STRONG COMMUTATIVITY PRESERVING BIDERIVATIONS ON PRIME RINGS

FAIZA SHUJAT¹, ABU ZAID ANSARI²∗, SHAHOOR KHAN³

¹Department of Mathematics, Taibah University, Madinah, Saudi Arabia
²Department of Mathematics, Faculty of Science, Islamic University, Madinah, Saudi Arabia
³Department of Mathematics, Government Degree College, Surankote 185121, Jammu and Kashmir, India

Abstract. In the present paper we investigate the commutativity in a prime ring R which admits biderivation 
\( D : R \times R \rightarrow R \) satisfying \( [D(x,x), D(y,y)] = [x,y] \) for all \( x, y \in R \). More precisely, we generalize the result of Bell et.al.[3] on strong commutativity preserving biderivations. Moreover, we obtain that generalized biderivation acts as left bimultiplier, whenever it behaves as right \( R \)-homomorphisms.

Keywords: prime (semiprime) ring; symmetric biderivation; generalized biderivation; bimultiplier.

2010 AMS Subject Classification: 16R50, 16W25, 16N60.

1. Introduction

Throughout the paper \( R \) will denote a ring with centre \( Z(R) \). A ring \( R \) is said to be prime (resp. semiprime) if \( aRb = \{0\} \) implies that either \( a = 0 \) or \( b = 0 \) (resp. \( aRa = \{0\} \) implies that \( a = 0 \)). We shall write \( [x,y] \) the commutator \( xy - yx \). An additive mapping \( d : R \rightarrow R \) is said to be a derivation if \( d(xy) = d(x)y + xd(y) \), for all \( x, y \in R \). A derivation \( d_a \) is inner if there exists

∗Corresponding author

E-mail address: ansari.abuzaid@gmail.com

Received May 25, 2016; Published March 1, 2017
a \in R \text{ such that } d_a(x) = [a,x], \text{ for all } x \in R. \text{ Maksa [6] introduced the concept of symmetric biderivations. A mapping } D : R \times R \rightarrow R \text{ is said to be symmetric if } D(x,y) = D(y,x), \text{ for all } x,y \in R. \text{ A mapping } f : R \rightarrow R \text{ defined by } f(x) = D(x,x), \text{ where } D : R \times R \rightarrow R \text{ is a symmetric and biadditive mapping, is called the trace of } D. \text{ The trace } f \text{ of } D \text{ satisfies the relation } f(x + y) = f(x) + f(y) + 2D(x,y), \text{ for all } x,y \in R. \text{ A biadditive mapping } D : R \times R \rightarrow R \text{ is called a biderivation if for every } x \in R, \text{ the map } y \mapsto D(x,y) \text{ as well as for every } y \in R, \text{ the map } x \mapsto D(x,y) \text{ is a derivation of } R, \text{ i.e., } D(xy,z) = D(x,z)y + xD(y,z) \text{ for all } x,y,z \in R \text{ and } D(x,yz) = D(x,y)z + yD(x,z) \text{ for all } x,y,z \in R.

It was shown in [6] that symmetric biderivations are related to general solution of some functional equations. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping } f : R \rightarrow R \text{ gives rise to a biderivation on } R. \text{ Namely linearizing } [x,f(x)] = 0 \text{ for all } x,y \in R, \text{ (x,y) \mapsto [f(x),y] \text{ is a biderivation. Typical examples are mapping of the form (x,y) \mapsto \lambda [x,y] \text{ for all } x,y \in R, \text{ where } \lambda \in C, \text{ the extended centroid of } R. \text{ We shall call such maps inner biderivations. There has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. Some results on symmetric biderivation in prime and semiprime rings can be found in [1, 5, 8].}

For any semiprime (prime) ring } R, \text{ one can construct Martindale ring of quotients } Q \text{ of } R \text{ (see [2]). As } R \text{ can be embedded isomorphically in } Q, \text{ we consider } R \text{ as a subring of } Q. \text{ If the element } q \in Q \text{ commutes with every element in } R, \text{ then } q \in C, \text{ the centre of } Q. \text{ C contains the centroid of } R \text{ and it is called the extended centroid of } R. \text{ In general, } C \text{ is a von Neumann regular ring and } C \text{ is a field if and only if } R \text{ is prime [2, Theorem 5]. For more details on Martindale ring of quotients, one can see [4].}

A map } f : R \rightarrow R \text{ is centralizing on } R \text{ if } [f(x),x] \in Z(R) \text{ for all } x \in R; \text{ in particular if } [f(x),x] = 0 \text{ for all } x \in R, \text{ then } f \text{ is called commuting on } R. \text{ A map } f : R \rightarrow R \text{ is called commutativity preserving on } R \text{ if } [f(x),f(y)] = 0 \text{ whenever } [x,y] = 0, \text{ for all } x,y \in R. \text{ In particular, if } [f(x),f(y)] = [x,y] \text{ for all } x,y \in R, \text{ then } f \text{ is called strong commutativity preserving on } R. \text{ In the sequel several results has been proved for strong commutativity preserving condition for}
example [3] and the references there in.

2. Strong Commutativity Preserving Biderivations

To prove our main results, we require the following lemmas which play key role in the proof of theorems.

**Lemma 2.1.** [5] Let $S$ be a set and $R$ be a semiprime ring. If functions $f$ and $g$ of $S$ into $R$ satisfy $f(s)xg(t) = g(s)xf(t)$ for all $s, t \in S$, $x \in R$, then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that $\varepsilon_i \varepsilon_j = 0$ for $i \neq j$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ and $\varepsilon_1 f(s) = \lambda \varepsilon_1 g(s)$, $\varepsilon_2 g(s) = 0$, $\varepsilon_3 f(s) = 0$ for all $s \in S$.

**Lemma 2.2.** [5] Let $R$ be a 2-torsionfree semiprime ring, and let $g : R \rightarrow R$ be a centralizing additive mapping. Then there exist $\lambda \in C$ and an additive mapping $\xi : R \rightarrow C$ such that $g(x) = \lambda x + \xi(x)$ for all $x \in R$.

**Theorem 2.1.** Let $R$ be a 2-torsionfree prime ring. If $R$ admits a symmetric biderivation $D$ with trace $f$ satisfying $[D(x,x), D(y,y)] = [x,y]$ for all $x, y \in R$, then there exist an idempotent $\varepsilon \in C$ and an element $\alpha \in C$ such that the algebra $(1 - \varepsilon)R$ is commutative and $\varepsilon D(x,y) = \alpha \varepsilon [x,y]$ for all $x, y \in R$.

**Proof** Consider $[D(x,x), D(y,y)] = [x,y]$ for all $x, y \in I$. If $D = 0$, then we have $[x,y] = 0$ and hence $R$ is commutative. Thus $D \neq 0$ and we have

$$[D(x,x), D(y,y)] = [x,y] \text{ for all } x, y \in R. \quad (2.1)$$

Linearization of (2.1) in $x$ yields that

$$2[D(x,z), f(y)] + [f(x), f(y)] + [f(z), f(y)] - [x,y] - [z,y] = 0 \text{ for all } x, y, z \in R. \quad (2.2)$$

Comparing (2.1) and (2.2), we get

$$2[D(x,z), f(y)] = 0 \text{ for all } x, y, z \in R. \quad (2.3)$$
Since $R$ is 2-torsionfree, we have

\[(2.4) \quad [D(x, z), f(y)] = 0 \text{ for all } x, y, z \in R.\]

Replacing $x$ by $xu$ in (2.4) and using (2.4), we find

\[(2.5) \quad D(x, z)[u, f(y)] + [z, f(y)]D(x, u) = 0 \text{ for all } u, x, y, z \in R.\]

Substitute $uf(y)$ for $u$ in (2.5) to get

\[(2.6) \quad D(x, z)[u, f(y)]f(y) + [z, f(y)]uD(x, f(y)) + [z, f(y)]D(x, u)f(y) = 0 \text{ for all } u, x, y, z \in R.\]

Application of (2.5) yields that

\[(2.7) \quad [z, f(y)]uD(x, f(y)) = 0 \text{ for all } u, x, y, z \in R.\]

Since $R$ is prime, we have either $D(x, f(y)) = 0$ or $[z, f(y)]$ for all $x, y, z \in R$. If $D(x, f(y)) = 0$ for all $x, y \in R$, then $D = 0$ by [7]. Which leads to a contradiction. Now consider the later case

\[(2.8) \quad [w, f(y)] = 0 \text{ for all } w, y \in R.\]

Linearizing (2.8) and using 2-torsion freeness of $R$, we find

\[(2.9) \quad [w, D(y, z)] = 0 \text{ for all } w, y, z \in R.\]

Replacing $y$ by $yu$ in (2.9) and using (2.9), we obtain

\[(2.10) \quad [w, y]D(u, z) + D(y, z)[w, u] = 0 \text{ for all } u, w, y, z \in R.\]

Substitute $uv$ for $u$ in (2.10) and use (2.10) to get

\[(2.11) \quad [w, y]uD(v, z) + D(y, z)u[w, v] = 0 \text{ for all } u, v, w, y, z \in R.\]

Define $S : R \times R \longrightarrow R$ by $S(x, y) = [x, y]$. Then (2.11) reduces to the form

\[(2.12) \quad S(y, w)uD(v, z) = D(y, z)uS(w, v) \text{ for all } u, v, w, y, z \in R.\]

Applying Lemma 2.1, there exist mutually orthogonal idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ and $\varepsilon_1 D(x, y) = \lambda \varepsilon_1 [x, y]$, $\varepsilon_2[x, y] = 0$, $\varepsilon_3 D(x, y) = 0$ for all $x, y \in R$. Setting $\varepsilon_1 + \varepsilon_3 = \varepsilon$ and $\alpha = \lambda \varepsilon_1$, then we have $\varepsilon D(x, y) = \alpha \varepsilon [x, y]$ for all $x, y \in R$. 
Theorem 2.2. Let $R$ be a 2-torsionfree prime ring and $g : R \rightarrow R$ be any additive map. If $R$ admits a symmetric biderivation $D$ with trace $f$ satisfying $[D(x, x), g(y)] = [x, y]$ for all $x, y \in R$, then one of the conditions hold

1. $D = 0$;
2. there exist $\lambda \in C$ and an additive mapping $\xi : R \rightarrow C$ such that $g(x) = \lambda x + \xi(x)$ for all $x \in R$.

Proof consider the given condition

\begin{equation}
[D(x, x), g(y)] = [x, y] \text{ for all } x, y \in R. \tag{2.13}
\end{equation}

Linearization of (2.13) in $x$ yields that

\begin{equation}
2[D(x, z), g(y)] + [f(x), g(y)] + [f(z), g(y)] - [x, y] - [z, y] = 0 \text{ for all } x, y, z \in R. \tag{2.14}
\end{equation}

Comparing (2.13) and (2.14), we get

\begin{equation}
2[D(x, z), g(y)] = 0 \text{ for all } x, y, z \in R. \tag{2.15}
\end{equation}

Since $R$ is 2-torsionfree, we have

\begin{equation}
[D(x, z), g(y)] = 0 \text{ for all } x, y, z \in R. \tag{2.16}
\end{equation}

Replacing $x$ by $xu$ in (2.16) and using (2.16), we find

\begin{equation}
D(x, z)[u, g(y)] + [z, g(y)]D(x, u) = 0 \text{ for all } u, x, y, z \in R. \tag{2.17}
\end{equation}

Substitute $f(v)z$ for $z$ in (2.17) and use (2.17) to obtain

\begin{equation}
D(x, f(v))[u, g(y)] + [f(v), g(y)]zD(x, u) = 0 \text{ for all } u, v, x, y, z \in R. \tag{2.18}
\end{equation}

Application of (2.16) enable us to have

\begin{equation}
D(x, f(v))[u, g(y)] = 0 \text{ for all } u, v, x, y, z \in R. \tag{2.19}
\end{equation}

Primeness of $R$ implies that either $D(x, f(v)) = 0$ or $[g(y), u] = 0$ for all $x, y, v, u \in R$. If $D(x, f(v)) = 0$ for all $x, v \in R$, then $D = 0$ by [7].

Next consider the case when $[g(y), u] = 0$ for all $u, y \in R$. Applying Lemma 2.2, we obtain a
\( \lambda \in C \) and an additive mapping \( \xi : R \rightarrow C \) such that \( g(y) = \lambda y + \xi(y) \) for all \( y \in R \). This completes the proof.

3. **Generalized Biderivation Acts as Homomorphism**

Let \( R \) be a ring and \( D : R \times R \rightarrow R \) be a biadditive map. A biadditive mapping \( \Delta : R \times R \rightarrow R \) is said to be a generalized biderivation if for every \( x \in R \), the map \( y \mapsto \Delta(x,y) \) is a generalized derivation of \( R \) associated with function \( y \mapsto D(x,y) \) as well as if for every \( y \in R \), the map \( x \mapsto \Delta(x,y) \) is a generalized derivation of \( R \) associated with function \( x \mapsto D(x,y) \) for all \( x, y \in R \). It also satisfies \( \Delta(x,yz) = \Delta(x,y)z + yD(x,z) \) and \( \Delta(xy,z) = \Delta(x,z)y + xD(y,z) \) for all \( x,y,z \in R \). For example consider a biderivation \( \theta \) of \( R \) and biadditive a function \( \phi : R \times R \rightarrow R \) such that \( \phi(x,yz) = \phi(x,y)z \) and \( \phi(xy,z) = \phi(x,z)y \) for all \( x,y,z \in R \). Then \( \theta + \phi \) is a generalized biderivation of \( R \). The trace \( g \) of \( \Delta \) is defined as \( \Delta(x,x) = g(x) \), which satisfies \( g(x+y) = g(x) + g(y) + \Delta(x,y) + \Delta(y,x) \) for all \( x,y \in R \).

A generalized biderivation \( \Delta \) is said to be a right \( R \)-homomorphism on an ideal \( I \) (or left \( R \)-homomorphism) if \( \Delta(x, yr) = \Delta(x,y)r \) and \( \Delta(xr, y) = \Delta(x,y)r \) (or \( \Delta(x, ry) = r\Delta(x,y) \) and \( \Delta(rx, y) = r\Delta(x,y) \) for all \( x, y \in I, r \in R \).

**Theorem 3.1.** Let \( R \) be a prime ring and \( I \) be a nonzero right ideal of \( R \). If \( \Delta \) is a generalized biderivation associated with \( D \) such that \( \Delta \) is a right \( R \)-homomorphism, then \( \Delta \) act as a left bimultiplier.

**Proof** Since \( \Delta \) is a right \( R \)-homomorphism, we have

\[
\Delta(x, yr) = \Delta(x,y)r \quad \text{for all } x, y \in I, \ r \in R.
\]

But we also have

\[
\Delta(x, yr) = \Delta(x,y)r + yD(x,r) \quad \text{for all } x, y \in I, \ r \in R.
\]
Comparing (3.1) and (3.2), we get

\[ yD(x, r) = 0 \text{ for all } x, y \in I, \ r \in R. \]

Since the right annihilator of \( I \) is zero, we get \( D(x, r) = 0 \) for all \( x \in I, \ r \in R \). This implies that \( D(x, z) = 0 \) for all \( x, z \in I \). We get \( \Delta(x, yw) = \Delta(x, y)w + yD(x, w) = \Delta(x, y)w \) for all \( x, y, w \in I \). Hence \( \Delta \) act as left bimultiplier.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**REFERENCES**


