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A NEW RELAXATION AND QUICK BOUNDS FOR LINEAR PROGRAMMING PROBLEM

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Abstract. In this paper a new relaxation is proposed for linear programming problem. Based on this relaxation very quick bounds are found for the problem and the associated integer programming restriction which can be used in a tree search algorithm to find the optimal IP solution. Some cost allocation strategies are described to allocate the column cost among nonzero entries of the column which leads to different bounds for the problem. A number of linear programming problems are randomly generated and computational results are presented.

Keywords: linear program; relaxation; cost allocation strategy.

2010 AMS Subject Classification: 90C05.

1. Introduction

The development of linear programming has been ranked among the most important scientific advances of the mid-20th century, and we must agree with this assessment. Its impact since just 1950 has been extraordinary. Today it is a standard tool that has saved many thousands or millions of dollars for most companies or businesses of even moderate size in the various industrialized countries of the world; and its use in other sectors of society has been spreading rapidly [4]. Linear programming problem can be solved either by simplex or one algorithm in

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the family of interior point methods. The simplex method [7, 8] - an exponential time (non-polynomial time) algorithm - or it's variation has been used and is being used to solve almost any linear programming problems for the last four decades. In 1979, Khachiyan proposed the ellipsoid method, the first polynomial - time(interior - point) algorithm, to solve linear programming problems[5]. In 1984 Karmarkar suggested the second polynomial time $(O(n^{3.5}))$ algorithm based on projective transformation [8]. Paulseng and *et al* [7] proposed a number of relaxation methods for linear programs. Their methods may be viewed as a generalized coordinate descent method where by the descent directions are chosen from a set. These methods may be alternatively used as an extension of the relaxation methods for network flow problem. Their computational results show that the relaxation method is faster than standard simplex [7]. EL-Darzi established that the set covering problem can be relaxed as assignment problem, shortest rout problem, maximal flow problem and minimal spanning tree [3]. Djannaty and *et al* [1, 2] proposed another relaxation for the set problems. The rest of the paper is organized as follows; In section 2 a new relaxation for linear programming problem is presented. In section 3 an upper bound is described for the relaxation and the original problem. In section 4 the optimality of the upper bound is proved. In section 5 a number of cost allocation strategies are explained. In section 6 computational results are presented. In section 7 future work and conclusions are discussed.

New relaxation for LP

Although linear programming problem in any form can be relaxed by the proposed method the following canonical form is considered:

$$\begin{aligned} \max \quad & z=\textstyle\sum_{j=1}^{n} c_j x_j \\ \text{s.t} \quad & \\ & \sum_{j=1}^{n} a_{ij}x_j \le b_i \qquad (i=1,2,\dots,m) \\ & x_j \ge 0 \qquad (j=1,2,\dots,n) \end{aligned} \tag{1}$$

where all coefficients of a_{ij}, c_j, b_i are assumed to be nonnegative integers and the problem is bounded feasible.

Let

$$R_i = \left\{ j | a_{ij} \neq 0 \right\} \qquad (i = 1, 2, ..., m)$$

$$H_j = \left\{ i | a_{ij} \neq 0 \right\} \qquad (j = 1, 2, ..., n)$$

Associated with each nonzero a_{ij} a nonnegative variable y_{ij} is introduced, in other word, each variable x_j is replaced by $|H_j|$ variables y_{ij}'s, where $|H_j|$ is the cardinal of H_j . The coefficient of y_{ij} in the relaxed problem is the same as the coefficient of x_j in the i^{th} constraint of (1) where $i \in H_j$ and it's coefficient in the objective function depends on how the column cost c_j is distributed between coefficients of y_{ij}'s $i \in H_j$ in the objective function of (2). Dividing the cost c_j into $c_{1j}, c_{2j}, ..., c_{kj}$, where $c_{ij} \geq 0, k = |H_j|$ and attributing the cost c_{ij} to variable y_{ij} in the relaxed problem should be done in such away that $\sum_{i \in H_j} c_{ij} = c_j, j = 1, ..., n$. The proposed relaxation can be stated as follows:

$$\max \quad y = \sum_{i=1}^{m} \sum_{j \in R_i} c_{ij} y_{ij}$$

$$\text{s.t} \tag{2}$$

$$\sum_{j \in R_i} a_{ij} y_{ij} \leq b_i \qquad (i = 1, 2, ..., m)$$

$$y_{ij} \geq 0 \qquad j = 1, 2, ..., n, \quad i \in H_j$$

in order to establish that the problem (2) is indeed a relaxation of problem (1), we demonstrate that the solution set of problem(1) is a subset of the solution set of problem (2). Let $X = (x_1, x_2, ..., x_n)$ be a feasible solution of problem (1), if we set $y_{ij} = x_j$ for all $i \in H_j$ and $j = 1, ..., n$ then:

$$\sum_{j \in R_i} a_{ij} y_{ij} = \sum_{j \in R_i} a_{ij} x_j = \sum_{j=1}^{j=n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, ..., m)$$

and obvoiusly $y_{ij} = x_j \geq 0$ for all $i \in H_j$ and $j = 1, ..., n$ which shows that X is included in the solution set of problem (2).

It can be proved that the objective function of (1) is the same as the objective function of (2) for the above solution.

The distribution of cost c_j over variables $y_{ij},\ i \in H_j$is valid if and only if $\sum_{i\in H_j} c_{ij} = c_j$ and there are infinitely many ways to perform such a cost allocation. A cost allocation strategy is a way of determining the cost c_{ij} and attaching this cost to variable y_{ij} . in the objective function of (2). Therefore we can write:

$$y = \sum_{j=1}^{n} \sum_{i\in H_j} c_{ij} y_{ij} = \sum_{j=1}^{n} \sum_{i\in H_j} c_{ij} x_j = \sum_{j=1}^{n} x_j \sum_{i\in H_j} c_{ij} = \sum_{j=1}^{n} c_j x_j = z$$

which states that the objective functions values of (1) and (2) are the same using this solution.

Obtaining bounds for LP

Let problem (1) be both feasible and bounded and all coefficients of problem (1) are nonnegative integers. Given that a valid cost allocation strategy is undertaken and the relaxed problem (2) is constructed using this strategy an upper bound for the LP is obtained as follows:

$$y = \sum_{i=1}^{m} \sum_{j\in R_i} c_{ij} y_{ij} = \sum_{i=1}^{m} \sum_{j\in R_i} \frac{c_{ij}}{a_{ij}} a_{ij} y_{ij}$$

If we let

$$M_i = \max_{j\in R_i} \left\{ \frac{c_{ij}}{a_{ij}}, a_{ij} \neq 0 \right\} \quad i = 1, \ldots, m$$

then we have

$$y \leq \sum_{i=1}^{m} \sum_{j\in R_i} M_i a_{ij} y_{ij} = \sum_{i=1}^{m} M_i \sum_{j\in R_i} a_{ij} y_{ij} \leq \sum_{i=1}^{m} M_i b_i$$

which shows that $UB = \sum_{i=1}^{m} M_i b_i$is an upper bound for the optimal objective fuction value of both (1) and (2). Note that the quality of the bounds depends on c_{ij}'s.Therefore they depend on the strategy undertaken to distribute the cost c_j among the coefficients of the variables y_{ij} $i \in R_j$ in the objective function of (2).

In order to find the lower bound we consider the dual problem and in a similar way the lower bound to the optimal objective value of the dual problem is found. It goes without saying that the minimum value of the objective funtion in the feasible region of (1) is zero and by lower bound we mean the lower bound to the optimal objective function vaue of (1) which is the same in both primal and dual problem.

Optimality of the upper Bound of (2)

$$\max \quad y = \sum_{i=1}^{m} \sum_{j \in R_i} c_{ij} y_{ij}$$

$$st \quad \sum_{j \in R_i} a_{ij} y_{ij} \leq b_i \qquad (i = 1, 2, ..., m)$$

$$y_{ij} \geq 0, \quad (j = 1, 2, ..., n), (i \in H_j)$$

The constraints of this problem are similar to the constraints of the transportation problem without the demand constraints, that is, the constraints are independent of each other in the sense that each variable y_{ij} appears only in constraint i, and the objective function is a separable function and can be classified into m groups. Solving problem (2) is equivalent to solving the following m linear programming problems:

$$\max \quad y_i = \sum_{j \in R_i} c_{ij} y_{ij}$$

s.t

$$\sum_{j \in R_i} a_{ij} y_{ij} \leq b_i$$

$$y_{ij} \geq 0, \quad (j \in R_i)$$

where $i = 1, ..., m$. The optimal objective value of problem (2) is $\sum_{i=1}^{m} y_i^*$ where y_i^* is the optimal objective value of the i^{th} problem. The solution to the i^{th} problem can be found simply by checking all extreme points, that is, letting $y_{ij} = \frac{b_i}{a_{ij}}$, setting all other variable to zero and finding the corresponding objective function value, as follows:

$$y_i = \frac{b_i}{a_{ij}} c_{ij} \rightarrow y_i^* = \max_{j \in R_i} \left\{ \frac{c_{ij}}{a_{ij}} b_i \right\} = b_i \max_{j \in R_i} \left\{ \frac{c_{ij}}{a_{ij}} \right\}$$

or $y_i^* = b_i M_i$ where $M_i = \max_{j \in R_i} \left\{ \frac{c_{ij}}{a_{ij}} \right\}$. Therefore, the optimal solution to (2)is $y^* = \sum_{i=1}^{m} b_i M_i$ which is the same as the upper bound earlier found in section 2 for both (1) and (2). Hence, we proved that the upper bound is indeed the optimal objective value to problem (2) but an upper bound to (1). The above theorem apply to lower bound as well.

Cost allocation strategies

Although, there are infinite ways to perform a cost allocation strategy a number of useful ones are proposed below:

Strategy 1 In this strategy the cost c_j is divided among the coefficients of y_{ij} in the objective function of (2) where $i \in H_j$ in proportion to the coefficients of a_{ij}, $i \in H_j$ in (1). In other word, given that $\sum_{i \in H_j} a_{ij} \neq 0$ which is always true, we set

$$c_{ij} = \frac{a_{ij}}{\sum_{i \in H_j} a_{ij}} \times c_j \quad i \in H_j \ , j = 1, ..., n$$

which is a valid strategy because

$$\sum_{i \in H_j} c_{ij} = \sum_{i \in H_j} \frac{a_{ij}}{\sum_{i \in H_j} a_{ij}} \times c_j = \frac{c_j}{\sum_{i \in H_j} a_{ij}} \times \sum_{i \in H_j} a_{ij} = c_j$$

this strategy is the fastest one and produces good bounds, the corresponding M_i is

$$M_i = \max_{j \in R_i} \left\{ \frac{\frac{a_{ij}}{\sum_{i \in H_j} a_{ij}} \times c_j}{a_{ij}}, a_{ij} \neq 0 \right\} = \max_{j \in R_i} \left\{ \frac{c_j}{\sum_{i \in H_j} a_{ij}} \right\} \quad i = 1...m$$

Therefore the upper bound is simply

$$UB = \sum_{i=1}^{m} M_i b_i$$

strategy 2

In this strategy the cost c_j is divided among the coefficients of y_{ij} in the objective function of (2) where $i \in H_j$ in proportion to the quantity $i * a_{ij}$, $i \in H_j$. In other word, given that $\sum_{i \in H_j} i \times a_{ij} \neq 0$ which is always true, we set

$$c_{ij} = \frac{i \times a_{ij}}{\sum_{i \in H_j} i \times a_{ij}} \times c_j \quad i \in H_j \ , j = 1, ..., n$$

which is a valid strategy because

$$\sum_{i \in H_j} c_{ij} = \sum_{i \in H_j} \frac{i \times a_{ij}}{\sum_{i \in H_j} i \times a_{ij}} \times c_j = \frac{c_j}{\sum_{i \in H_j} i \times a_{ij}} \times \sum_{i \in H_j} i \times a_{ij} = c_j$$

this strategy is compare able with strategy 1. M_i can be found in a similar way to the other strategies. **Strategy 3**

In this strategy c_j is divided equally among the coefficients of variables $y_{ij}, i \in H_j$ in the objective function of (2), that is,

$$c_{ij} = \frac{c_j}{|H_j|}, i \in H_j$$

therefore

$$M_i = \max_{j \in R_i} \left\{ \frac{c_j}{|H_j| a_{ij}} \right\} \quad i = 1, ..., m$$

which depends on i. We show that this is a valid strategy because:

$$\sum_{i \in H_j} c_{ij} = \sum_{i \in H_j} \frac{c_j}{|H_j|} = \frac{c_j}{|H_j|} \sum_{i \in H_j} 1 = \frac{c_j}{|H_j|} \times |H_j| = c_j$$

Thus

$$UB = \sum_{i=1}^{m} M_i b_i$$

Strategy 4

In this strategy the cost c_j is divided among the coefficients of y_{ij} in the objective function of (2) where $i \in H_j$ in proportion to the quantity $i * a_{ij}/b_i$, $i \in H_j$. In other word, given that $\sum_{i\in H_j} i \times a_{ij}/b_i \neq 0$ which is always true, we set

$$c_{ij} = \frac{i \times a_{ij}/b_i}{\sum_{i\in H_j} i \times a_{ij}/b_i} \times c_j \quad i \in H_j\ , j = 1, \ldots, n$$

which is a valid strategy because

$$\sum_{i\in H_j} c_{ij} = \sum_{i\in H_j} \frac{i \times a_{ij}/b_i}{\sum_{i\in H_j} i \times a_{ij}/b_i} \times c_j = \frac{c_j}{\sum_{i\in H_j} i \times a_{ij}/b_i} \times \sum_{i\in H_j} i \times a_{ij}/b_i = c_j$$

this strategy is relatively stronger than strategy 1 and 2. M_i can be found in a similar way to the other strategies.

Numerical example

Let take the following simple linear programming problem and find the lower and upper bound using one cost allocation strategy:

$$\begin{aligned}
\min \quad & z = 3x_1 + 5x_2 \\
st \quad & 2x_1 + x_2 \geq 8 \\
& x_1 + 4x_2 \geq 10 \\
& 7x_1 + 6x_2 \geq 42 \\
& 4x_1 + 9x_2 \geq 36 \\
& x_1, x_2 \geq 0
\end{aligned}$$

The optimal solution is $(x_1, x_2) = (\frac{54}{13}, \frac{28}{13})$ and the optimal value is 23.2. Replacing x_1 by variables x_{11}, x_{12}, x_{13} and x_{14} and replacing x_2 by variables x_{12}, x_{22}, x_{32} and x_{42} the relaxed problem can be stated as follows; where c_{ij}'s are chosen such that $c_{11} + c_{21} + c_{31} + c_{41} = 3$ and

$c_{12}+c_{22}+c_{32}+c_{42}=5$. Although there are infinite ways to choose c_{ij}'s,

$$\min z=(c_{11}x_{11}+c_{21}x_{21}+c_{31}x_{31}+c_{41}x_{41})+(c_{12}x_{12}+c_{22}x_{22}+c_{32}x_{32}+c_{42}x_{42})$$

$s.t$

$$\begin{array}{lllllllll}
2x_{11}+ & & & & x_{12} & & & & \geq 8\\
 & x_{21}+ & & & & 4x_{22} & & & \geq 10\\
 & & 7x_{31}+ & & & & 6x_{32} & & \geq 42\\
 & & & 4x_{41}+ & & & & 9x_{42} & \geq 36\\
x_{11}, & x_{21}, & x_{31}, & x_{41}, & x_{12}, & x_{22}, & x_{32}, & x_{42} & \geq 0
\end{array}$$

Lower bound using Strategy 1

$$\min z=(\tfrac{3}{14}2x_{11}+\tfrac{3}{14}x_{21}+\tfrac{3}{14}7x_{31}+\tfrac{3}{14}4x_{41})+(\tfrac{5}{20}x_{12}+\tfrac{5}{20}4x_{22}+\tfrac{5}{20}6x_{32}+\tfrac{5}{20}9x_{42})$$

$$=\min\left\{\tfrac{3}{14},\tfrac{5}{20}\right\}=\tfrac{3}{14}$$

$$z\geq\tfrac{3}{14}2x_{11}+\tfrac{3}{14}x_{21}+\tfrac{3}{14}7x_{31}+\tfrac{3}{14}4x_{41}+\tfrac{3}{14}x_{12}+\tfrac{3}{14}4x_{22}+\tfrac{3}{14}6x_{32}+\tfrac{3}{14}9x_{42}$$

$$=\tfrac{3}{14}(2x_{11}+x_{12})+\tfrac{3}{14}(x_{21}+4x_{22})+\tfrac{3}{14}(7x_{31}+6x_{32})+\tfrac{3}{14}(4x_{41}+9x_{42})$$

$$z\geq\tfrac{3}{14}\times(8+10+42+36)=\tfrac{3}{14}\times 96=20.5$$

Upper bound using strategy 1

Let write the dual of the problem:

$$\max\quad y_0=8y_1+10y_2+42y_3+36y_4$$

s.t

$$\begin{aligned}
&2y_1+y_2+7y_3+4y_4\leq 3\\
&y_1+4y_2+6y_3+9y_4\leq 5\\
&y_1,\quad y_2,\quad y_3,\quad y_4\ \geq 0
\end{aligned}$$

The relaxed problem is:

$$\max \quad y_0 = 8(y_{11}+y_{21})+10(y_{12}+y_{22})+42(y_{13}+y_{23})+36(y_{14}+y_{24})$$

s.t

$$2y_{11}+y_{12}+7y_{13}+4y_{14} \le 3$$
$$y_{21}+4y_{22}+6y_{23}+9y_{24} \le 5$$
$$y_{11},y_{21},y_{31},y_{41},y_{12},y_{22},y_{32},y_{42} \ge 0$$

$$\max y_0 = (\tfrac{8}{3}2y_{11}+\tfrac{8}{3}y_{21})+(\tfrac{10}{5}y_{12}+\tfrac{10}{5}4y_{22})+(\tfrac{42}{13}7y_{13}+\tfrac{42}{13}6y_{23})+(\tfrac{36}{13}4y_{14}+\tfrac{36}{13}9y_{24})$$

$$\max\left\{\tfrac{8}{3},\tfrac{10}{5},\tfrac{42}{13},\tfrac{36}{13}\right\}=\tfrac{42}{13} \quad \text{and} \quad \max\left\{\tfrac{8}{3},\tfrac{10}{5},\tfrac{42}{13},\tfrac{36}{13}\right\}=\tfrac{42}{13}$$

$$y_0 \le \tfrac{42}{13}(2y_{11}+y_{12}+7y_{13}+4y_{14})+\tfrac{42}{13}(y_{21}+4y_{22}+6_{23}+9_{24}) \le \tfrac{42}{13}(3+5)=25.8$$

Computational experiment

Test problems

A number of linear programming problems are randomly generated where all the coefficients are assumed to be nonnegative integers. The details of these problems are presented in Table 1. All computations are done using Fortran 95 compiler on an Intel Core (TM)i 7-3537U CPU@ 2.00GHz 2.50GHz processor.

Obtaining bounds

The lower bounds and upper bounds are found by the proposed relaxation using strategy 1. Execution times are also inserted in columns 3 and 5 of Table 2.

In comparison to the optimal LP execution times, the time rquired to find the upper and lower bounds are negligible.

Comparison of different strategies

The above 4 strategies were applied on test problems and the upper bound were found. As is seen from table 3 strategy 3 is the worst and strategy 4 is relatively better than the other three.

Problem name	Problem Size	Density	Optimal Value	Iterations to Optimality	Execution time
100by70	100×70	0.965	5701.74	56	2.31
100by701	100×70	0.966	4956.95	36	2.112
100by702	100×70	0.967	6030.33	30	2.393
100by100	100×100	0.952	5055.71	76	2.106
100by1001	100×100	0.949	4940.59	79	2.068
100by1002	100×100	0.952	4950.12	89	1.915
100by2001	100×200	0.964	8266.64	207	4.596
100by2002	100×200	0.967	7931.37	154	3.136
100by2003	100×200	0.965	8586.50	148	2.315
100by2004	100×200	0.966	8106.30	165	2.538

TABLE 1. Charactrisitcs of the generated problems

Problem name	Upper bound	Execution time	Lower bound	Execution time
100by70	9114.72	0.024	2586.93	0.024
100by701	7864.18	0.029	2655.11	0.03
100by702	8983.73	0.066	2931.43	0.066
100by100	6888.48	0.019	2707.37	0.033
100by101	6805.98	0.052	2329.60	0.049
100by102	7382.05	0.075	2321.69	0.075
100by2001	11229.92	0.035	4022.65	0.053
100by2002	11094.73	0.128	3508.36	0.086
100by2003	11873.35	0.081	3952.96	0.047
100by2004	10930.28	0.065	3832.97	0.045

TABLE 2. problem bounds

Problem name	U.bound by st.1	U.bound by st.2	U.bound by st.3	U.bound by st.4	Average ex. time
100by70	9114.72	8900.76	85945.94	8762.11	0.023
100by701	7864.18	7399.56	72965.53	7216.51	0.033
100by702	8983.73	8988.21	86449.29	8700.99	0.013
100by100	6888.48	7119.93	54316.24	7046.80	0.033
100by101	6805.98	6834.13	52198.62	6732.36	0.052
100by102	7382.05	7878.19	54585.31	7822.49	0.075
100by201	11229.92	11111.21	133931.66	11027.02	0.035
100by202	11094.73	11156.76	127872.32	10978.0	0.014
100by203	11873.35	12434.82	135344.34	12186.63	0.034
100by204	10930.28	11226.19	133041.65	10957.06	0.047

TABLE 3. Upper bounds for different strategies

The more problem specific knowledge is employed in the cost allocation the better bounds are obtained.

Conclusions

A new relaxation was proposed for linear programming problem and quick bounds were found which suggests to be used in a branch and bound algorithm to find the optimal solution for the corresponding integer programing problems. Further investigation can be done to find stronger cost allocation strategies to approach the optimal objective value. As the parameter M_i's in $UB = \sum_{i=1}^{i=m} M_i b_i$ play a similar role to shadow prices in $y^* = \sum_{i=1}^{i=m} y_i^* b_i$, therefore, if we can find an upper bound strong enough to approach the optimal objective value, M_i tends to the i^{th} shadow price y_i^*.

Conflict of Interests

The authors declare that there is no conflict of interests.

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