barreto-Souzaa et. al. [7] in (2011) and with a different parametrization, the same law has been studied by Tojeiro et al. [17], which called it the complementary Weibull-geometric distribution. The W-G distribution generalizes the exponential-geometric (EG) distribution (proposed by Adamidis and Loukas [2]) and Weibull distributions. The hazard function of the EG distribution is monotone

decreasing but the hazard function of the W-G distribution can take more general forms. Unlike the Weibull distribution, the W-G distribution is useful for modeling unimodal failure rates.
The W-G distribution with the scale parameter $\beta > 0$, shape parameter $\alpha > 0$ and $p \in (0,1)$ has the following probability density function (pdf) and cumulative distribution function (cdf)

$$f(x;\alpha,\beta,p) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}\{1-pe^{-(\beta x)^{\alpha}}\}^{-2}, \quad x>0, \tag{3.1}$$

$$F(x) = (1-e^{-(\beta x)^{\alpha}})(1-pe^{-(\beta x)^{\alpha}})^{-1}, \quad x>0, \tag{3.2}$$

respectively.
As it can be seen from (3.1) when p approaches zero we obtain the two-parameter Weibull distribution. Another special case is obtained for $\alpha = 1$, which corresponds to the exponential-geometric (EG) distribution with parameter $\beta > 0$.
The corresponding survival or reliability function is

$$S(x) = 1-F(x) = ((1-p)e^{-(\beta x)^{\alpha}})(1-pe^{-(\beta x)^{\alpha}})^{-1}, \quad x>0, \tag{3.3}$$

and the hazard rate function is given by

$$H(x) = \frac{f(x)}{S(x)} = \alpha\beta^{\alpha}x^{\alpha-1}\{1-pe^{-(\beta x)^{\alpha}}\}^{-2}, \quad x>0. \tag{3.4}$$

The hazard function (3.4) is decreasing for $0 < \alpha \leq 1$. However, for $\alpha > 1$ it can take different forms.
Hamedani and Ahsanullah [10] presented various characterizations of the W-G distribution. Jodra and Jimnez-Gamero [11] obtain explicit expressions for the moments of order statistics from the half-logistic distribution, the Weibull-geometric distribution and the long-term Weibull-geometric distribution.
In this paper, the maximum likelihood and Bayes methods of estimation are used for estimating the three unknown parameters α, β and p of the model based on progressive first-failure censored data. Two cases are considered. In the first case, the parameter α is assumed to be known while in the second one, the three parameters α, β and p are assumed to be unknown. Monte Carlo simulation study is used to compare the different estimates.

4. Maximum likelihood estimation

In this section we derive the maximum likelihood estimates (MLEs) of the unknown parameters α, β and p of the W-G(α, β, p) with pdf and cdf given by (3.1) and (3.2), respectively. Thus, from (2.1) the likelihood function for progressive first-failure censored scheme take the following form

$$L(\alpha,\beta,p;\underline{x}) = Ck^m \prod_{i=1}^{m} \alpha\beta^{\alpha}(1-p)^{k(R_i+1)} x_i^{\alpha-1} e^{-(\beta x_i)^{\alpha}(k(R_i+1))} \times M(1-pe^{-(\beta x_i)^{\alpha}}M)^{-(k(R_i+1)+1)}, \tag{4.1}$$

where

$$C = n(n-R_1-1)(n-R_1-R_2-2)...(n-R_1-R_2-...-R_{m-1}-m+1)$$

The logarithm of (4.1) can be written as

$$\begin{aligned} l(\alpha,\beta,p;\underline{x}) = mM[\ln\alpha + \alpha\ln\beta + \ln(1-p)M] + (\alpha-1)\sum_{i=1}^{m}\ln x_i \\ -\sum_{i=1}^{m}(\beta x_i)^{\alpha}M(k(R_i+1)M) + \ln(1-p)\sum_{i=1}^{m}M(k(R_i+1)-1M) \\ -\sum_{i=1}^{m}M(k(R_i+1)+1M)\ln(1-pe^{-(\beta x_i)^{\alpha}}). \end{aligned} \tag{4.2}$$

Taking the derivatives with respect to α, β and p of (4.2) and putting them equal to zero we get

$$\frac{\partial l}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\ln(\beta x_i) - \sum_{i=1}^{m}(\beta x_i)^{\alpha}\lambda_i\ln(\beta x_i) + p\sum_{i=1}^{m}(\beta x_i)^{\alpha}\eta_i\ln(\beta x_i) = 0, \tag{4.3}$$

$$\frac{\partial l}{\partial\beta} = \frac{m\alpha}{\beta} - \alpha\beta^{\alpha-1}\sum_{i=1}^{m}x_i^{\alpha}\lambda_i - \alpha p\beta^{\alpha-1}\sum_{i=1}^{m}x_i^{\alpha}\eta_i = 0, \tag{4.4}$$

$$\frac{\partial l}{\partial p} = \frac{-1}{1-p}M[m + \sum_{i=1}^{m}(\lambda_i-1)M] + \sum_{i=1}^{m}\eta_i = 0. \tag{4.5}$$

where

$$\lambda_i = k(R_i+1),$$

$$\eta_i = e^{-(\beta x_i)^\alpha} M(\lambda_i + 1M)M(1 - pe^{-(\beta x_i)^\alpha} M)^{-1}.$$

We have here two cases. In the first case we considered α is known and solved equations (4.4) and (4.5) together, numerically. In the second one we considered the three parameters as unknown and by solving the non-linear equations (4.3), (4.4) and (4.5)together, numerically, we get the maximum likelihood estimates of α, β and p.

5. Bayesian estimation

Assume that the prior densities for the parameters α and β are the Gamma distribution such that α is dependent on β, Gamma(u, β) and Gamma(a, b), respectively. Thus the proposed priors for parameters α and β may be taken as

$$g_1(\beta) = \frac{\beta^{a-1} e^{-\beta/b}}{\Gamma(a) b^a}, \quad a, b > 0, \quad \beta > 0, \tag{5.1}$$

$$g_2(\alpha|\beta) = \frac{\alpha^{u-1} e^{-\alpha/\beta}}{\Gamma(u) \beta^u}, \quad u, \beta > 0, \quad \alpha > 0. \tag{5.2}$$

Assume that the prior density for the parameter p is the Beta distribution B(v, w) which take the form

$$g_3(p) = \frac{p^{v-1}(1-p)^{w-1}}{B(v,w)}, \quad 0 \le p \le 1. \tag{5.3}$$

Hence, the joint prior distribution for α, β and p is

$$g(\alpha, \beta, p) = \frac{b^{-a}}{B(v,w)\Gamma(u)\Gamma(a)} \alpha^{u-1} \beta^{a-u-1} p^{v-1} (1-p)^{w-1} e^{-(\alpha/\beta + \beta/b)}. \tag{5.4}$$

From (4.1) and (5.4) we get the joint posterior $q(\alpha, \beta, p|\underline{x})$ as

$$\begin{aligned} q(\alpha, \beta, p|\underline{x}) = KCk^m \prod_{i=1}^{m} \alpha^u \beta^{a+\alpha-u-1} p^{v-1} (1-p)^{\lambda_i + w - 1} x_i^{\alpha-1} \\ \times M(1 - pe^{-(\beta x_i)^\alpha} M)^{-(\lambda_i+1)} e^{-\lambda_i (\beta x_i)^\alpha - (\alpha/\beta + \beta/b)}. \end{aligned} \tag{5.5}$$

where K is the normalizing constant given from

$$K^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty q(\alpha, \beta, p|\underline{x}) dp d\alpha d\beta.$$

Under squared error loss function, the Bayes estimator of a function $u(\alpha, \beta, p)$ is the posterior mean of the function and is given by a ratio of three integrals as follows

$$\hat{u}_B(\alpha,\beta,p) = E(u(\alpha,\beta,p|\underline{x})) = \int_\alpha \int_\beta \int_p u(\alpha,\beta,p) q(\alpha,\beta,p \mid \underline{x}) \quad dp d\beta d\alpha. \tag{5.6}$$

Under Linex loss function, the Bayes estimator of $u(\alpha,\beta,p)$ is given by

$$\begin{aligned} \hat{u}_B(\alpha,\beta,p) &= -\frac{1}{\xi} lnM(E(e^{-\xi u(\alpha,\beta,p)} \mid \underline{x})M) \\ &= -\frac{1}{\xi} lnM(\int_\alpha \int_\beta \int_p e^{-\xi u(\alpha,\beta,p)} q(\alpha,\beta,p|\underline{x}) dp d\beta d\alpha M) \end{aligned} \tag{5.7}$$

It is clear from Equations (5.6) and (5.7), that both of the integrals can not be obtained in a simple closed form and hence numerical methods of integration must be used. Therefore, we use the Monte Carlo integration sampling procedure to compute Bayes estimate under two different types of loss functions.

5.1 Bayes estimation using Monte Carlo integration

The Bayes estimators of the parameters α, β and p can not be obtained in simple closed form. So, we can use MCI procedure to get the Bayes estimators of the parameters.

We can obtain Bayes estimation using MCI by generating α_i, β_i and p_i, $i = 1,2,\ldots,s$, from the prior distribution given by (5.1), (5.2) and (5.3), respectively. Then, we have the Bayes estimators under squared error and Linex loss functions as the following.

5.1.1 The Bayes estimators under the squared error loss function

We can write the Bayes estimates of α, β and p under the squared error loss function as

$$\hat{\alpha}_{BS} = \frac{\sum_{i=1}^{s} \alpha_i L(\alpha_i,\beta_i,p_i;\underline{x})}{\sum_{i=1}^{s} L(\alpha_i,\beta_i,p_i;\underline{x})}, \tag{5.8}$$

$$\hat{\beta}_{BS} = \frac{\sum_{i=1}^{s} \beta_i L(\alpha_i,\beta_i,p_i;\underline{x})}{\sum_{i=1}^{s} L(\alpha_i,\beta_i,p_i;\underline{x})}, \tag{5.9}$$

and

$$\hat{p}_{BS} = \frac{\sum_{i=1}^{s} p_i L(\alpha_i,\beta_i,p_i;\underline{x})}{\sum_{i=1}^{s} L(\alpha_i,\beta_i,p_i;\underline{x})}, \tag{5.10}$$

respectively.

5.1.2 The Bayes estimators under Linex loss function

The posterior expectation with respect to the posterior density of α, β and p are given, respectively, by

$$E_{\alpha}(e^{-\xi\alpha}|\underline{x}) = \frac{\sum_{i=1}^{s} e^{-\xi\alpha_i} L(\alpha_i, \beta_i, p_i; \underline{x})}{\sum_{i=1}^{s} L(\alpha_i, \beta_i, p_i; \underline{x})}, \tag{5.11}$$

$$E_{\beta}(e^{-\xi\beta}|\underline{x}) = \frac{\sum_{i=1}^{s} e^{-\xi\beta_i} L(\alpha_i, \beta_i, p_i; \underline{x})}{\sum_{i=1}^{s} L(\alpha_i, \beta_i, p_i; \underline{x})}, \tag{5.12}$$

and

$$E_{p}(e^{-\xi p}|\underline{x}) = \frac{\sum_{i=1}^{s} e^{-\xi p_i} L(\alpha_i, \beta_i, p_i; \underline{x})}{\sum_{i=1}^{s} L(\alpha_i, \beta_i, p_i; \underline{x})}. \tag{5.13}$$

Hence, the Bayes estimates of α, β and p under Linex loss function are given, respectively, by

$$\hat{\alpha}_{BL} = -\frac{1}{\xi} \ln M[\frac{\sum_{i=1}^{s} e^{-\xi\alpha_i} L(\alpha_i, \beta_i, p_i; \underline{x})}{\sum_{i=1}^{s} L(\alpha_i, \beta_i, p_i; \underline{x})} M], \tag{5.14}$$

$$\hat{\beta}_{BL} = -\frac{1}{\xi} \ln M[\frac{\sum_{i=1}^{s} e^{-\xi\beta_i} L(\alpha_i, \beta_i, p_i; \underline{x})}{\sum_{i=1}^{s} L(\alpha_i, \beta_i, p_i; \underline{x})} M], \tag{5.15}$$

and

$$\hat{p}_{BL}(t) = -\frac{1}{\xi} \ln M[\frac{\sum_{i=1}^{s} e^{-\xi p_i} L(\alpha_i, \beta_i, p_i; \underline{x})}{\sum_{i=1}^{s} L(\alpha_i, \beta_i, p_i; \underline{x})} M]. \tag{5.16}$$

6. Numerical computations

The performance of the different methods cannot be compared theoretically. Based on Progressive First-Failure Censoring Scheme, the different estimators are computed and compared numerically for different combinations of n, m, k and random censoring scheme R. All computations were performed using Mathematica 7.0. We mainly compare the performance of the MLEs and Bayes estimators of the unknown parameters p and β, where α is known, under two different losses by using the Monte Carlo simulation.

The comparison between the estimates is taking place according to the following steps.

(1) For a given prior parameters we generate α, β and p from the joint prior density given by (5.4).

(2) For different combinations of n, m, k with the generated parameters α, β and p in step (1), we generate a progressive first-failure censored sample of size m from the density function with pdf (3.1).

(3) The likelihood estimators are then obtained by solving the two nonlinear equations given by (4.4) and (4.5) numerically.

(4) The Bayes estimators are then obtained by applying the Monte Carlo integration technique (MCI) under squared error loss function , given by (5.9) and (5.10), and under Linex loss function, given by (5.15) and (5.16).

(5) The above four steps are repeated and the quantities $(\hat{\theta} - \theta)^2$ are computed where $\hat{\theta}$ stands for an estimate of θ (ML or Bayes).

Progressive first-failure censored data with random removals were generated using the algorithm described in Balakrishnan and Sandhu [4].

From the priors $Gamma(2,3)$ we generate $\beta = 1.1099$, $Gamma(3, \frac{1}{\beta})$ we generate $\alpha = 2.5276$ and from $Beta(2,4)$ we generate $p = 0.5629$.

TABLE 1. Mean square error of the estimator $\hat{\beta}$

			ML	MCI	
k	n	m	$ER(\hat{\beta}_{ML})$	$ER(\hat{\beta}_{BS})$	$ER(\hat{\beta}_{BL})$
3	30	10	0.2545	0.0954	0.0702
		20	0.1992	0.0509	0.0408
	50	30	0.1689	0.0406	0.0337
		40	0.1343	0.0318	0.0269
5	30	10	0.2391	0.0781	0.0595
		20	0.1820	0.0438	0.0359
	50	30	0.2155	0.0314	0.0267
		40	0.1672	0.0239	0.0207

TABLE 2. Mean square error of the estimator $\hat{p}$

			ML	MCI	
k	n	m	$ER(\hat{p}_{ML})$	$ER(\hat{p}_{BS})$	$ER(\hat{p}_{BL})$
3	30	10	0.1065	0.0297	0.0351
		20	0.1042	0.0273	0.0329
	50	30	0.0877	0.0244	0.0303
		40	0.0826	0.0173	0.0233
5	30	10	0.1102	0.0288	0.0342
		20	0.0952	0.0252	0.0309
	50	30	0.1034	0.0238	0.0296
		40	0.0860	0.0158	0.0217

7. Conclusions

From the results, in tables (1) and (2), it can be observed that the Bayes estimates under the symmetric (SEL) and the asymmetric (Linex) loss functions are generally better than their corresponding MLEs. It can also be seen that the mean squared errors decrease as the sample sizes increase.

Conflict of Interests

The authors declare that there is no conflict of interests.

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